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Kyoto University
THE ENRICHED RIEMANN SPHERE AND STABILITY

TZEE-CHAR KUO AND LAURENTIU PAUNESCU

In this presentation we will discuss a few suggestive examples, indicating our new approach to Singularity Theory (more details will appear elsewhere).

A general principle which we believe in is that the study of analytic function germs in \( n+1 \) variables is \textit{Global Analysis} of polynomials in \( n \) variables.

This is illustrated here in the case \( n=1 \). Loosely speaking, the classical Morse Stability Theorem in one variable, properly formulated, is "transplanted" into Algebraic Geometry as theorems on equi-singularities in \( \mathbb{C}^2 \) (equivalence of singularities); it also suggests a \textit{stronger} definition for "equi-singular deformation".

For example, in contemporary Algebraic Geometry, the following deformations

\[
Q(x, y; t) := x^4 - t^2 x^2 y^2 + y^4, \quad P(x, y; t) := x^3 - y^4 - 3t^2 xy^k, \quad k \geq 3, \tag{0.1}
\]

are equi-singular, because their zero sets are topologically trivial (Milnor \( \mu \)-constant).

However, \( Q \) is \textit{not} equi-singular from our point of view. The hypothesis of our Equi-singularity Theorem is not satisfied. The associated family of polynomials \( x^4 - t^2 x^2 + 1 \) is not Morse stable (\( x = 0 \) splits into three critical points when \( t \neq 0 \)).

On the other hand, the Pham family \( P \) \textit{is equi-singular in our sense}. (Even though the "polar" \( \partial P/\partial x \) splits into distinct factors \( x \pm ty^d \) if \( k = 2d \).) The associated family \( x^3 - 1 \), being independent of \( t \), is obviously Morse stable. By our Equi-singularity Theorem, \( P \) itself, \textit{not merely} the zero set, admits a trivialization.

1. Morse Stability

When does a given family \( F(x, y; t) \), like \( Q, P \) above, admit a trivialization, and of what kind? This is answered in our Equi-singularity Theorem, modelled on the classical Morse Theorem. The Morse Stability Theorem over \( \mathbb{F} \) is also geometrized.

**Definition 1.1.** Given \( p_t(x) := a_0(t)x^n + \cdots + a_n(t) \in \mathbb{K}(t)[x] \), as a deformation of \( p_0(x) \), \( a_0(t) \neq 0, t \in I_K \), where \( \mathbb{K} := \mathbb{C} \) or \( \mathbb{R} \). A critical point \( c \in \mathbb{K} \) of \( p_0(x) \) is \textit{stable} if it admits a \textit{continuous} deformation \( c_t \in \mathbb{K} \), a critical point of \( p_t(x) \), with \( m_{\text{cr}}(c_t) = m_{\text{cr}}(c) \). (See Example (1.2).)

The deformation \( \{p_t\} \) is \textit{Morse stable} if the following hold.

(1) Every critical point \( c \in \mathbb{K} \) of \( p_0(x) \) is stable;
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(2) If \( p_0(c) = p_0(c') \), \( c, c' \) critical points of \( p_0(z) \), then \( p_t(c_t) = p_t(c'_t) \), \( t \in I_K \);
(3) If \( p_0(c) = p_0'(c) = 0 \), i.e., \( c \) is a multiple root of \( p_0(z) \), then \( p_t(c_t) = 0 \), \( t \in I_K \).

Conditions (1), (2) come from Morse Theory; (3) is new, needed for Algebraic Geometry.

A version of the classical Morse Stability Theorem is the following.

**The Morse Theorem.** Suppose \( \{p_t(x)\} \) is Morse stable. There exist \( t \)-level preserving homeomorphisms \( D : \mathbb{K} \times I_K \to \mathbb{K} \times I_K \), and \( \delta : \mathbb{K} \times I_K \to \mathbb{K} \times I_K \),

\[
D : (x, t) \mapsto (D_t(x), t); \quad \delta : (v, t) \mapsto (d_t(v), t), \quad d_t(0) = 0,
\]

where \( D_0(x) = x, d_0(v) = v \), such that \( p_t(D_t(x)) = d_t(p_0(x)) \), and \( c \) is a critical point of \( p_0 \) iff \( D_t(c) \) is one of \( p_t \). (Note that \( p_0(a) = 0 \) iff \( p_t(D_t(a)) = 0 \).)

\[
I_R := \{ t \in \mathbb{R} \mid |t| < \epsilon \}, \quad I_C := \{ t \in \mathbb{C} \mid |t| < \epsilon \}, \quad I_F := \{ t \in \mathbb{D} \mid |t| < \epsilon \}, 1 > \epsilon > 0.
\]

Here \( \mathbb{K} = \mathbb{R}, \mathbb{C} \) or the Newton-Puiseux field \( \mathbb{F} \). The "disk" \( D \subset \mathbb{F} \) is described in the next section.

**Example 1.2.** Take \( \mathbb{K} = \mathbb{R} \). For \( p_t(x) = x^2(x^2 + t^2) \in \mathbb{R}[x] \), \( 0 \) is a critical point of \( p_0 \) which splits into 3 critical points in \( \mathbb{C} \), one remains in \( \mathbb{R} \). Thus 0 admits a unique continuous deformation \( c_t \equiv 0 \) in \( \mathbb{R} \). But \( m_{crit}(c_t) \) is not constant, 0 is unstable.

2. **The Enriched Riemann Sphere**

The Riemann sphere \( \mathbb{C}P^1 \) is "enriched" to \( \mathbb{C}P^1_* \) with "infinitesimals", which are irreducible curve germs; and \( \mathbb{C} \) enriched to \( \mathbb{C}_* \). The Newton-Puiseux field \( \mathbb{F} \) provides coordinate systems, in terms of which several structures are defined.

The Cauchy Integral Theorem, Taylor expansions, critical points, stability, etc., are generalized to \( \mathbb{F} \); and so is the classical Morse Stability Theorem.

Take a holomorphic map germ \( A : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0), A(z) \neq 0 \) if \( z \neq 0 \). The image set germ, \( Im(A) \), or the geometric locus of \( A \), has a well-defined tangent line, \( T(A) \), at \( 0 \). We call \( Im(A) \) an infinitesimal at \( T(A) \in \mathbb{C}P^1 \). The set of infinitesimals is denoted by \( \mathbb{C}P^1_* \).

The geometric locus of \( z \mapsto (az, bz) \) is identified with \( [a : b] \in \mathbb{C}P^1 \); hence \( \mathbb{C}P^1 \subset \mathbb{C}P^1_* \).

For example, the curve germ \( x^2 - y^3 = 0 \), as the geometric locus of \( z \mapsto (z^3, z^2) \), is an infinitesimal at \( [0 : 1] \). It is "closer" to \( [0 : 1] \) than any \( [a : 1], a \neq 0 \).

As in Projective Geometry, \( \mathbb{C}P^1_* \) is a union \( \mathbb{C}P^1_* = \mathbb{C}_* \cup \mathbb{C}_* \), where

\[
\mathbb{C}_* := \{ Im(A) \mid T(A) \neq [1 : 0] \}, \quad \mathbb{C}_* := \{ Im(A) \mid T(A) \neq [0 : 1] \}.
\]

The classical Newton-Puiseux Theorem asserts that the field \( \mathbb{F} \) of convergent fractional power series in an indeterminate \( y \) is algebraically closed.

Recall that a non-zero element of \( \mathbb{F} \) is (finite or infinite) convergent series

\[
\alpha : \alpha(y) = a_0y^{n_0/N} + a_1y^{n_1/N} + \cdots, \quad a_i \neq 0, \quad n_0 < n_1 < \cdots, \quad (2.1)
\]
The following $D$ is an integral domain with quotient field $F$ and maximal ideal $M$,

$$D := \{ \alpha \in F \mid O_{y}(\alpha) \geq 0 \}$$

$M_{1} := \{ \alpha \mid O_{y}(\alpha) > 0 \}$. $M_{1} := \{ \alpha \mid O_{y}(\alpha) \geq 1 \}$;

$M_{1}$ is an ideal. Define $|\alpha| := \sum 2^{-n_{i}/N}|a_{i}|(1 + |a_{i}|)^{-1}, d(\alpha, \beta) := |\alpha - \beta|$ is a metric on $D$. Thus, $\lim_{m \to \infty} \sum a_{i}(m)y^{n_{i}/N} = 0$ if each $a_{i}(m) \to 0$, the point-wise convergence.

Given $\alpha \in M_{1}$, let $A(z) := (\alpha(z^{N}), z^{N}), N := m_{\text{puis}}(\alpha)$. We define $\alpha_{*} := \pi_{*}(\alpha) := Im(A), \alpha_{*} := \pi_{*}(\alpha) := Im(A), A(z) := (z^{N}, \alpha(z^{N}))$. Let $C_{*}$ (resp. $C_{*}'$) be furnished with the quotient topology of $\pi_{*}$ (resp. $\pi_{*}'$). As for the transition function in the overlap $C_{*} \cap C_{*}'$, take $x = \alpha(y)$, $n_{0}/N = 1$, we then "solve $y$ in terms of $x$", obtaining $y = \beta(x) := b_{0}x + b_{1}x^{n_{1}'/N'} + \cdots$, $a_{0}b_{0} = 1$, each $b_{i}$ is a polynomial in finitely many of $(\sqrt[n]{a_{0}})^{-1}, a_{i}a_{0}, a_{2}/a_{0}, \ldots$. Hence the topologies coincide in $C_{*} \cap C_{*}'$.

The quotient topology on $\mathbb{C}P_{*}^{1}$ is well-defined.

Next, let $X, Y \subset \mathbb{R}^{n}$ be germs of sub-analytic sets at 0, $X \cap Y = \{0\}, X \neq \{0\} \neq Y$. The contact order $O(X, Y)$ is, by definition, the smallest number $L$ (the Lojasiewicz exponent) such that $d(x, y) \geq a||x - y||^{L}$, where $x \in X, y \in Y, ||x|| = ||y||, a > 0$ a constant.

Hence $O(\alpha_{*}, \beta_{*})$ is well-defined, $O(\alpha_{*}, \alpha_{*}) := \infty$. (Example: for $\alpha, \beta \in M_{1}, O(\pi_{*}(\alpha), \pi_{*}(\beta)) =$ max$_{k,j}\{O_{y}(\alpha_{*}^{(k)} - \beta_{*}^{(j)})\}$.) This is the contact order structure on $\mathbb{C}P_{*}^{1}$.

The enriched Riemann Sphere is $\mathbb{C}P_{*}^{1}$ furnished with the above structures; $C_{*}$ is the enriched complex plane.

### 3. Equi-singularity Theorem

Given $f(x, y) \in \mathbb{C}{x, y}$, mini-regular in $x$ of order $m$, i.e.,

$$f(x, y) = H_{m}(x, y) + H_{m+1}(x, y) + \cdots, H_{m}(1, 0) \neq 0, H_{i}(x, y) \text{ i-form.}$$

Take a deformation $F(x, y; t) = \sum_{i+j \geq m} c_{ij}(t)x^{i}y^{j} \in \mathbb{C}{x, y, t}, F(x, y; 0) = f(x, y)$.

Define $\phi_{t}(\xi) := F(\xi, y; t), \xi \in M_{1}, \Phi := \{ \phi_{t} \}, t \in I_{C}$. The Equi-singularity Theorem. Suppose $\Phi$ is Morse stable. There exists a map germ

$$H : (\mathbb{C}^{2} \times I_{C}, 0 \times I_{C}) \to (\mathbb{C}^{2} \times I_{C}, 0 \times I_{C}), ((x, y), t) \mapsto (\eta_{t}(x, y), t),$$

which is a homeomorphism, real bi-analytic outside $\{0\} \times I_{C}$, such that

1. $F(\eta_{t}(x, y); t) = f(x, y), t \in I_{C}, \text{ i.e., } F(x, y; t)$ is "trivialized" by $H;
(2) $H_* : \mathbb{C}P_*^1 \times \mathbb{C} \rightarrow \mathbb{C}P_*^1 \times \mathbb{C}, (\alpha_*, t) \mapsto (\eta_t(\alpha_*), t)f$ is a homeomorphism, where $\eta_t(\alpha_*)$ as a set germ is a point of $\mathbb{C}P_*^1$ (we do not claim that if $A$ is holomorphic then so is $\eta_t \circ A$);

(3) The contact order is preserved: $O(\alpha_*, \beta_*) = O(\eta_t(\alpha_*), \eta_t(\beta_*));$

(4) The Puiseux pairs is preserved: $\chi_{\text{puis}}(\eta_t(\alpha_*)) = \chi_{\text{puis}}(\alpha_*);$ 

(5) There exists a constant $\epsilon > 0$, $\epsilon \leq \|\eta_t(x, y)\|/\|(x, y)\| \leq 1/\epsilon$, $t \in I_{\mathbb{C}}$;

(6) If $\mathcal{R} : (\mathbb{R}, 0) \rightarrow (\mathbb{C}^2, 0)$ is (real-)analytic then so is $\eta_t \circ \mathcal{R}$, i.e., $\eta_t$ is arc-analytic.

The proof of the Equi-singularity Theorem above, uses a vector field $\vec{F}(x, y, t)$, $(x, y, t) \in U \times I_{\mathbb{C}}$.

There exists $\gamma(y) := \gamma_0(y) + \cdots$, $F_z(\gamma(y), y; 0) = 0$; i.e., $F_z(x, y; 0)$ vanishes on the curve germ $\Delta := \pi_*(\gamma)$ which is customarily called a “polar” of $F(x, y; 0)$.

Let $\Delta_t$ denote the image of $\Delta$ at time $t$ in the flow. Note that the above does not imply that $\Delta_t$ is a polar of $F(x, y; t)$.

The set $\mathcal{P}(\Gamma) := \{\Delta \in \mathbb{C}_* | O(\Delta, \Gamma) > O(\gamma_0)\}$ contains at least one polar of $F(x, y; 0)$.

Hence we call $\mathcal{P}(\Gamma)$ a blurred polar, and $\Gamma$ its canonical representative.

As we shall prove, the flow preserves the contact order, hence induces a bijection between $\mathcal{P}(\Gamma)$ and $\mathcal{P}(\Gamma_t)$. The flow only carries one blurred polar to another.

The Pham family $P(x, y; t)$ in $(0.1)$, $k = 2d$, has two polars when $t \neq 0$, but only one blurred polar. The blurred polar is invariant under the flow; the polars are not. Nevertheless this suffices for showing the triviality of the Pham family.

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