THE ENRICHED RIEMANN SPHERE AND STABILITY

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In this presentation we will discuss a few suggestive examples, indicating our new approach to Singularity Theory (more details will appear elsewhere).

A general principle which we believe in is that the study of analytic function germs in $n + 1$ variables is Global Analysis of polynomials in $n$ variables.

This is illustrated here in the case $n = 1$. Loosely speaking, the classical Morse Stability Theorem in one variable, properly formulated, is "transplanted" into Algebraic Geometry as theorems on equi-singularities in $\mathbb{C}^2$ (equivalence of singularities); it also suggests a stronger definition for "equi-singular deformation".

For example, in contemporary Algebraic Geometry, the following deformations

$$Q(x, y; t) := x^4 - t^2 x^2 y^2 + y^4, \quad P(x, y; t) := x^3 - y^4 - 3t^2 xy^k, \quad k \geq 3,$$

are equi-singular, because their zero sets are topologically trivial (Milnor $\mu$-constant).

However, $Q$ is not equi-singular from our point of view. The hypothesis of our Equi-singularity Theorem is not satisfied. The associated family of polynomials $x^4 - t^2 x^2 + 1$ is not Morse stable ($x = 0$ splits into three critical points when $t \neq 0$).

On the other hand, the Pham family $P$ is equi-singular in our sense. (Even though the "polar" $\partial P/\partial x$ splits into distinct factors $x \pm ty^d$ if $k = 2d$.) The associated family $x^3 - 1$, being independent of $t$, is obviously Morse stable. By our Equi-singularity Theorem, $P$ itself, not merely the zero set, admits a trivialization.

1. MORSE STABILITY

When does a given family $F(x, y; t)$, like $Q$, $P$ above, admit a trivialization, and of what kind? This is answered in our Equi-singularity Theorem, modelled on the classical Morse Theorem. The Morse Stability Theorem over $\mathbb{F}$ is also geometrized.

**Definition 1.1.** Given $p_t(x) := a_0(t)x^n + \cdots + a_n(t) \in K\{t\}[x]$, as a deformation of $p_0(x)$, $a_0(t) \neq 0$, $t \in I_K$, where $K := \mathbb{C}$ or $\mathbb{R}$. A critical point $c \in K$ of $p_0(x)$ is stable if it admits a continuous deformation $c_t \in K$, a critical point of $p_t(x)$, with $m_{\mathfrak{cr}H}(c_t) = m_{\mathfrak{cr}H}(c)$. (See Example (1.2).)

The deformation $\{p_t\}$ is Morse stable if the following hold.

1. Every critical point $c \in K$ of $p_0(x)$ is stable;

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T.-C. Kuo and L. Paunescu

(2) If $p_0(c) = p_0(c')$, $c$, $c'$ critical points of $p_0(z)$, then $p_t(c_t) = p_t(c'_t)$, $t \in I_K$;
(3) If $p_0(c) = p_0'(c) = 0$, i.e., $c$ is a multiple root of $p_0(z)$, then $p_t(c_t) = 0$, $t \in I_K$.

Conditions (1), (2) come from Morse Theory; (3) is new, needed for Algebraic Geometry.
A version of the classical Morse Stability Theorem is the following.

**The Morse Theorem.** Suppose $\{p_t(x)\}$ is Morse stable. There exist $t$-level preserving homeomorphisms $D : \mathbb{K} \times I_K \rightarrow \mathbb{K} \times I_K$, and $\delta : \mathbb{K} \times I_K \rightarrow \mathbb{K} \times I_K$,

$$D : (x, t) \mapsto (D_t(x), t); \quad \delta : (v, t) \mapsto (d_t(v), t), \quad d_t(0) = 0,$$

where $D_0(x) = x$, $d_0(v) = v$, such that $p_t(D_t(x)) = d_t(p_0(x))$, and $c$ is a critical point of $p_0$ iff $D_t(c)$ is one of $p_t$. (Note that $p_0(a) = 0$ iff $p_t(D_t(a)) = 0$.)

$$I_R := \{t \in \mathbb{R} \mid |t| < \epsilon\}, \quad I_C := \{t \in \mathbb{C} \mid |t| < \epsilon\}, \quad I_F := \{t \in \mathbb{D} \mid |t| < \epsilon\}, 1 >> \epsilon > 0. \quad (1.2)$$

Here $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or the Newton-Puiseux field $\mathbb{F}$. The “disk” $D \subset \mathbb{F}$ is described in the next section.

**Example 1.2.** Take $\mathbb{K} = \mathbb{R}$. For $p_t(x) = x^2(x^2 + t^2) \in \mathbb{R}[x]$, 0 is a critical point of $p_0$ which splits into 3 critical points in $\mathbb{C}$, one remains in $\mathbb{R}$. Thus 0 admits a unique continuous deformation $c_t \equiv 0$ in $\mathbb{R}$. But $m_{crit}(c_t)$ is not constant, 0 is unstable.

2. **The Enriched Riemann Sphere**

The Riemann sphere $\mathbb{C}P^1$ is “enriched” to $\mathbb{C}P^1_*$ with “infinitesimals”, which are irreducible curve germs; and $\mathbb{C}$ enriched to $\mathbb{C}_*$. The Newton-Puiseux field $\mathbb{F}$ provides coordinate systems, in terms of which several structures are defined.

The Cauchy Integral Theorem, Taylor expansions, critical points, stability, etc., are generalized to $\mathbb{F}$; and so is the classical Morse Stability Theorem.

Take a holomorphic map germ $\mathcal{A} : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^2, 0)$, $\mathcal{A}(z) \neq 0$ if $z \neq 0$. The image set germ, $Im(\mathcal{A})$, or the geometric locus of $\mathcal{A}$, has a well-defined tangent line, $T(\mathcal{A})$, at 0. We call $Im(\mathcal{A})$ an infinitesimal at $T(\mathcal{A}) \in \mathbb{C}P^1$. The set of infinitesimals is denoted by $\mathbb{C}P^1_*$.

The geometric locus of $z \mapsto (az, bz)$ is identified with $[a : b] \in \mathbb{C}P^1_*$; hence $\mathbb{C}P^1 \subset \mathbb{C}P^1_*$. For example, the curve germ $x^3 - y^3 = 0$, as the geometric locus of $z \mapsto ((x^3), (x^2))$, is an infinitesimal at $[0 : 1]$. It is “closer” to $[0 : 1]$ than any $[a : 1], a \neq 0$.

As in Projective Geometry, $\mathbb{C}P^1_*$ is a union $\mathbb{C}P^1_* = \mathbb{C}_* \cup \mathbb{C}_*$, where

$$\mathbb{C}_* := \{Im(\mathcal{A}) \mid T(\mathcal{A}) \neq [1 : 0]\}, \quad \mathbb{C}_* := \{Im(\mathcal{A}) \mid T(\mathcal{A}) \neq [0 : 1]\}.$$

The classical Newton-Puiseux Theorem asserts that the field $\mathbb{F}$ of convergent fractional power series in an indeterminate $y$ is algebraically closed.

Recall that a non-zero element of $\mathbb{F}$ is (finite or infinite) convergent series

$$\alpha : \alpha(y) = a_0y^{n_0/N} + a_1y^{n_1/N} + \cdots, \quad a_i \neq 0, \quad n_0 < n_1 < \cdots, \quad (2.1)$$
THE ENRICHED RIEMANN SPHERE AND STABILITY

where \( n_i \in \mathbb{Z}, N \in \mathbb{Z}^+ \), \( a_i \in \mathbb{C} \). The order of \( \alpha \) is \( O_y(\alpha) := n_0/N \); \( O_y(0) := +\infty \).

We can assume \( GCD(N, n_0, n_1, \ldots) = 1 \). The Puiseux multiplicity of \( \alpha \) is \( m_{\text{puis}}(\alpha) := N \).

The conjugates of \( \alpha \) are \( \alpha_{\text{conj}}^{(k)}(y) := \sum a_i \theta^{kn_0} y^{n_i/N} \), \( 0 \leq k \leq N - 1 \), where \( \theta := e^{2\pi i/N} \).

The following \( D \) is an integral domain with quotient field \( \mathbb{F} \) and maximal ideal \( M \).

Thus, \( \lim_{m \to \infty} \sum a_i(m) y^{n_i/N} = 0 \) iff each \( a_i(m) \to 0 \), the point-wise convergence.

Given \( \alpha \in M \), let \( \mathcal{A}(z) := (\alpha(z^N), z^N) \), \( N := m_{\text{puis}}(\alpha) \). We define \( \alpha_* := \pi_*(\alpha) := Im(\mathcal{A}) \), and use \( \pi_* : M \to \mathbb{F} \), a many-to-one surjective mapping, as a coordinate system on \( \mathbb{C} \).

A coordinate system on \( \mathbb{C}' \) is \( \pi'_* : M \to \mathbb{C}' \), \( \alpha_* := \pi'_*(\alpha) := Im(\mathcal{A}) \), \( A(z) := (z^N, \alpha(z^N)) \).

Let \( \mathbb{C} \) (resp. \( \mathbb{C}' \)) be furnished with the quotient topology of \( \pi_* \) (resp. \( \pi'_* \)). As for the transition function in the overlap \( \mathbb{C} \cap \mathbb{C}' \), take \( x = \alpha(y) \), \( n_0/N = 1 \), we then "solve y in terms of x", obtaining \( y = \beta(x) := b_0 x + b_1 x^{n_0/N} + \cdots, a_0 b_0 = 1 \), each \( b_i \) is a polynomial in finitely many of \( (\sqrt[n_i]{a_0})^{-1}, a_1/a_0, a_2/a_0, \ldots \). Hence the topologies coincide in \( \mathbb{C} \cap \mathbb{C}' \).

The quotient topology on \( \mathbb{C}P_*^1 \) is well-defined.

Next, let \( X, Y \subset \mathbb{R}^n \) be germs of sub-analytic sets at 0, \( X \cap Y = \{0\}, X \neq \{0\} \neq Y \). The contact order \( O(X, Y) \) is, by definition, the smallest number \( L \) (the Lojasiewicz exponent) such that \( d(x, y) \geq a ||(x, y)||^L \), where \( x \in X, y \in Y, ||x|| = ||y||, a > 0 \) a constant.

Hence \( O(\alpha_*, \beta_*) := \infty \). (Example: for \( \alpha, \beta \in M \), \( O(\pi_*(\alpha), \pi_*(\beta)) = \max_{k,j} \{O(\alpha_{\text{conj}}^{(k)} - \beta_{\text{conj}}^{(j)})\} \). This is the contact order structure on \( \mathbb{C}P_*^1 \).

The enriched Riemann Sphere is \( \mathbb{C}P_*^1 \) furnished with the above structures; \( \mathbb{C} \) is the enriched complex plane.

3. Equi-singularity Theorem

Given \( f(x, y) \in \mathbb{C}\{x, y\} \), mini-regular in \( x \) of order \( m \), i.e.,

\[
f(x, y) = H_0(x, y) + H_{m+1}(x, y) + \cdots, H_m(1, 0) \neq 0, H_i(x, y) \text{ i-form.}
\]

Take a deformation \( F(x, y; t) = \sum_{i+j \geq m} c_{ij}(t)x^i y^j \in \mathbb{C}\{x, y, t\} \), \( F(x, y; 0) = f(x, y) \).

Define \( \phi_t(\xi) := F(\xi, y; t), \xi \in M, \Phi := \{\phi_t\}, t \in I_\mathbb{C} \).

The Equi-singularity Theorem. Suppose \( \Phi \) is Morse stable. There exists a map germ

\[
H : (\mathbb{C}^2 \times I_\mathbb{C}, 0 \times I_\mathbb{C}) \to (\mathbb{C}^2 \times I_\mathbb{C}, 0 \times I_\mathbb{C}), ((x, y), t) \mapsto (\eta_t(x, y), t),
\]

which is a homeomorphism, real bi-analytic outside \( \{0\} \times I_\mathbb{C} \), such that

(1) \( F(\eta_t(x, y), t) = f(x, y), t \in I_\mathbb{C}, \) i.e., \( F(x, y; t) \) is "trivialized" by \( H \);
(2) $H_\ast : \mathbb{C}P^1_\ast \times I_C \to \mathbb{C}P^1_\ast \times I_C$, $(\alpha_\ast, t) \mapsto (\eta_t(\alpha_\ast), t)f$ is a homeomorphism, where $\eta(\alpha_\ast)$ as a set germ is a point of $\mathbb{C}P^1_\ast$ (we do not claim that if $A$ is holomorphic then so is $\eta \circ A$);
(3) The contact order is preserved: $O(\alpha_\ast, \beta_\ast) = O(\eta_t(\alpha_\ast), \eta_t(\beta_\ast))$;
(4) The Puiseux pairs is preserved: $\chi_{puis}(\eta_t(\alpha_\ast)) = \chi_{puis}(\alpha_\ast)$;
(5) There exists a constant $\varepsilon > 0$, $\varepsilon \leq \|\eta_t(x, y)/\|(x, y)\| \leq 1/\varepsilon$, $t \in I_C$;
(6) If $\mathcal{R} : (\mathbb{R}, 0) \to (\mathbb{C}^2, 0)$ is (real-)analytic then so is $\eta_t \circ \mathcal{R}$, i.e., $\eta_t$ is arc-analytic.

The proof of the Equi-singularity Theorem above, uses a vector field $\vec{F}(x, y, t)$, $(x, y, t) \in U \times I_C$.
There exists $\gamma(y) := \gamma_y(y) + \cdots, F_y(\gamma(y), y; 0) = 0$; i.e., $F_y(x, y; 0)$ vanishes on the curve germ $\Delta := \pi_\ast(\gamma)$ which is customarily called a “polar” of $F(x, y; 0)$.
Let $\Delta_t$ denote the image of $\Delta$ at time $t$ in the flow. Note that the above does not imply that $\Delta_t$ is a polar of $F(x, y; t)$.
The set $\mathcal{P}(\Gamma) := \{\Delta \in \mathbb{C}_\ast \mid O(\Delta, \Gamma) > O(\gamma_y)\}$ contains at least one polar of $F(x, y; 0)$. Hence we call $\mathcal{P}(\Gamma)$ a blurred polar, and $\Gamma$ its canonical representative.
As we shall prove, the flow preserves the contact order, hence induces a bijection between $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma_t)$. The flow only carries one blurred polar to another.
The Pham family $P(x, y; t)$ in (0.1), $k = 2d$, has two polars when $t \neq 0$, but only one blurred polar. The blurred polar is invariant under the flow; the polars are not. Nevertheless this suffices for showing the triviality of the Pham family.