On pseudo-immersions of a surface into the plane

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1 Introduction

In this paper, all manifolds and maps are differentiable of class $C^\infty$. Let $M$ be a compact connected oriented surface with exactly one boundary component. For a map $F : M \to \mathbb{R}^2$, we define the set of singularities of $F$ as $\Sigma(F) = \{ q \in M \mid \text{rank } dF_q < 2 \}$. The map $F : M \to \mathbb{R}^2$ is called a pseudo-immersion if the following set of conditions is fulfilled:

1. There is some open neighborhood $U$ of $\partial M$, such that $F|U : U \to \mathbb{R}^2$ is an orientation preserving immersion.

2. In the neighborhood of every singularity $x \in M$, $F$ can be represented, in appropriate coordinate systems, by: $y_1 = x_1, y_2 = x_2^2$. We call this type of singularity a fold singularity.

Note that if $F : M \to \mathbb{R}^2$ is a pseudo-immersion, then $\Sigma(F)$ is a union of circles and $F|\Sigma(F)$ is an immersion. A pseudo-immersion was defined by Poénaru [6] for a smooth map $F : M^n \to N^n$ between $n$-manifolds. In his definition, he added a condition for the position of a singular set. In this paper, we do not consider an immersion as a pseudo-immersion.

Let $M$ be a compact connected oriented surface with exactly one boundary component. The boundary $\partial M$ has the induced orientation of $M$. That is, let $n$ be the outward normal vector field of $\partial M$ in $M$ then, $\partial M$ is oriented by the unit tangent vector $\tau$ such that the frame $(n, \tau)$ represents the positive orientation of $M$. Let $F : M \to \mathbb{R}^2$ be an orientation preserving immersion. The winding number $W(F|\partial)$ of the restricted immersion $F|\partial M$ is the degree of the map $dF(\tau) : \partial M = S^1 \to S^1$.

By the Poincaré–Hopf’s theorem, we have

$$(1.1) \quad W(F|\partial M) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic class of $M$.

Our problem is the following: if $F : M \to \mathbb{R}^2$ is a pseudo-immersion, then what is the relation between $W(F|\partial), \chi(M)$ and $\#\Sigma(F)$? Here $\#\Sigma(F)$ is the number of connected components of $\Sigma(F)$.

Before stating the main theorem, we should define an invariant which relates to the number of singular set components.
**Definition 1.1.** For two odd integers $\chi$ and $W$, we define

\[
m(\chi, W) = \begin{cases} 
\frac{\chi + W}{2} + 1 & \text{if } W > 0, \\
\frac{\chi - W}{2} & \text{if } W < 0.
\end{cases}
\]

The main theorems are the following.

**Theorem 1.2.** Let $F : M \to \mathbb{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary component of the plane.

1. If $\chi(M) - W(F|\partial) \equiv 0 \pmod{4}$, then $\#\Sigma(F) \geq \max\{m(\chi(M), W(F|\partial)), 2\}$.
2. If $\chi(M) - W(F|\partial) \equiv 2 \pmod{4}$, then $\#\Sigma(F) \geq \max\{m(\chi(M), W(F|\partial)), 1\}$.

**Theorem 1.3.** For any fixed odd integer $W$ and odd integer $\chi \leq 1$, there exists a pseudo-immersion $F : M \to \mathbb{R}^2$ of a compact connected oriented surface with exactly one boundary component such that

\[
\chi(M) = \chi, \ W(F|\partial) = W
\]

and such that

1. $\#\Sigma(F) = \max\{m(\chi, W), 2\}$ if $\chi - W \equiv 0 \pmod{4}$
2. or
3. $\#\Sigma(F) = \max\{m(\chi, W), 1\}$ if $\chi - W \equiv 2 \pmod{4}$.

**Remark 1.4.** Concerning Theorems 1.2 and 1.3, we note the following.

1. Nagase [5] introduced a folding-map. The singularity of a folding-map is the same as that of a pseudo-immersion, but it may attach the boundary of a source manifold. Nagase proved that any immersion of $S^2$ into the interior of a homotopy 3-ball $V$ extends to a folding-map of $D^3$ into $V$ whose fold-set consists of mutually disjoint disks.

2. Ekholm and Larsson [1] defined an admissible map. The singularity of an admissible map has not only fold singularities but also cusp singularities. For an admissible map of $D^2$ to the plane, Ekholm and Larsson expressed the minimal number of singular set components as a function of cusps and the normal degree of the image of the boundary curve of $D^2$.

3. Eliashberg [2] proved the existence of stable maps between oriented surfaces. Similar results of Theorems 1.2 and 1.3 for fold maps between oriented closed surfaces were found by the author [7].

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2 Preliminaries

In this section, we state an important tool to prove Theorem 1.2.

Let $F : M \to \mathbb{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary. Note that $\Sigma(F) \subset M$ is two colourable. Here, to say that a 1-dimensional submanifold $V \subset M$ is two colourable means that $V$ divides $M$ into a pair of nonempty open surfaces $(B, R)$ of $M$ such that $B \cap R = \emptyset$, $B \cup R = M \setminus V$ and the closures $\overline{B}$ and $\overline{R}$ of $B$ and $R$ in $M$ respectively both contain $V$.

For a connected component $\gamma \subset \Sigma(F)$, we define the normal vector field $\nu_{\gamma}$ of $F(\gamma)$ as follows: $\nu_{\gamma}$ points towards the direction in which the number of preimages of the regular value near $F(\gamma)$ decreases. Since $F|_{\gamma} : \gamma \hookrightarrow \mathbb{R}^2$ is an immersion, $\gamma$ is oriented by the tangent vector field $\tau_{\gamma}$ such that the frame $(\nu_{\gamma}, dF(\tau_{\gamma}))$ represents the positive orientation of $\mathbb{R}^2$. The winding number $W(F|_{\gamma})$ is the degree of the map $dF(\tau_{\gamma}) : \gamma = S^1 \to S^1$ in which the source has the above orientation.

Let $N(\gamma) = \gamma \times [-1, 1]$ be a tubular neighborhood of $\gamma \subset \Sigma(F)$ such that $\gamma = \gamma \times \{0\}$ and we set $N(\Sigma(F)) = \bigcup_{\gamma \subset \Sigma(F)} N(\gamma)$. Let $E$ be a connected open surface of $M \setminus N(\Sigma(F))$ such that $\overline{E} \cap N(\gamma) \neq \emptyset$. Since $E$ is orientable and $F|E$ is an immersion, we define the orientation of $E$ such that $F|E : E \leftrightarrow \mathbb{R}^2$ is an orientation preserving immersion. Each connected component of $\partial E$ has the induced orientation of $E$. Note that if $E$ contains $\partial M$, the induced orientations of $\partial M$ from that of $M$ and $E$ are the same. Suppose that $\gamma \times \{i\}$ ($i = -1$ or 1) belongs to $\partial E$. Since the orientation of $\gamma \times \{i\}$ is the same as that of $\gamma \times \{0\}$, we have

\begin{equation}
W(F|_{\gamma \times \{i\}}) = W(F|_{\gamma \times \{0\}}).
\end{equation}

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let $F : M \to \mathbb{R}^2$ be a pseudo-immersion of a compact connected oriented surface with exactly one boundary component. Since $\Sigma(F) \subset M$ is two colourable, we set $(B, R)$ as a two colour decomposition of the pair $(M, \Sigma(F))$ such that $\partial \overline{B}$ contains $\partial M$. By (3.1) and the fact that $\Sigma(F)$ is a closed 1-dimensional submanifold, we have

\begin{align}
(3.1) \quad & \chi(\overline{B}) = W(\Sigma(F)) + W(F|\partial), \\
(3.2) \quad & \chi(\overline{R}) = W(\Sigma(F)), \\
(3.3) \quad & \chi(M) = \chi(\overline{B}) + \chi(\overline{R}).
\end{align}

Therefore, we have

\begin{align}
(3.4) \quad & W(F|\partial) = \chi(M) - 2W(\Sigma(F)), \\
(3.5) \quad & W(F|\partial) = \chi(\overline{B}) - \chi(\overline{R}).
\end{align}

Since $\chi(M)$ is odd, we have the following proposition.
Proposition 3.1. The winding number $W(F|\partial)$ of the restricted immersion $F|\partial M$ is odd.

Suppose that the number of connected components of $\overline{B}$ (resp. $\overline{R}$) is $n_B$ (resp. $n_R$), the sum of the genuses of each connected component of $B$ (resp. $R$) is $g_B$ (resp. $g_R$) and the genus of $M$ is $g$. Since the number of boundary components of $B$ is equal to $\#\Sigma(F) + 1$ and the number of boundary components of $R$ is equal to $\#\Sigma(F)$, (3.3) and (3.5) are written as;

\begin{align*}
(3.6) & \quad \chi(M) = 2n_B - 2g_B + 2n_R - 2g_R - 2\#\Sigma(F) - 1, \\
(3.7) & \quad W(F|\partial) = 2n_B - 2g_B - 2n_R + 2g_R - 1.
\end{align*}

Thus, we have

\begin{equation}
(3.8) \quad \chi(M) - W(F|\partial) = 4n_R - 4g_R - 2\#\Sigma(F).
\end{equation}

By this equation, we have the following.

Proposition 3.2. If $\chi(M) - W(F|\partial) \equiv 0 \pmod{4}$, then the number of singular set components $\#\Sigma(F)$ is even. If $\chi(M) - W(F|\partial) \equiv 2 \pmod{4}$, then the number of singular set components $\#\Sigma(F)$ is odd.

Suppose that $W(F|\partial) > 0$. Then by (3.8), we have

\begin{align*}
\#\Sigma(F) & = \frac{-\chi(M) + W(F|\partial)}{2} + 2n_R - 2g_R \\
& \geq \frac{-\chi(M) + W(F|\partial)}{2} + 2 - 2g \\
& = \frac{\chi(M) + W(F|\partial)}{2} + 1.
\end{align*}

(3.9)

Here, $g$ is the genus of $M$.

Suppose that $W(F|\partial) < 0$. Then instead of (3.8), we have

\begin{equation}
(3.10) \quad \chi(M) + W(F|\partial) = 4n_B - 4g_B - 2\#\Sigma(F) - 2.
\end{equation}

Therefore,

\begin{align*}
\#\Sigma(F) & = \frac{-\chi(M) - W(F|\partial)}{2} + 2n_B - 2g_B - 1 \\
& \geq \frac{-\chi(M) - W(F|\partial)}{2} + 2 - 2g - 1 \\
& = \frac{\chi(M) - W(F|\partial)}{2}.
\end{align*}

(3.11)

Combining (3.9), (3.11) and Proposition 3.2, we have the desired inequalities. This completes the proof of Theorem 1.2.
To prove Theorem 1.3, it is necessary to construct the desired pseudo-immersions concretely by using Francis' theorem [4]. Instead of giving such pseudo-immersions in all the cases, in this section, we give typical examples.

4.1 The case of $\chi = 1 - 2g$ and $W = 2g - 1$

Let $M_g$ be a closed oriented surface of the genus $g$ and $M_{g,1} = M_g \setminus D^2$. It is obvious that $\chi(M_{g,1}) = 1 - 2g$. In this subsection, we construct a pseudo-immersion $F : M_{g,1} \rightarrow \mathbb{R}^2$ such that $W(F|\partial) = 2g - 1$ and $\#\Sigma(F) = m(1 - 2g, 2g - 1) = 1$.

Let $N(\partial M_{g,1}) = \partial M_{g,1} \times [-1,0]$ be a tubular neighborhood of $\partial M_{g,1}$ such that $\partial M_{g,1} = \partial M_{g,1} \times \{0\}$. Let $F_1 : M_{g,1} \setminus N(\partial M_{g,1}) \leftrightarrow \mathbb{R}^2$ be an orientation preserving immersion and $F_2 : N(\partial M_{g,1}) \leftrightarrow \mathbb{R}^2$ an orientation preserving immersion such that $F_1|\partial M_{g,1} \times \{-1\} = F_2|\partial M_{g,1} \times \{-1\}$. Then, by attaching $F_1$ and $F_2$ and by changing the orientation of $M_{g,1} \setminus N(\partial M_{g,1})$, we have a desired pseudo-immersion $F = F_1 \cup F_2 : M_{g,1} \rightarrow \mathbb{R}^2$ such that $W(F|\partial) = 2g - 1$, $\Sigma(F) = \partial M_{g,1} \times \{-1\}$. See Figure 1.

4.2 The case of $\chi = 1$ and $W = -2n + 1$

Let $n$ be a positive integer. In this subsection, we construct a pseudo-immersion $\tilde{F} : M_{0,1} \rightarrow \mathbb{R}^2$ such that $W(\tilde{F}|\partial) = -2n + 1$ and $\#\Sigma(\tilde{F}) = m(1, -2n + 1) = n$.

Before constructing the desired pseudo-immersion, we will explain a boundary connected sum of two pseudo-immersions. Let $F : M \rightarrow \mathbb{R}^2$ and $G : N \rightarrow \mathbb{R}^2$ be two pseudo-immersions such that $F(M) \cap G(N) = \emptyset$. Let $I_a \times I_b$ be a rectangle of
two closed intervals \( I_a = I_b = [0, 1] \) and \( H : I_a \times I_b \to \mathbb{R}^2 \) an orientation preserving embedding. Let \( i_M : \{0\} \times I_b \to \partial M \) and \( i_N : \{1\} \times I_b \to \partial N \) be orientation reversing embeddings such that \( F \circ i_M = G \circ i_N = H \). Then \( F \cup_{i_M} H \cup_{i_N} G : M \cup_{i_M} I_a \times I_b \cup_{i_N} N \to \mathbb{R}^2 \) is a pseudo-immersion. We denote \( F \cup_{i_M} H \cup_{i_N} G \) as \( F \# G \) and \( M \cup_{i_M} I_a \times I_b \cup_{i_N} N \) as \( M \# N \) and we call \( F \# G \) a boundary connected sum of \( F \) and \( G \). Note that \( W(F \# G | \partial) = W(F | \partial) + W(G | \partial) - 1 \). See Figure 2.

Let \( F_i : M_{0,1} \to \mathbb{R}^2 \) be a copy of the pseudo-immersion which is constructed in Subsection 4.1. We take a boundary connected sum of \( F_1, F_2, \ldots, F_n \). We set \( \overline{F} = F_1 \# F_2 \# \cdots \# F_n \) and we have \( M_{0,1} \# M_{0,1} \# \cdots \# M_{0,1} = M_{0,1} \). Because \( W(\overline{F} | \partial) = -n - (n - 1) = -2n + 1 \) and \( \#(\overline{F}) = n \), the pseudo-immersion \( \overline{F} : M_{0,1} \to \mathbb{R}^2 \) is the desired one. See Figures 3 and 4 in the case \( n = 3 \).

5 Supplement

5.1 Position of the singular set

In this section, we remark on the positions of the singular set of a pseudo-immersion \( F : M \to \mathbb{R}^2 \).

**Proposition 5.1.** Let \( F_1 \) and \( F_2 : M \to \mathbb{R}^2 \) be two pseudo-immersions of a compact connected oriented surface with exactly one boundary component. If \( \Sigma(F_1) = \Sigma(F_2) = 1 \), then an orientation preserving diffeomorphism \( \Phi : M \to M \) such that \( \Phi(\Sigma(F_1)) = \Sigma(F_2) \) exists.
Figure 3: Pseudo-immersions $F_i : M_{0,1} \rightarrow \mathbb{R}^2$ ($i = 1, 2, 3$).

Figure 4: A pseudo-immersion $\tilde{F} : M_{0,1} \rightarrow \mathbb{R}^2$. 
This proposition is obvious. If the number of singular set components is more than one, the above proposition is not true. For example, let $\Sigma_1$ and $\Sigma_2$ be two simple closed curves in $M_{1,1}$ that splits $M_{1,1}$ into three connected surfaces. Two of them are annuli and the other is one punctured torus. Let $\Sigma_3$ and $\Sigma_4$ be two simple closed curves in $M_{1,1}$ that splits $M_{1,1}$ into two connected surfaces. Both of them are annuli. By using Francis' theorem [4], we have two pseudo-immersions $F_1$ and $F_2 : M_{1,1} \to \mathbb{R}^2$ such that $\Sigma(F_1) = \Sigma_1 \cup \Sigma_2$, $\Sigma(F_2) = \Sigma_3 \cup \Sigma_4$, $F_1(\Sigma_1) = F_2(\Sigma_3)$, $F_1(\Sigma_2) = F_2(\Sigma_4)$, $F_1(\partial M_{1,1}) = F_2(\partial M_{1,1})$ and $W(F_1|\partial) = W(F_2|\partial) = -1$. See Figure 5.

![Diagram of pseudo-immersions](image)

Figure 5: Two pseudo-immersions $F_1$ and $F_2 : M_{1,1} \to \mathbb{R}^2$ such that $F_1(\Sigma(F_1)) = F_2(\Sigma(F_2))$.

### 5.2 Image of the boundary of a pseudo-immersion

In this subsection, we state the existence of a pseudo-immersion such that the given plane curve is the image of the boundary of the map.

Applying Eliashberg and Francis' theorem [2, 3], we have the following theorem.

**Theorem 5.2.** Let $M$ be a compact connected oriented surface with exactly one boundary component. If $f : \partial M \to \mathbb{R}^2$ is an oriented immersion such that $W(f)$ is
odd, then there exists a pseudo-immersion $F : M \to \mathbb{R}^2$ such that

(5.1) \[ F|\partial M = f \]

and

(5.2) \[ \#\Sigma(F) = \max\{m(\chi(M), W(f)), 2\} \quad \text{if} \quad \chi(M) - W(f) \equiv 0 \pmod{4} \]

or

(5.3) \[ \#\Sigma(F) = \max\{m(\chi(M), W(f)), 1\} \quad \text{if} \quad \chi(M) - W(f) \equiv 2 \pmod{4}. \]

The details of Theorem 5.2 are in [8].

References


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