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THE ENRICHED RIEMANN SPHERE AND STABILITY

TZEE-CHAR KUO AND LAURENTIU PAUNESCU

In this presentation we will discuss a few suggestive examples, indicating our new approach to Singularity Theory (more details will appear elsewhere).

A general principle which we believe in is that the study of analytic function germs in $n + 1$ variables is Global Analysis of polynomials in $n$ variables.

This is illustrated here in the case $n = 1$. Loosely speaking, the classical Morse Stability Theorem in one variable, properly formulated, is "transplanted" into Algebraic Geometry as theorems on equi-singularities in $\mathbb{C}^2$ (equivalence of singularities); it also suggests a stronger definition for "equi-singular deformation".

For example, in contemporary Algebraic Geometry, the following deformations

$$Q(x, y; t) := x^4 - t^2 x^2 y^2 + y^4, \quad P(x, y; t) := x^3 - y^4 - 3t^2 xy^k, \quad k \geq 3,$$

are equi-singular, because their zero sets are topologically trivial (Milnor $\mu$-constant).

However, $Q$ is not equi-singular from our point of view. The hypothesis of our Equi-singularity Theorem is not satisfied. The associated family of polynomials $x^4 - t^2 x^2 + 1$ is not Morse stable ($x = 0$ splits into three critical points when $t \neq 0$).

On the other hand, the Pham family $P$ is equi-singular in our sense. (Even though the "polar" $\partial P/\partial x$ splits into distinct factors $x \pm ty^d$ if $k = 2d$.) The associated family $x^3 - 1$, being independent of $t$, is obviously Morse stable. By our Equi-singularity Theorem, $P$ itself, not merely the zero set, admits a trivialization.

1. Morse Stability

When does a given family $F(x, y; t)$, like $Q$, $P$ above, admit a trivialization, and of what kind? This is answered in our Equi-singularity Theorem, modelled on the classical Morse Theorem. The Morse Stability Theorem over $\mathbb{K}$ is also geometrized.

**Definition 1.1.** Given $p_t(x) := a_0(t)x^n + \cdots + a_n(t) \in \mathbb{K}\{t\}[x]$, as a deformation of $p_0(x)$, $a_0(t) \neq 0, t \in I_K$, where $\mathbb{K} := \mathbb{C}$ or $\mathbb{R}$. A critical point $c \in \mathbb{K}$ of $p_0(x)$ is stable if it admits a continuous deformation $c_t \in \mathbb{K}$, a critical point of $p_t(x)$, with $m_{c_0}(c_t) = m_{c_0}(c)$. (See Example (1.2).

The deformation $\{p_t\}$ is Morse stable if the following hold.

1. Every critical point $c \in \mathbb{K}$ of $p_0(x)$ is stable;
(2) If \( p_0(c) = p_0(c') \), \( c, c' \) critical points of \( p_0(z) \), then \( p_t(c_t) = p_t(c'_t), \ t \in I_K \); 
(3) If \( p_0(c) = p'_0(c) = 0 \), i.e., \( c \) is a multiple root of \( p_0(z) \), then \( p_t(c_t) = 0, \ t \in I_K \).

Conditions (1), (2) come from Morse Theory; (3) is new, needed for Algebraic Geometry. A version of the classical Morse Stability Theorem is the following.

**The Morse Theorem.** Suppose \( \{p_t(z)\} \) is Morse stable. There exist \( t \)-level preserving homeomorphisms \( D : \mathbb{K} \times I_K \to \mathbb{K} \times I_K \), and \( \delta : \mathbb{K} \times I_K \to \mathbb{K} \times I_K \),

\[
D : (x, t) \mapsto (D_t(x), t); \quad \delta : (v, t) \mapsto (d_t(v), t), \quad d_t(0) = 0,
\]

where \( D_0(x) = x, d_0(v) = v, \) such that \( p_t(D_t(x)) = d_t(p_0(x)) \), and \( c \) is a critical point of \( p_0 \) iff \( D_t(c) \) is one of \( p_t \). (Note that \( p_0(a) = 0 \) iff \( p_t(D_t(a)) = 0 \).)

\[
I_R := \{ t \in R \mid |t| < \epsilon \}, \quad I_C := \{ t \in C \mid |t| < \epsilon \}, \quad I_F := \{ t \in D \mid |t| < \epsilon \}, 1 > > \epsilon > 0.
\]

Here \( K = \mathbb{R}, C \) or the Newton-Puiseux field \( F \). The "disk" \( D \subset F \) is described in the next section.

**Example 1.2.** Take \( K = \mathbb{R} \). For \( p_t(x) = x^2(x^2 + t^2) \in \mathbb{R}[x], \) 0 is a critical point of \( p_0 \) which splits into 3 critical points in \( C \), one remains in \( \mathbb{R} \). Thus 0 admits a unique continuous deformation \( c_t \equiv 0 \) in \( \mathbb{R} \). But \( m_{crit}(c_t) \) is not constant, 0 is unstable.

2. **The Enriched Riemann Sphere**

The Riemann sphere \( CP^1 \) is "enriched" to \( CP^1_* \) with "infinitesimals", which are irreducible curve germs; and \( C \) enriched to \( C_* \). The Newton-Puiseux field \( F \) provides coordinate systems, in terms of which several structures are defined.

The Cauchy Integral Theorem, Taylor expansions, critical points, stability, etc., are generalized to \( F \); and so is the classical Morse Stability Theorem.

Take a holomorphic map germ \( A : (C, 0) \to (C^2, 0), \) \( A(z) \neq 0 \) if \( z \neq 0 \). The image set germ, \( Im(A) \), the geometric locus of \( A \), has a well-defined tangent line, \( T(A) \), at 0. We call \( Im(A) \) an infinitesimal at \( T(A) \subset CP^1 \). The set of infinitesimals is denoted by \( CP^1_* \).

The geometric locus of \( z \mapsto (az, bz) \) is identified with \( [a : b] \in CP^1 \); hence \( CP^1 \subset CP^1_* \).

For example, the curve germ \( x^2 - y^3 = 0 \), as the geometric locus of \( z \mapsto (z^3, z^2) \), is an infinitesimal at \( [0:1] \). It is "closer" to \( [0:1] \) than any \([a : 1], a \neq 0 \).

As in Projective Geometry, \( CP^1_* \) is a union \( CP^1 = C_* \cup C'_*, \) where

\[
C_* := \{ Im(A) \mid T(A) \neq [1:0] \}, \quad C'_* := \{ Im(A) \mid T(A) \neq [0:1] \}.
\]

The classical Newton-Puiseux Theorem asserts that the field \( F \) of convergent fractional power series in an indeterminate \( y \) is algebraically closed.

Recall that a non-zero element of \( F \) is a (finite or infinite) convergent series

\[
\alpha : \alpha(y) = a_0y^{n_0/N} + a_1y^{n_1/N} + \cdots, \quad a_i \neq 0, \quad n_0 < n_1 < \cdots,
\]

\[(2.1)\]
where \( n_{i} \in \mathbb{Z}, N \in \mathbb{Z}^{+}, a_{i} \in \mathbb{C}. \) The order of \( \alpha \) is \( O_{y}(\alpha) := n_{0}/N; \) \( O_{y}(0) := +\infty. \)

We can assume \( \gcd(N, n_{0}, n_{1}, \ldots) = 1. \) The Puiseux multiplicity of \( \alpha \) is \( m_{\text{puise}}(\alpha) := N. \)

The conjugates of \( \alpha \) are \( \alpha^{(k)}_{\text{conj}}(y) := \sum a_{i} \theta^{kn_{i}} y^{n_{i}/N}, 0 \leq k \leq N - 1, \) where \( \theta := e^{2\pi i/N}. \)

The following \( D \) is an integral domain with quotient field \( \mathbb{F} \) and maximal ideal \( \mathcal{M}, \)

\[ D := \{ \alpha \in \mathbb{F} \mid O_{y}(\alpha) \geq 0 \}, \quad \mathcal{M}_{1} := \{ \alpha \mid O_{y}(\alpha) > 0 \}, \quad \mathcal{M}_{1} := \{ \alpha \mid O_{y}(\alpha) \geq 1 \}; \]

\( \mathcal{M}_{1} \) is an ideal. Define \( |\alpha| := \sum 2^{-n_{i}/N} |a_{i}|(1 + |a_{i}|)^{-1}, \) \( d(\alpha, \beta) := |\alpha - \beta| \) is a metric on \( D. \)

Thus, \( \lim_{m \to \infty} \sum a_{i}(m)y^{n_{i}/N} = 0 \) iff each \( a_{i}(m) \to 0, \) the point-wise convergence.

Given \( \alpha \in \mathcal{M}_{1}, \) let \( \mathcal{A}(z) := (\alpha(z^{N}), z^{N}), N := m_{\text{puise}}(\alpha). \) We define \( \alpha_{*} := \pi_{*}(\alpha) := \text{Im}(\mathcal{A}), \) and use \( \pi_{*} : \mathcal{M}_{1} \to \mathbb{C}_{*}, \) a many-to-one surjective mapping, as a coordinate system on \( \mathbb{C}_{*}. \)

A coordinate system on \( \mathbb{C}_{*} \) is \( \pi'_{*} : \mathcal{M}_{1} \to \mathbb{C}_{*}, \alpha_{*} := \pi'_{*}(\alpha) := \text{Im}(\mathcal{A}), \mathcal{A}(z) := (z^{N}, \alpha(z^{N})). \)

Let \( \mathbb{C}_{*} \) (resp. \( \mathbb{C}_{*}' \)) be furnished with the quotient topology of \( \pi_{*} \) (resp. \( \pi'_{*} \)). As for the transition function in the overlap \( \mathbb{C}_{*} \cap \mathbb{C}_{*}', \) take \( x = \alpha(y), n_{0}/N = 1, \) we then "solve \( y \) in terms of \( x", \) obtaining \( y = \beta(x) := b_{0}x + b_{1}x^{n_{0}/N} + \cdots, a_{0}b_{0} = 1, \) each \( b_{i} \) is a polynomial in finitely many of \( (\sqrt[n]{a_{0}})^{-1}, a_{1}/a_{0}, a_{2}/a_{0}, \ldots. \) Hence the topologies coincide in \( \mathbb{C}_{*} \cap \mathbb{C}_{*}'. \)

The quotient topology on \( \mathbb{C}P_{1}^{1} \) is well-defined.

Next, let \( X, Y \subset \mathbb{R}^{n} \) be germs of sub-analytic sets at 0, \( X \cap Y = \{0\}, X \neq \{0\} \neq Y. \) The contact order \( O(X, Y) \) is, by definition, the smallest number \( L \) (the Lojasiewicz exponent) such that \( d(x, y) \geq a||x||^{L} \) where \( x \in X, y \in Y, ||x|| = ||y||, a > 0 \) a constant.

Hence \( O(\alpha_{*}, \beta_{*}) \) is well-defined, \( O(\alpha_{*}, \alpha_{*}) := \infty. \) (Example: for \( \alpha, \beta \in \mathcal{M}_{1}, O(\pi_{*}(\alpha), \pi_{*}(\beta)) = \max_{k,j} \{O_{y}(\alpha^{(k)}_{\text{conjugate}} - \beta^{(j)}_{\text{conjugate}})\}. \) This is the contact order structure on \( \mathbb{C}P_{1}^{1}. \)

The enriched Riemann Sphere is \( \mathbb{C}P_{1}^{1} \) furnished with the above structures; \( \mathbb{C}_{*} \) is the enriched complex plane.

### 3. Equi-singularity Theorem

Given \( f(x, y) \in \mathbb{C}\{x, y\}, \) mini-regular in \( x \) of order \( m, \) i.e.,

\[ f(x, y) = H_{m}(x, y) + H_{m+1}(x, y) + \cdots, \quad H_{m}(1, 0) \neq 0, \quad H_{i}(x, y) \text{ i-form}. \]

Take a deformation \( F(x, y; t) = \sum_{i+j \geq m} c_{ij}(t)x^{i}y^{j} \in \mathbb{C}\{x, y, t\}, F(x, y; 0) = f(x, y). \)

Define \( \phi_{t}(\xi) := F(\xi, y; t), \xi \in \mathcal{M}_{1}, \Phi := \{ \phi_{t} \}, t \in \mathcal{I}_{C}. \)

The Equi-singularity Theorem. Suppose \( \Phi \) is Morse stable. There exists a map germ

\[ H : (\mathbb{C}^{2} \times \mathcal{I}_{C}, 0 \times \mathcal{I}_{C}) \to (\mathbb{C}^{2} \times \mathcal{I}_{C}, 0 \times \mathcal{I}_{C}), ((x, y), t) \mapsto (\eta_{t}(x, y), t), \tag{3.1} \]

which is a homeomorphism, real bi-analytic outside \( \{0\} \times \mathcal{I}_{C}, \) such that

1. \( F(\eta_{t}(x, y); t) = f(x, y), t \in \mathcal{I}_{C}, \text{ i.e., } F(x, y; t) \text{ is "trivialized" by } H; \)
(2) $H_* : \mathbb{C}P_*^1 \times I_C \to \mathbb{C}P_*^1 \times I_C$, $(\alpha_*, t) \mapsto (\eta_t(\alpha_*), t)$, is a homeomorphism, where $\eta_t(\alpha_*)$ as a set germ is a point of $\mathbb{C}P_*^1$ (we do not claim that if $A$ is holomorphic then so is $\eta \circ A$);

(3) The contact order is preserved: $O(\alpha_*, \beta_*) = O(\eta_0(\alpha_*), \eta_0(\beta_*));$

(4) The Puiseux pairs is preserved: $\chi_{\text{puis}}(\eta_t(\alpha_*)) = \chi_{\text{puis}}(\alpha_*);$ (5) There exists a constant $\epsilon > 0, \epsilon \leq \|\eta(x, y)\|/\|(x, y)\| \leq 1/\epsilon, t \in I_C;$(6) If $\mathcal{R} : (\mathbb{R}, 0) \to (\mathbb{C}^2, 0)$ is (real-)analytic then so is $\eta \circ \mathcal{R}$, i.e., $\eta$ is arc-analytic.

The proof of the Equi-singularity Theorem above, uses a vector field $\vec{F}(x, y, t), (x, y, t) \in U \times I_C.$

There exists $\gamma(y) := \gamma_\phi(y) + \cdots, F_\ell(\gamma(y), y; 0) = 0; i.e., F_\ell(x, y; 0)$ vanishes on the curve germ $\Delta := \pi_*(\gamma)$ which is customarily called a “polar” of $F(x, y; 0)$.

Let $\Delta_t$ denote the image of $\Delta$ at time $t$ in the flow. Note that the above does not imply that $\Delta_t$ is a polar of $F(x, y; t)$.

The set $\mathcal{P}(\Gamma) := \{ \Delta \in \mathbb{C}_* | O(\Delta, \Gamma) > O(\gamma_\phi) \}$ contains at least one polar of $F(x, y; 0).$ Hence we call $\mathcal{P}(\Gamma)$ a blurred polar, and $\Gamma$ its canonical representative. As we shall prove, the flow preserves the contact order, hence induces a bijection between $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma_t).$ The flow only carries one blurred polar to another.

The Pham family $P(x, y; t)$ in (0.1), $k = 2d,$ has two polars when $t \neq 0,$ but only one blurred polar. The blurred polar is invariant under the flow; the polars are not. Nevertheless this suffices for showing the triviality of the Pham family.

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