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On pseudo-immersions of a surface into the plane

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1 Introduction

In this paper, all manifolds and maps are differentiable of class C^{∞}. Let M be a compact connected oriented surface with exactly one boundary component. For a map $F : M \to \mathbf{R}^2$, we define the set of singularities of F as $\Sigma(F) = \{q \in M \mid \mathrm{rank}\, dF_q < 2\}$. The map $F : M \to \mathbf{R}^2$ is called a *pseudo-immersion* if the following set of conditions is fulfilled:

1. There is some open neighborhood U of ∂M, such that $F|U : U \looparrowright \mathbf{R}^2$ is an orientation preserving immersion.
2. In the neighborhood of every singularity $x \in M$, F can be represented, in appropriate coordinate systems, by: $y_1 = x_1, y_2 = x_2^2$. We call this type of singularity a *fold singularity*.

Note that if $F : M \to \mathbf{R}^2$ is a pseudo-immersion, then $\Sigma(F)$ is a union of circles and $F|\Sigma(F)$ is an immersion. A pseudo-immersion was defined by Poénaru [6] for a smooth map $F : M^n \to N^n$ between n-manifolds. In his definition, he added a condition for the position of a singular set. In this paper, we do not consider an immersion as a pseudo-immersion.

Let M be a compact connected oriented surface with exactly one boundary component. The boundary ∂M has the induced orientation of M. That is, let n be the outward normal vector field of ∂M in M then, ∂M is oriented by the unit tangent vector τ such that the frame (n, τ) represents the positive orientation of M. Let $F : M \looparrowright \mathbf{R}^2$ be an orientation preserving immersion. The winding number $W(F|\partial)$ of the restricted immersion $F|\partial M$ is the degree of the map $dF(\tau) : \partial M = S^1 \to S^1$. By the Poincaré–Hopf's theorem, we have

$$W(F|\partial M) = \chi(M), \tag{1.1}$$

where $\chi(M)$ is the Euler characteristic class of M.

Our problem is the following: if $F : M \to \mathbf{R}^2$ is a pseudo-immersion, then what is the relation between $W(F|\partial), \chi(M)$ and $\sharp\Sigma(F)$? Here $\sharp\Sigma(F)$ is the number of connected components of $\Sigma(F)$.

Before stating the main theorem, we should define an invariant which relates to the number of singular set components.

Definition 1.1. For two odd integers χ and W, we define

$$m(\chi, W) = \begin{cases} \dfrac{\chi + W}{2} + 1 & \text{if } W > 0, \\ \dfrac{\chi - W}{2} & \text{if } W < 0. \end{cases} \tag{1.2}$$

The main theorems are the following.

Theorem 1.2. *Let $F : M \to \mathbf{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary component of the plane.*

(1.3) *If* $\chi(M) - W(F|\partial) \equiv 0 \pmod 4$, *then* $\sharp\Sigma(F) \geq \max\{m(\chi(M), W(F|\partial)), 2\}$.

(1.4) *If* $\chi(M) - W(F|\partial) \equiv 2 \pmod 4$, *then* $\sharp\Sigma(F) \geq \max\{m(\chi(M), W(F|\partial)), 1\}$.

Theorem 1.3. *For any fixed odd integer W and odd integer $\chi \leq 1$, there exists a pseudo-immersion $F : M \to \mathbf{R}^2$ of a compact connected oriented surface with exactly one boundary component such that*

$$\chi(M) = \chi, W(F|\partial) = W \tag{1.5}$$

and such that

$$\sharp\Sigma(F) = \max\{m(\chi, W), 2\} \quad \text{if } \chi - W \equiv 0 \pmod 4 \tag{1.6}$$

or

$$\sharp\Sigma(F) = \max\{m(\chi, W), 1\} \quad \text{if } \chi - W \equiv 2 \pmod 4. \tag{1.7}$$

Remark 1.4. Concerning Theorems 1.2 and 1.3, we note the following.

1. Nagase [5] introduced a folding-map. The singularity of a folding-map is the same as that of a pseudo-immersion, but it may attach the boundary of a source manifold. Nagase proved that any immersion of S^2 into the interior of a homotopy 3-ball V extends to a folding-map of D^3 into V whose fold-set consists of mutually disjoint disks.

2. Ekholm and Larsson [1] defined an admissible map. The singularity of an admissible map has not only fold singularities but also cusp singularities. For an admissible map of D^2 to the plane, Ekholm and Larsson expressed the minimal number of singular set components as a function of cusps and the normal degree of the image of the boundary curve of D^2.

3. Eliashberg [2] proved the existence of stable maps between oriented surfaces. Similar results of Theorems 1.2 and 1.3 for fold maps between oriented closed surfaces were found by the author [7].

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2 Preliminaries

In this section, we state an important tool to prove Theorem 1.2.

Let $F : M \to \mathbf{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary. Note that $\Sigma(F) \subset M$ is two colourable. Here, to say that a 1-dimensional submanifold $V \subset M$ is two colourable means that V divides M into a pair of nonempty open surfaces (B, R) of M such that $B \cap R = \emptyset$, $B \cup R = M \setminus V$ and the closures $\overline{B}$ and $\overline{R}$ of B and R in M respectively both contain V.

For a connected component $\gamma \subset \Sigma(F)$, we define the normal vector field ν_γ of $F(\gamma)$ as follows: ν_γ points towards the direction in which the number of preimages of the regular value near $F(\gamma)$ decreases. Since $F|\gamma : \gamma \hookrightarrow \mathbf{R}^2$ is an immersion, γ is oriented by the tangent vector field τ_γ such that the frame $(\nu_\gamma, dF(\tau_\gamma))$ represents the positive orientation of $\mathbf{R}^2$. The winding number $W(F|\gamma)$ is the degree of the map $dF(\tau_\gamma) : \gamma = S^1 \to S^1$ in which the source has the above orientation.

Let $N(\gamma) = \gamma \times [-1, 1]$ be a tubular neighborhood of $\gamma \subset \Sigma(F)$ such that $\gamma = \gamma \times \{0\}$ and we set $N(\Sigma(F)) = \bigcup_{\gamma \subset \Sigma(F)} N(\gamma)$. Let E be a connected open surface of $M \setminus N(\Sigma(F))$ such that $\overline{E} \cap N(\gamma) \neq \emptyset$. Since E is orientable and $F|E$ is an immersion, we define the orientation of E such that $F|E : E \hookrightarrow \mathbf{R}^2$ is an orientation preserving immersion. Each connected component of ∂E has the induced orientation of E. Note that if E contains ∂M, the induced orientations of ∂M from that of M and E are the same. Suppose that $\gamma \times \{i\}$ $(i = -1$ or $1)$ belongs to ∂E. Since the orientation of $\gamma \times \{i\}$ is the same as that of $\gamma \times \{0\}$, we have

$$W(F|\gamma \times \{i\}) = W(F|\gamma \times \{0\}). \tag{2.1}$$

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let $F : M \to \mathbf{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary component. Since $\Sigma(F) \subset M$ is two colourable, we set (B, R) as a two colour decomposition of the pair $(M, \Sigma(F))$ such that $\partial\overline{B}$ contains ∂M. By (3.1) and the fact that $\Sigma(F)$ is a closed 1-dimensional submanifold, we have

$$\chi(\overline{B}) = W(\Sigma(F)) + W(F|\partial), \tag{3.1}$$

$$\chi(\overline{R}) = W(\Sigma(F)), \tag{3.2}$$

$$\chi(M) = \chi(\overline{B}) + \chi(\overline{R}). \tag{3.3}$$

Therefore, we have

$$W(F|\partial) = \chi(M) - 2W(\Sigma(F)), \tag{3.4}$$

$$W(F|\partial) = \chi(\overline{B}) - \chi(\overline{R}). \tag{3.5}$$

Since $\chi(M)$ is odd, we have the following proposition.

Proposition 3.1. *The winding number $W(F|\partial)$ of the restricted immersion $F|\partial M$ is odd.*

Suppose that the number of connected components of $\overline{B}$ (resp. $\overline{R}$) is n_B (resp. n_R), the sum of the genuses of each connected component of B (resp. R) is g_B (resp. g_R) and the genus of M is g. Since the number of boundary components of B is equal to $\sharp\Sigma(F)+1$ and the number of boundary components of R is equal to $\sharp\Sigma(F)$, (3.3) and (3.5) are written as;

$$\chi(M) = 2n_B - 2g_B + 2n_R - 2g_R - 2\sharp\Sigma(F) - 1, \tag{3.6}$$

$$W(F|\partial) = 2n_B - 2g_B - 2n_R + 2g_R - 1. \tag{3.7}$$

Thus, we have

$$\chi(M) - W(F|\partial) = 4n_R - 4g_R - 2\sharp\Sigma(F). \tag{3.8}$$

By this equation, we have the following.

Proposition 3.2. *If* $\chi(M) - W(F|\partial) \equiv 0 \pmod 4$, *then the number of singular set components* $\sharp\Sigma(F)$ *is even. If* $\chi(M) - W(F|\partial) \equiv 2 \pmod 4$, *then the number of singular set components* $\sharp\Sigma(F)$ *is odd.*

Suppose that $W(F|\partial) > 0$. Then by (3.8), we have

$$\begin{aligned} \sharp\Sigma(F) &= \frac{-\chi(M) + W(F|\partial)}{2} + 2n_R - 2g_R \\ &\geq \frac{-\chi(M) + W(F|\partial)}{2} + 2 - 2g \\ &= \frac{\chi(M) + W(F|\partial)}{2} + 1. \end{aligned} \tag{3.9}$$

Here, g is the genus of M.

Suppose that $W(F|\partial) < 0$. Then instead of (3.8), we have

$$\chi(M) + W(F|\partial) = 4n_B - 4g_B - 2\sharp\Sigma(F) - 2. \tag{3.10}$$

Therefore,

$$\begin{aligned} \sharp\Sigma(F) &= \frac{-\chi(M) - W(F|\partial)}{2} + 2n_B - 2g_B - 1 \\ &\geq \frac{-\chi(M) - W(F|\partial)}{2} + 2 - 2g - 1 \\ &= \frac{\chi(M) - W(F|\partial)}{2}. \end{aligned} \tag{3.11}$$

Combining (3.9), (3.11) and Proposition 3.2, we have the desired inequalities. This completes the proof of Theorem 1.2.

4 Examples

To prove Theorem 1.3, it is necessary to construct the desired pseudo-immersions concretely by using Francis' theorem [4]. Instead of giving such pseudo-immersions in all the cases, in this section, we give typical examples.

4.1 The case of $\chi = 1 - 2g$ and $W = 2g - 1$

Let M_g be a closed oriented surface of the genus g and $M_{g,1} = M_g \setminus D^2$. It is obvious that $\chi(M_{g,1}) = 1-2g$. In this subsection, we construct a pseudo-immersion $F : M_{g,1} \to \mathbf{R}^2$ such that $W(F|\partial) = 2g - 1$ and $\sharp\Sigma(F) = m(1 - 2g, 2g - 1) = 1$.

Let $N(\partial M_{g,1}) = \partial M_{g,1} \times [-1, 0]$ be a tubular neighborhood of $\partial M_{g,1}$ such that $\partial M_{g,1} = \partial M_{g,1} \times \{0\}$. Let $F_1 : \overline{M_{g,1} \setminus N(\partial M_{g,1})} \looparrowright \mathbf{R}^2$ be an orientation preserving immersion and $F_2 : N(\partial M_{g,1}) \looparrowright \mathbf{R}^2$ an orientation preserving immersion such that $F_1|\partial M_{g,1} \times \{-1\} = F_2|\partial M_{g,1} \times \{-1\}$. Then, by attaching F_1 and F_2 and by changing the orientation of $\overline{M_{g,1} \setminus N(\partial M_{g,1})}$, we have a desired pseudo-immersion $F = F_1 \cup F_2 : M_{g,1} \to \mathbf{R}^2$ such that $W(F|\partial) = 2g - 1$, $\Sigma(F) = \partial M_{g,1} \times \{-1\}$. See Figure 1.

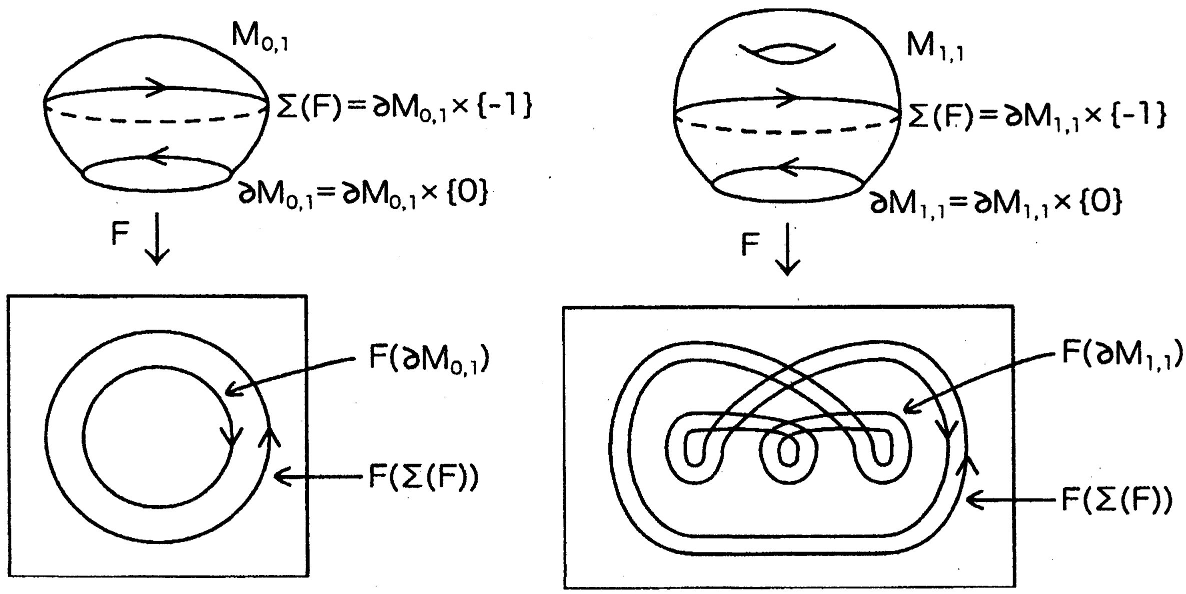

Figure 1: The cases of $g = 0, 1$.

4.2 The case of $\chi = 1$ and $W = -2n + 1$

Let n be a positive integer. In this subsection, we construct a pseudo-immersion $\widetilde{F} : M_{0,1} \to \mathbf{R}^2$ such that $W(\widetilde{F}|\partial) = -2n + 1$ and $\sharp\Sigma(\widetilde{F}) = m(1, -2n + 1) = n$.

Before constructing the desired pseudo-immersion, we will explain a boundary connected sum of two pseudo-immersions. Let $F : M \to \mathbf{R}^2$ and $G : N \to \mathbf{R}^2$ be two pseudo-immersions such that $F(M) \cap G(N) = \emptyset$. Let $I_a \times I_b$ be a rectangle of two closed intervals $I_a = I_b = [0, 1]$ and $H : I_a \times I_b \to \mathbf{R}^2$ an orientation preserving embedding. Let $i_M : \{0\} \times I_b \to \partial M$ and $i_N : \{1\} \times I_b \to \partial N$ be orientation reversing embeddings such that $F \circ i_M = G \circ i_N = H$. Then $F \cup_{i_M} H \cup_{i_N} G : M \cup_{i_M} I_a \times I_b \cup_{i_N} N \to \mathbf{R}^2$ is a pseudo-immersion. We denote $F \cup_{i_M} H \cup_{i_N} G$ as $F \natural G$ and $M \cup_{i_M} I_a \times I_b \cup_{i_N} N$ as $M \natural N$ and we call $F \natural G$ a boundary connected sum of F and G. Note that $W(F \natural G|\partial) = W(F|\partial) + W(G|\partial) - 1$. See Figure 2.

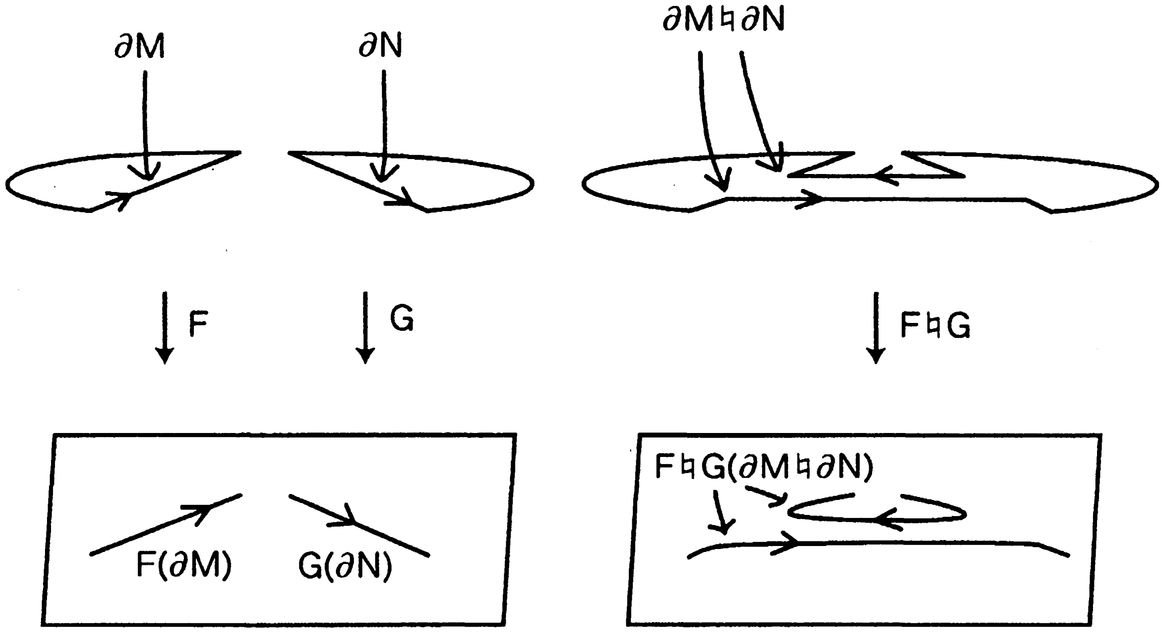

Figure 2: A boundary connected sum of two pseudo-immersions.

Let $F_i : M_{0,1} \to \mathbf{R}^2$ $(i = 1, 2, \ldots, n)$ be a copy of the pseudo-immersion which is constructed in Subsection 4.1. We take a boundary connected sum of $F_1, F_2, \ldots, F_n$. We set $\widetilde{F} = F_1 \natural F_2 \natural \cdots \natural F_n$ and we have $M_{0,1} \natural M_{0,1} \natural \cdots \natural M_{0,1} = M_{0,1}$. Because $W(\widetilde{F}|\partial) = -n - (n-1) = -2n + 1$ and $\sharp \Sigma(\widetilde{F}) = n$, the pseudo-immersion $\widetilde{F} : M_{0,1} \to \mathbf{R}^2$ is the desired one. See Figures 3 and 4 in the case $n = 3$.

5 Supplement

5.1 Position of the singular set

In this section, we remark on the positions of the singular set of a pseudo-immersion $F : M \to \mathbf{R}^2$.

Proposition 5.1. *Let F_1 and $F_2 : M \to \mathbf{R}^2$ be two pseudo-immersions of a compact connected oriented surface with exactly one boundary component. If $\Sigma(F_1) = \Sigma(F_2) = 1$, then an orientation preserving diffeomorphism $\Phi : M \to M$ such that $\Phi(\Sigma(F_1)) = \Sigma(F_2)$ exists.*

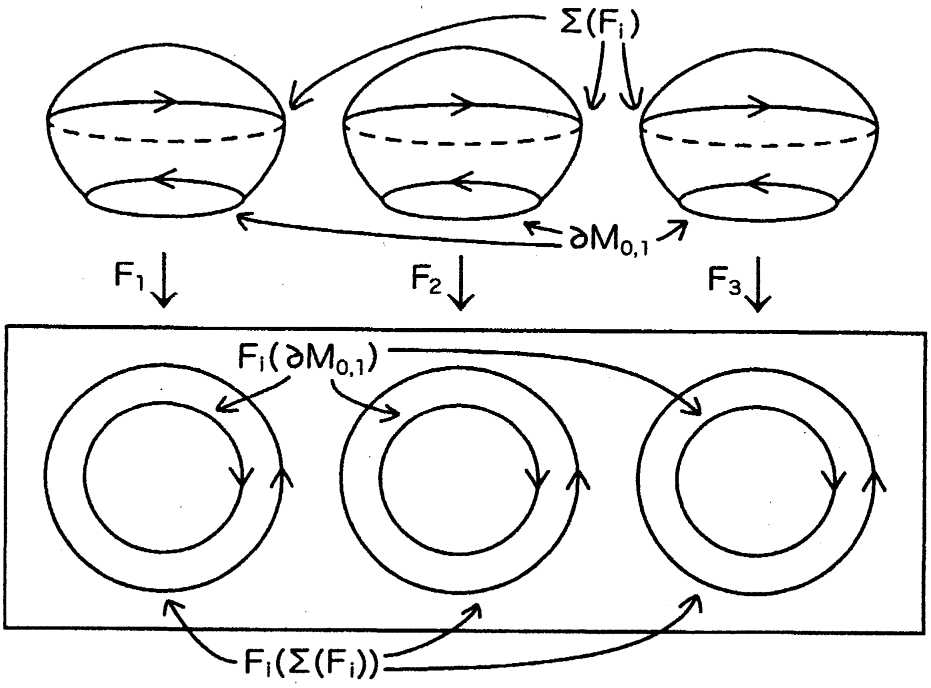

Figure 3: Pseudo-immersions $F_i : M_{0,1} \to \mathbf{R}^2$ $(i = 1, 2, 3)$.

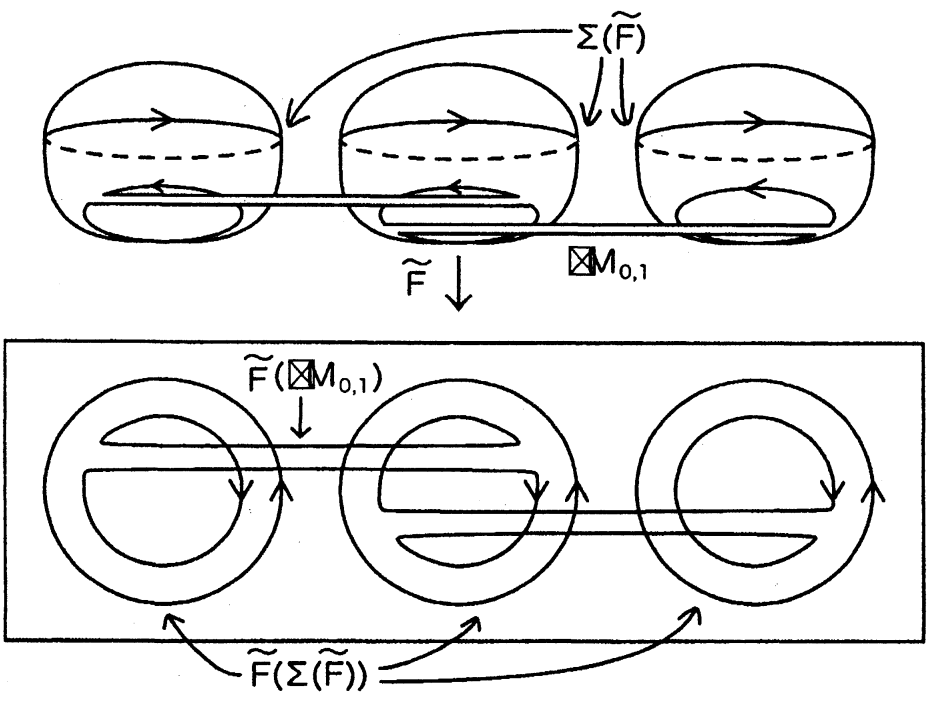

Figure 4: A pseudo-immersion $\widetilde{F} : M_{0,1} \to \mathbf{R}^2$.

This proposition is obvious. If the number of singular set components is more than one, the above proposition is not true. For example, let Σ_1 and Σ_2 be two simple closed curves in $M_{1,1}$ that splits $M_{1,1}$ into three connected surfaces. Two of them are annuli and the other is one punctured torus. Let Σ_3 and Σ_4 be two simple closed curves in $M_{1,1}$ that splits $M_{1,1}$ into two connected surfaces. Both of them are annuli. By using Francis' theorem [4], we have two pseudo-immersions F_1 and $F_2 : M_{1,1} \to \mathbf{R}^2$ such that $\Sigma(F_1) = \Sigma_1 \cup \Sigma_2$, $\Sigma(F_2) = \Sigma_3 \cup \Sigma_4$, $F_1(\Sigma_1) = F_2(\Sigma_3)$, $F_1(\Sigma_2) = F_2(\Sigma_4)$ $F_1(\partial M_{1,1}) = F_2(\partial M_{1,1})$ and $W(F_1|\partial) = W(F_2|\partial) = -1$. See Figure 5.

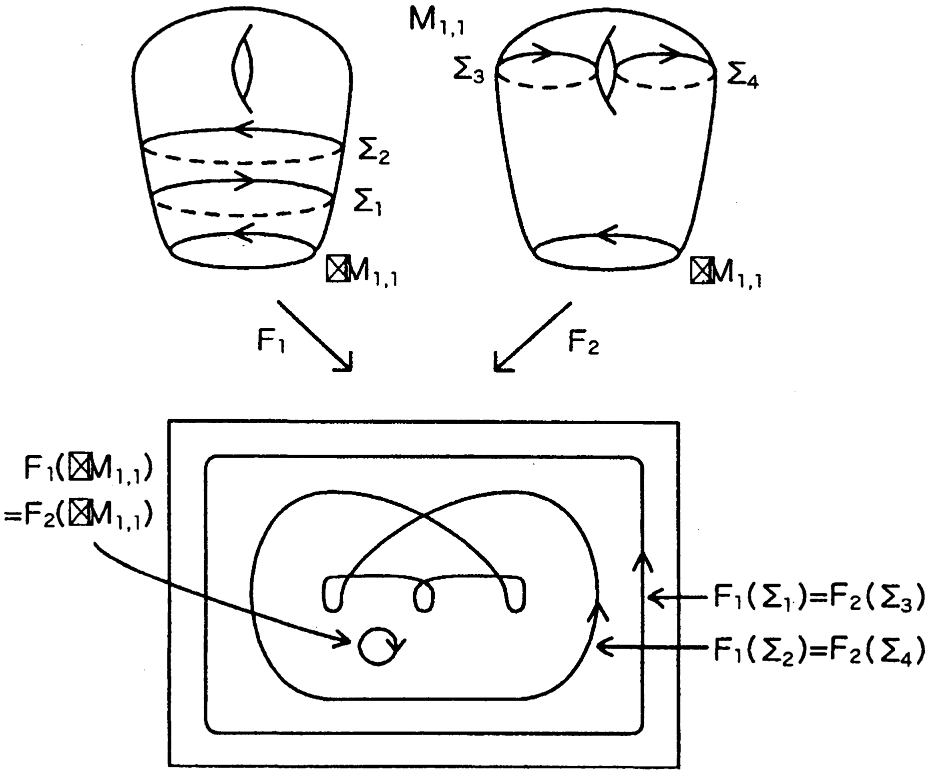

Figure 5: Two pseudo-immersions F_1 and $F_2 : M_{1,1} \to \mathbf{R}^2$ such that $F_1(\Sigma(F_1)) = F_2(\Sigma(F_2))$.

5.2 Image of the boundary of a pseudo-immersion

In this subsection, we state the existence of a pseudo-immersion such that the given plane curve is the image of the boundary of the map.

Applying Eliashberg and Francis' theorem [2, 3], we have the following theorem.

Theorem 5.2. *Let M be a compact connected oriented surface with exactly one boundary component. If $f : \partial M \looparrowright \mathbf{R}^2$ is an oriented immersion such that $W(f)$ is* *odd, then there exists a pseudo-immersion* $F : M \to \mathbf{R}^2$ *such that*

$$F|\partial M = f \tag{5.1}$$

and

$$\sharp\Sigma(F) = \max\{m(\chi(M), W(f)), 2\} \quad \text{if } \chi(M) - W(f) \equiv 0 \pmod 4 \tag{5.2}$$

or

$$\sharp\Sigma(F) = \max\{m(\chi(M), W(f), 1\} \quad \text{if } \chi(M) - W(f) \equiv 2 \pmod 4. \tag{5.3}$$

The details of Theorem 5.2 are in [8].

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