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ON INJECTIVITY OF TAME MAPPINGS

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In this short note, we give a criterion for the injectivity of tame mappings. This was part of the talk given by the third named author at the second Australian-Japanese meeting on real and complex singularities, held in Kyoto in November 2007. For a more comprehensive study and also for a list of relevant articles on this topic, we send the reader to our paper [1].

Let $U$ be a convex open set of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ a tame map. Let $B$ denote the set where $f$ is not $C^1$, and let $\hat{B}$ denote the set where $f$ is not a $C^1$-immersion. For a subset $V$ of $U$ we put

$$C(f, V) = \text{the convex hull of } \{df(x) : x \in V - B\}.$$ 

We denote its closure by $\overline{C}(f, V)$, and define $\overline{CC}(f, V)$ by

$$\overline{CC}(f, V) = \text{the cone of } \overline{C}(f, V) \text{ with vertex } 0.$$ 

We also use the notations $C(f) = C(f, U)$, $CC(f) = CC(f, U)$, etc., for shortness.

Take $x, x' \in U$, $x \neq x'$, and put $v = \frac{x' - x}{|x - x'|}$. We denote the segment connecting $x$ and $x'$ by $[x, x']$. If $f$ is tame, then we have the following well known facts,

- $[x, x'] \cap \hat{B}$ is a finite set, or
- a sub-segment of $[x, x']$ is subset of $\hat{B}$.

Setting $g(t) = f(t + tv)$, we have

$$g'(t) = df(x + tv)v, \quad \text{whenever } x + tv \notin B.$$ 

Let $\Sigma$ denote the set of singular matrices and $\Sigma_v$ denote the set of singular matrices annihilating $v$.

**Theorem.** A tame map $f : U \to \mathbb{R}^n$ is injective, if the following conditions hold:

- $C(f, U - \hat{B})$ does not contain singular matrices.
- If $\hat{B}$ contains a segment with direction $v$, then $f$ is not constant on this segment, and $\Sigma_v$ is an extremal set of $\overline{CC}(f)$.

It is not hard to see that the theorem follows from Lemmas 1 and 3 below.

**Lemma 1.** Assume that $[x, x'] \cap \hat{B}$ is a finite set. If $C(f, U - \hat{B})$ does not contain singular matrices, then $f(x) \neq f(x')$.

**Proof.** By supposition, the set $\hat{C} = \{t \in [0, 1] : x + tv \in \hat{B}\}$ is finite. If $C(f, U - \hat{B}) \cap \Sigma = \emptyset$, then $C(f, [x, x'] - \hat{B}) \cap \Sigma_v = \emptyset$. Then we obtain that

$$C(g, [0, 1] - \hat{C}) = C(f, [x, x'] - \hat{B}) \cdot v \neq 0.$$ 

We employ the following lemma to complete the proof. □
Lemma 2. Let $g : [a, b] \to \mathbb{R}^n$ be a tame map and let \( \hat{C} \) be a finite subset of \([a, b]\). If \( C(g, [a, b] - \hat{C}) \) does not contain 0, then \( g(a) \neq g(b) \).

**Proof.** The proof is similar to the proof of Lemma 5.3, in [1], and we omit it. \( \square \)

**Lemma 3.** Assume that \([x, x'] \cap \bar{B}\) contains a sub-segment with direction \( v \). If \( \Sigma_v \) is an extremal set of the closure of \( CC(f) \), then \( f(x) \neq f(x') \), or \( f|_{[x,x']} \) is constant.

**Proof.** Since \( f \) is tame, we may choose a vector \( u \) such that
\[
C_u = \{ t \in [0, 1] : x + tv + su \in B \}
\]
are finite sets when \( 0 < s < \varepsilon \). Set \( g_s(t) = f(x + tv + su) \). We remark that
\[
g'_s(t) \in C(g_s, [0, 1] - C_u) = C(f, [x + su, x' + su] - B) \cdot v \subset C(f) \cdot v.
\]
We assume that \( g_0(t) \) is not constant. This means that \( g_0(t) \) is not zero on some subinterval of \([0, 1]\). We then obtain that \( g_s(t) \) is not constant for sufficiently small \( s > 0 \), and \( \langle w, g'_s(t) \rangle > 0 \) for some \( t \in [0, 1] \). We thus have
\[
\langle w, g_s(1) \rangle - \langle w, g_s(0) \rangle = \int_0^1 \langle w, g'_s(t) \rangle dt > 0.
\]
When \( s \to 0 \), we obtain
\[
\langle w, g_0(1) \rangle - \langle w, g_0(0) \rangle = \int_0^1 \langle w, g'_0(t) \rangle dt \geq 0.
\]
Assuming that the equality holds, we will have that \( \langle w, g'_0(t) \rangle = 0 \) for almost all \( t \).
We will conclude that \( g'_0(t) = 0 \), which is a contradiction. By Lemma 4.3 in [1], we have
\[
g'_0(t) = \lim_{s \to 0} g'_s(t) = \lim_{s \to 0} df(x + tv + su) \cdot v
\]
and this is in the closure of \( C(f) \cdot v \). Remark that \( df(x + tv + su) \) goes to infinity, even though the right hand side of \((*)\) stays in a compact set.

We now remark that the closure of \( C(f) \cdot v \) is the image of the closure of the cone of \( C(f) \) by the map defined by \( A \mapsto Av \), that is,
\[
\overline{C(f)} \cdot v = \overline{C(f)} \cdot v.
\]
Since \( \Sigma_v \) is an extremal set of \( \overline{CC(f)} \), 0 is an extremal point of \( \overline{C(f)} \cdot v \). This means that \( \langle w, z \rangle \geq 0 \) for any \( z \in \overline{C(f)} \cdot v \) and the equality holds only if \( z = 0 \). Since \( \langle w, g'_0(t) \rangle = 0 \) for almost all \( t \), we conclude that \( g'_0(t) = 0 \) for almost all \( t \). \( \square \)

**Remark.** If \( f \) is locally Lipschitz, one can replace the assumption in Lemma 3 by the following condition:
\( \Sigma_v \) is an extremal set of the closure of \( CC(f) \) (since \( C(f) \) is a bounded set).

**References**

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