数理解析研究所講究録
第 1611 巻 2008 年 172-179

Entropy and recurrent dimensions of discrete dynamical systems given by the Gauss map

熊本大学・工学部 内藤 幸一郎 (Koichiro Naito)
Faculty of Engineering,
Kumamoto University

1. Introduction

The Gauss map $G:[0,1)\rightarrow[0,1)$ is defined by

$$G(x)=\begin{cases} 0 & \text{if } x=0\\ \frac{1}{x}-[\frac{1}{x}] & \text{otherwise}\end{cases}$$

where $[\cdot]$ denotes the integer part. In [4] D. Barrow has shown that the Mixmaster universe model exhibits chaotic behavior as time goes to 0 near the initial singularity state of the universe by connecting the Einstein equations with the dynamical system $\{x, Gx, G^{2}x, \ldots, G^{n}x, \ldots\}$, which has positive entropies.

For a sequence of partial quotients in the continued fractions of an irrational number $\alpha=[a_{0}, a_{1}, a_{2}, \ldots, a_{n}, \ldots]$ the Gauss map G is a shift map for this squence:

$$G\alpha=[a_{1}, a_{2}, \ldots], \quad \ldots, G^{n}\alpha=[a_{n}, a_{n+1}, \ldots].$$

Khintchine's conjecture is as follows: the sequence of partial quotients in the continued fraction expansions of an algebraic real number of degree ≥ 3 is unbounded and "random"(aperiodic), but almost nothing has been proved yet. Our main purpose is to investigate chaotic behaviors of the partial quotients of these irrational numbers. In this paper we give some partial results by estimating recurrent dimensions and topological entropy of these continued fraction sequences.

First, introducing symbolic dynamical systems, we give inequality relations between recurrent dimensions and entropy of an alphabets sequence. The recurrent dimensions have been introduced in our previous paper [9] as the parameters, which evaluate recurrent properties, defined by using ε-neighborhood recurrent times. For a sequence $u=\{a_{i}\}_{i\geq 1}$ of the partial quotients of α and a shift map σ, defined by $(\sigma u)_{n}=u_{n+1}=a_{n+1}$, we consider a discrete orbit $\Sigma=\{u, \sigma u, \sigma^{2}u, \ldots, \sigma^{n}u, \ldots\}$, which corresponds to the discrete dynamical system $\{G^{n}\alpha : n=1,2, \ldots\}$. We define the lower recurrent dimension by the following limit infimum value as $\varepsilon\rightarrow 0$, using the infimum of the first ε-neighborhood recurrent times in the orbit Σ, which is denoted by $\underline{M}_{\Sigma}(\varepsilon)$:

$$\underline{D}(\Sigma)=\liminf_{\varepsilon\rightarrow 0}\frac{\log\underline{M}_{\Sigma}(\varepsilon)}{-\log\varepsilon}$$

and we also define the upper recurrent dimension by using their supremum values:

$$\overline{D}(\Sigma)=\limsup_{\varepsilon\rightarrow 0}\frac{\log\overline{M}_{\Sigma}(\varepsilon)}{-\log\varepsilon}.$$

We give the following inequality relations between these recurrent dimensions of Σ and the topological entropy $\mathcal{H}_p(u)$, which is defined by using the complexity function of the sequence u.

$$\underline{D}(\Sigma) \leq \mathcal{H}_p(u) \leq \overline{D}(\Sigma).$$

Recently, in [1] B.Adamczewski and Y. Bugeaud gave a class of transcendental numbers, the partial quotients sequences of which have some recurrent properties. For this class numbers we call them τ_0-transcendental numbers or τ_0-recurrent numbers with the recurrent order value $\tau_0 > 0$. The complement of the set of τ_0-transcendental numbers in the set of irrational numbers contains algebraic numbers of degree ≥ 3 and 0-recurrent or non-recurrent transcedental numbers.

For the sequence u of partial quotients of an irrational number α we can show that

$$\underline{D}(\Sigma) = \overline{D}(\Sigma) = 0$$

if α is a τ_0-transcendental number for $\tau_0 > 0$ with some unifomly recurrent conditions.

In our previous papers [10], [12], [13] we introduce the gap value $\mathcal{G}(\Sigma)$ of recurrent dimensions by

$$\mathcal{G}(\Sigma) = \overline{D}(\Sigma) - \underline{D}(\Sigma)$$

as the parameter which specifies the levels of unpredictability of a sequence u or a discrete orbit. Thus, if we could prove the converse statement: $\overline{D}(\Sigma) = 0$ yields that α is a transcendental number (or a quadratic irrational), we could show that the sequence u of partial quotients of an algebraic number, which has its degree ≥ 3, satisfies

$$0 = \underline{D}(\Sigma) < \overline{D}(\Sigma) \quad \text{or} \quad 0 < \underline{D}(\Sigma) \leq \mathcal{H}_p(u),$$

that is, the sequence u is unpredictable, since its gap value $\mathcal{G}(\Sigma)$ of recurrent dimensions is positive, or chaotic, since its topological entropy $\mathcal{H}_p(u)$ is positive. Nothing has yet been proved for the converse statement, but here we give an example of continued fractions which has positive gaps of recurrent dimensions.

This paper is an anouncement of our recent results and so their complete proves will be given in the forthcoming paper.

2. Symbolic dynamical systems

In this section, introducing notations of symbolic dynamical systems, we show some inequality relations between the recurrent dimensions and the topological entropy of an alphabet sequence.

Let $\mathcal{A} = \{a_1, a_2, ..., a_d\}$ be a finite set of symbols and a word $V = v_1v_2..v_r$ be a finite string of elements of $\mathcal{A}$ with its length r, denoted by $|V| = r$. The set of nonnegative integers is denoted by $\mathbf{N}_0 = \mathbf{N} \cup \{0\} = \{0, 1, 2, ...\}$ and we consider a (one-sided) sequence of elements of $\mathcal{A}$, $u = (u_n)_{n \in \mathbf{N}_0} = u_0u_1u_2... \in \mathbf{A}^{\mathbf{N}_0}$. A word $W = w_1w_2...w_r$ is called a factor of u if $u_m = w_1, u_{m+1} = w_2, ..., u_{m+r-1} = w_r$ for some $m \in \mathbf{N}_0$. $\mathcal{L}(u)$ denotes the set of all factors of u, which is called the language of the sequence u and $\mathcal{L}_n(u)$ denotes the set of all factors with its length n.

We denote the complexity function of u by $P_u(n) = \#\mathcal{L}_n(u)$, which is the number of different words of length n occurring in u. We consider the following metric on $\mathcal{A}^{\mathbf{N}_0}$:

$$d(u,v) = 2^{-\min\{n\in\mathbf{N}_0: u_n\neq v_n\}}$$

for $u, v \in \mathcal{A}^{\mathbf{N}_0} : u \neq v$. The one-sided shift $\sigma : \mathcal{A}^{\mathbf{N}_0} \to \mathcal{A}^{\mathbf{N}_0}$ is defined by

$$(\sigma u)_n = u_{n+1}, \quad n \in \mathbf{N}_0$$

and its discrete orbit is denoted by

$$\Sigma := \Sigma_u = \{u, \sigma u, \sigma^2 u, ..., \sigma^n u, ...\}.$$

Denote the recurrency function of u by $R_u(n)$, which is the least integer $m(:= R_u(n))$ such that each m-factor of u contains every n-factor of u.

We define the first ε-recurrent times by

$$\underline{M}_\Sigma(\varepsilon) = \inf_{l\in\mathbf{N}_0} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\},$$

$$\overline{M}_\Sigma(\varepsilon) = \sup_{l\in\mathbf{N}_0} \min\{m \in \mathbf{N} : \ d(\sigma^{m+l}u, \sigma^l u) < \varepsilon\}.$$

Then we can obtain the following relations

Lemma 2.1. *For* $\varepsilon_n = 2^{-n}$, $n = 1, 2, ...$, *we have*

$$\underline{M}_\Sigma(\varepsilon_n) \leq P_u(n), \tag{2.1}$$

$$\overline{M}_\Sigma(\varepsilon_n) = R_u(n) - n + 1. \tag{2.2}$$

In [6] Morse and Hedlund have given the following inequality

$$P_u(n) + n \leq R_u(n). \tag{2.3}$$

Now we have the following sequence of inequalities:

$$\underline{M}_\Sigma(\varepsilon_n) \leq P_u(n), \quad P_u(n) + n \leq R_u(n), \quad R_u(n) - n + 1 = \overline{M}_\Sigma(\varepsilon_n).$$

It follows that

$$\underline{M}_\Sigma(\varepsilon_n) \leq P_u(n) \leq \overline{M}_\Sigma(\varepsilon_n). \tag{2.4}$$

The topological entropy $\mathcal{H}_P(u)$ is given by the complexity function:

$$\mathcal{H}_P(u) = \lim_{n\to\infty} \frac{\log_d P_u(n)}{n}.$$

Here we also put

$$\mathcal{H}_R(u) = \lim_{n\to\infty} \frac{\log_d R_u(n)}{n}$$

and we define the recurrent dimensions

$$\overline{D}_r(\Sigma) = \limsup_{\varepsilon\to 0} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon},$$

$$\underline{D}_r(\Sigma) = \liminf_{\varepsilon\to 0} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log\varepsilon}.$$

In [10] we have shown that these recurrent dimensions are given by

$$\overline{D}_r(\Sigma) = \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon},$$
$$\underline{D}_r(\Sigma) = \liminf_{n\to\infty} \inf_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log\varepsilon}$$

for any sequence $\{\varepsilon_n\} : \varepsilon_n \downarrow 0$ as $n \to \infty$.

Then we have the following inequality relations.

Theorem 2.2. *For a sequence $u \in \mathcal{A}^{\mathbf{N}_0}$ and $\Sigma = \{\sigma^n u : n \in \mathbf{N}_0\}$ we have*

$$\frac{\log 2}{\log d} \cdot \underline{D}_r(\Sigma) \le \mathcal{H}_P(u) \le \mathcal{H}_R(u) = \frac{\log 2}{\log d} \cdot \overline{D}_r(\Sigma).$$

Proof. The first (left side) inequality can be estimated by the definitions and Eq.(2.1) in Lemma 2.1.

$$\begin{aligned}
\underline{D}_r(\Sigma) &= \liminf_{n\to\infty} \inf_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \underline{M}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\le \liminf_{n\to\infty} \frac{\log \underline{M}_\Sigma(\varepsilon_n)}{-\log\varepsilon_n} \\
&\le \frac{\log d}{\log 2} \lim_{n\to\infty} \frac{\log P_u(n)}{n\log d} = \frac{\log d}{\log 2} \cdot \mathcal{H}_P(u).
\end{aligned}$$

The second inequality is obvious from the definitions and Eq.(2.3). The right hand side equality is also obtained by the following estimates, using Eq.(2.2),

$$\begin{aligned}
\overline{D}_r(\Sigma) &= \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\le \limsup_{n\to\infty} \frac{\log \overline{M}_\Sigma(\varepsilon_{n+1})}{-\log\varepsilon_n} \\
&= \limsup_{n\to\infty} \frac{\log(R_u(n+1) - (n+1) + 1)}{n\log 2} \\
&\le \frac{\log d}{\log 2} \lim_{n\to\infty} \frac{\log R_u(n+1)}{(n+1)\log d} \cdot \frac{n+1}{n} = \frac{\log d}{\log 2} \cdot \mathcal{H}_R(u)
\end{aligned}$$

and

$$\begin{aligned}
\overline{D}_r(\Sigma) &= \limsup_{n\to\infty} \sup_{\varepsilon_{n+1}\le\varepsilon\le\varepsilon_n} \frac{\log \overline{M}_\Sigma(\varepsilon)}{-\log\varepsilon} \\
&\ge \limsup_{n\to\infty} \frac{\log \overline{M}_\Sigma(\varepsilon_n)}{-\log\varepsilon_n} \\
&= \lim_{n\to\infty} \frac{\log(R_u(n) - n + 1)}{n\log 2} \\
&\ge \frac{\log d}{\log 2} \lim_{n\to\infty} \frac{\log \frac{1}{2} R_u(n)}{n\log d} = \frac{\log d}{\log 2} \cdot \mathcal{H}_R(u).
\end{aligned}$$

□

3. τ_0-TRANSCENDENTAL NUMBERS

In this section we investigate the recurrent properties and the entropy of discrete orbits given by partial quotients sequences of continued fraction expansions.

For an irrational positive number $\alpha : 0 < \alpha < 1$ we consider the sequence $u = \{a_i\}_{i\geq 1}$ of its partial quotients: $\alpha = [0 : a_1, a_2, ...]$. Let U_j, V_j be words of u: $U_j = a_1 a_2 \cdots a_{n_j}$ and $V_j = a_1 a_2 \cdots a_{m_j}$. We denote the concatenation of the two words U_j and V_j by $U_j V_j$.

We say that a real number α has a linearly recurrent continued fraction sequence (abr. c.f.s.) if there exists an infinite sequence of prefix words $U_j V_j$ in the c.f.s u of α which satisfy the following conditions for a positive constant τ_0:

(i) $|U_j|$ is increasing,

(ii) $\dfrac{|V_j|}{|U_j|} \geq \tau_0$ for all j.

Hereafter we also consider the case where the sequence of the partial quotients are unbounded, but, for simplicity, we assume the following condition on the denominators of convergents $\{p_n/q_n\}$:

$$\lim_{l\to\infty} (q_l)^{\frac{1}{l}} = e^{\frac{\pi^2}{12\log 2}} := K_L(\text{Khinchin-Lévy Constant})$$

It is well known that almost all numbers satisfy this condition.

In [1] Adamczewski and Bugeaud proved that, if an irrational number α has a linearly recurrent c.f.s., then α is transcendental. So we also call a real number, which has the linearly recurrent c.f.s with the conditions (i) and (ii), a τ_0-transcendental number.

We need the further definitions on recurrent dimensions. For an element $\sigma^l u \in \Sigma$, $l \in \mathbf{N}_0$, define the first ε-recurrent time by

$$M_{\sigma^l u}(\varepsilon) = \min\{m \in \mathbf{N} : \ d_2(\sigma^{m+l}u, \sigma^l u) < \varepsilon\},$$

and the upper and the lower recurrent dimensions of $\sigma^l u \in \Sigma$ by

$$\overline{D}_r(\sigma^l u) = \limsup_{\varepsilon\to 0} \frac{\log M_{\sigma^l u}(\varepsilon)}{-\log\varepsilon},$$

$$\underline{D}_r(\sigma^l u) = \liminf_{\varepsilon\to 0} \frac{\log M_{\sigma^l u}(\varepsilon)}{-\log\varepsilon}.$$

It follows from the definitions that we have

$$\underline{D}_r(\Sigma) \leq \underline{D}_r(\sigma^l u) \leq \overline{D}_r(\sigma^l u) \leq \overline{D}_r(\Sigma) \tag{3.1}$$

for every $l \in \mathbf{N}_0$.

We can obtain the following relation between the recurrent dimensions and the τ_0-transcendental numbers.

Theorem 3.1. *We assume that α is a τ_0-transcendental number for some $\tau_0 > 0$, which satisfies the conditions (i), (ii) and the following condition (iii) in addition:*

(iii) $\displaystyle\lim_{j\to\infty} \frac{\log|V_{j+1}|}{|V_j|} = 0.$

Then $\overline{D}_r(\sigma^n u) = \underline{D}_r(\sigma^n u) = 0$ holds for all $n \in \mathbf{N}_0$.

We can prove Theorem 3.1 by using the following Lemmas.

Lemma 3.2. *If α is a τ_0-transcendental number for some $\tau_0 > 0$ with satisfying the condition (iii), $\overline{D}_r(u) = \underline{D}_r(u) = 0$ holds.*

Lemma 3.3. *If α is a τ_0-transcendental number for $\tau_0 > 0$ with (iii), $G^n\alpha$, $\forall n \in \mathbf{N}$, is a τ_0-transcendental number for $\tau_0 > 0$ with (iii).*

Lemma 3.4. *For $n \in \mathbf{N}$, assume that $G^n\alpha$ is a τ_0-transcendental number for some $\tau_0 > 0$ with (iii). Then $\overline{D}_r(\sigma^n u) = \underline{D}_r(\sigma^n u) = 0$ holds.*

Adamczewski and Bugeaud have also given the following transcendental criterion in [1].

Let $\alpha : 0 < \alpha < 1$, an irrational number and consider the sequence $u = \{a_i\}_{i \geq 1}$, the sequence of the partial quotients of $\alpha = [0 : a_1, a_2, ...]$. $\Sigma = \{\sigma^n u : n \in \mathbf{N}_0\}$.

Here we say that α is a weakly τ_0-transcendental number if there exists a sequence $\{W_j U_j V_j\}$ of prefixes of u, which satisfies the following conditions:

(i') each V_j is a prefix of $U_j V_j$ and $|W_j|$ is increasing,

(ii') there exists a constant $\tau_0 > 0 : \dfrac{|V_j|}{|U_j|} \geq \tau_0$ for all j,

(iii') there exists a constant $\tau_1 > 0 : \ \tau_0 > \tau_1, \ \dfrac{|W_j|}{|U_j|} \leq \tau_1$ for all j.

Theorem 3.5. *Assume that α is a weakly τ_0-transcendental number. Then we have*

$$\underline{D}_r(\Sigma) = 0.$$

By using the same argument as in the proof of Theorem 3.5 we obtain the following corollary.

Corollary 3.6. *Assume that α is a τ_0-transcendental number. Then we have*

$$\underline{D}_r(\Sigma) = 0.$$

To obtain the estimate

$$\overline{D}_r(\Sigma) = \underline{D}_r(\Sigma) = 0$$

we need the following uniform assumption on the recurrent property.

We say that α has a uniformly recurrent c.f.s. if there exist two increasing sequences of integers $\{l_j\}$, $\{m_j\}$ such that for every $l \in \mathbf{N}$ the c.f.s. of a real number $G^l\alpha$ has a sequence of prefix words $\{U_j' V_j'\}$, which satisfies

$$|V_j'| \geq l_j, \quad |U_j'| \leq m_j.$$

Theorem 3.7. *Assume that α has a uniformly recurrent c.f.s. for the sequences $\{l_j\}$, $\{m_j\}$ and assume that*

$$\lim_{j \to \infty} \frac{\log m_{j+1}}{l_j} = 0. \tag{3.2}$$

Then $\overline{D}_r(\Sigma) = \underline{D}_r(\Sigma) = 0$ holds.

If the sequence $\{G^l\alpha\}$ is almost periodic, then α has a uniformly recurrent c.f.s.. The definition of almost periodicity in this case is as follows.

For every small $\varepsilon > 0$, there exists $l_\varepsilon > 0$ (ε-inclusion length) such that for each integer m, there exists a integer $p \in [m, m + l_\varepsilon]$ (ε-almost period), which satisfies

$$\sup_{l \in \mathbf{N}} d(G^l\alpha, G^{l+p}\alpha) \leq \varepsilon.$$

Putting $\varepsilon_j = 2^{-l_j}$, $l_{\varepsilon_j} = m_j$, we can take an ε_j-almost period $p_j \in [1, l_{\varepsilon_j}]$, which satisfies

$$\sup_{l \in \mathbf{N}} d(G^l\alpha, G^{l+p_j}\alpha) \leq \varepsilon_j.$$

Thus we can admit the c.f.s. of $G^l\alpha$, which has prefixes $\{U_j'V_j'\}$ such that

$$|U_j'| = p_j \leq l_{\varepsilon_j} = m_j, \quad |V_j'| \geq l_j.$$

4. Continued fractions of Sturmian sequences

Let $\mathcal{A} = \{1, 2\}$ and $u = u_0u_1u_2...$ be a 1-type sturmian sequence, which does not contain a word 22. Then it is well known that the complexity function $P_u(n) = n + 1$ and the frequency value of the letter 1 is given by

$$\tau = \lim_{n \to \infty} \frac{|u_0u_1...u_{n-1}|_1}{n}$$

where $|U|_1$ is the number of occurrences of the letter 1 in a word U.

For the discrete orbit

$$\Sigma := \Sigma_u = \{u, \sigma u, \sigma^2 u, ..., \sigma^n u, ...\}$$

it follows from Theorem 3.5 that the lower recurrent dimension $\underline{D}_r(\Sigma) = 0$, since the irrational number $\alpha = [u_0, u_1, \cdots]$, which has a Sturmian c.f. sequence, is τ_0-tarascendental (cf. [2], [3]). Since the complexity function $P_u(n) = n + 1$ for Sturmian sequences, we can also show that $\underline{D}_r(\Sigma) = 0$ by using the definition of the topological entropy $\mathcal{H}_p(u)$ and the inequality relation in Theorem 2.2.

Here we estimate its upper recurrent dimensions $\overline{D}_r(\Sigma)$ according to the algebraic properties, parametrized Diophantine conditions, of the frequency value τ. In our previous papers [10],[11] we introduce d_0-(D) condition, which specifies the (good or bad) levels of approximation by rational numbers.

If an irrational number τ satisfies d_0-(D) condition for $0 \leq d_0 < \infty$, then τ is a Roth number with its order $d_0 + \varepsilon$ for every $\varepsilon > 0$ and also τ is a weak Liouville number with its order $d_0 - \varepsilon$ for every $\varepsilon > 0$. In case where an irrational number τ does not satisfy the Diophantine condition for a finite value d_0, we say that τ is a Liouville number or $d_0 = \infty$.

Theorem 4.1. *For a discrete orbit Σ given by a Sturmian sequence, assume that the frequency τ satisfies d_0-(D) condition for $0 \leq d_0 < \infty$. Then we have*

$$\overline{D}_r(\Sigma) = 0. \tag{4.1}$$

Theorem 4.2. *For a discrete orbit Σ given by a Sturmian sequence, let $\{r_n/s_n\}$ be the convergents of the frequency τ and assume that τ is a Liouville number such that there exists a subsequence $\{s_{n_j}\} \subset \{s_n\}$, which satisfies*

$$s_{n_j+1} \geq L^{s_{n_j}} \tag{4.2}$$

for a constant $L > 1$. Then we have

$$\overline{D}_r(\Sigma) \geq \frac{\log L}{\log 2}. \tag{4.3}$$

Consequently, the gap value of the recurrent dimensions is positive:

$$\mathcal{G}(\Sigma) \geq \frac{\log L}{\log 2}.$$

Remark 4.3. It follows from(4.2) that

$$|\tau - \frac{r_{n_j}}{s_{n_j}}| \leq \frac{1}{s_{n_j} L^{s_{n_j}}},$$

which gives the extremely good approximation property ($d_0 = \infty$) by rational numbers.

References

1. B.Adamczewski and Y.Bugeaud, *On the complexity of algebraic numbers II. Continued fractions*, Acta Math. **195** (2005), 1-20.
2. B.Adamczewski, Y.Bugeaud and J.L.Davison, *Continued fractions and transcendental numbers*, Ann. Inst. Fourier (Grenoble) **56** (2006), 2093–2113.
3. J.-P.Allouche, J.L.Davison, M.Queffe'lec and L.Q.Zamboni, *Transcendence of Sturmian or morphic continued fractions*, J.Number Theory. **91** (2001), 39–66.
4. J.D.Barrow, *Chaos in the Einstein Equations*, Phys. Rev. Lett. **46** (1981), 963–966.
5. Y.A.Khinchin, "Continued Fractions", the University of Chicago Press 1964. 28 # 5037
6. M.Morse and G.A.Hedlund, *Symbolic dynamics II- Sturmian trajectories*, American Journal Math., **62** (1940), 1–42.
7. K.Naito, *Dimension estimate of almost periodic attractors by simultaneous Diophantine approximation*, J. Differential Equations, **141** (1997), 179–200.
8. ———, *Correlation dimensions of quasi-periodic orbits with frequencies given by quasi Roth numbers*, J. Korean Math. Soc. **37** (2000), 857–870.
9. ———, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations*, Trans. Amer. Math. Soc. **354** no. 3 (2002), 1137–1151.
10. ———, *Recurrent dimensions of quasi-periodic solutions for nonlinear evolution equations II: Gaps of dimensions and Diophantine conditions*, Discrete and Continuous Dynamical Systems **11** (2004), 449–488.
11. ———, *Classifications of Irrational Numbers and Recurrent Dimensions of Quasi-Periodic Orbits*, J. Nonlinear Anal. Convex Anal. **5** (2004), 169–185.
12. K.Naito and Y.Nakamura, *Recurrent dimensions and Diophantine conditions of discrete dynamical systems given by circle mappings*, J. Nonlinear and Convex Analysis, **8** (2007), 105–120.
13. ———, *Recurrent dimensions and Diophantine conditions of discrete dynamical systems given by circle mappings II*, Yokohama M.J. **54** (2007), 13–30.
14. W.M.Schmidt, "Diophantine Approximation", Springer Lecture Notes in Math. 785, 1980.