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Generalized James constant and fixed point theorems for multivalued nonexpansive mappings*

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Abstract. We present the concept of the generalized James constant and its use concerning the uniform normal structure. Then the Domínguez - Lorenzo condition will be introduced. A relationship on the constant and the condition is considered. As a consequence, a fixed point theorem for multivalued nonexpansive mappings is obtained.

Keywords: Generalized James constant, Domínguez - Lorenzo condition, Fixed point theorem.

Mathematics Subject Classification: 47H09, 54H25.

1 Introduction

Let X be a Banach space and E be a weakly compact convex subset of X. Let $T : E \to KC(E)$ be a nonexpansive mappings with values are compact and convex subsets of E. Since we are considering self-valued nonexpansive mappings, we may assume throughout that the domain E is also separable (see [16]).

Since the publication of Nadler [18] in 1969 on the extension of the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces many authors have tried to do the same for classical fixed point theorems for single-valued nonexpansive mappings.

By using Edelstien's method of asymptotic centers, Lim [17] proved in 1974 the existence of a fixed point for a multivalued nonexpansive self-mapping $T : E \to K(E)$ where E is a nonempty bounded closed convex subset of a uniformly convex Banach space. In 1990, Kirk and Massa [14] extended this theorem of Lim by proving that every multivalued nonexpansive self-mapping $T : E \to KC(E)$ has a

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fixed point where E is a nonempty bounded closed convex subset of a Banach space X for which the asymptotic center in E of each bounded sequence of X is nonempty and compact. Xu [20] in 2001 extended Kirk-Massa's theorem to a multivalued nonself-mapping $T : E \to KC(X)$ which satisfies the inwardness condition.

Following the idea in Domínguez and Lorenzo [10], Dhompongsa et al.[5] introduced the so-called the Domínguez - Lorenzo condition ((DL)-condition), i.e., an inequality concerning the asymptotic radius and the Chebyshev radius of the asymptotic center for some types of sequences and proved a fixed point theorem for a nonself multivalued nonexpansive mapping on a Banach space which satisfies the (DL)-condition. It is known that [7, Theorem 3.6], the (DL)-condition implies the weak multivalued fixed point property (w-MFPP)(i.e., every nonexpansive mapping $T : E \to KC(E)$ has a fixed point, where E is a weakly compact convex subset of X). Indeed, in [7], we introduced another property, namely, property (D), which is stricly weaker than the (DL)-condition, the property that implies w-MFPP.

Recently, Domínguez and Gavira [8] proved that every uniformly smooth Banach space has w-MFPP by showing that the condition $\xi_X(\beta) < \frac{1}{1-\beta}$ for some $\beta \in (0,1)$ satisfies the (DL)-condition. Here ξ_X is the modulus of squareness of the space X. They also showed in [8] that the condition $r_X(1) > 0$ impies the (DL)-condition, where r_X is the Opial modulus associated to the space X.

The purpose of this paper is devoted to finding more properties that implies the (DL)-condition.

2 Preliminaries

Let X and E be as above , let $FB(E)$ be the family of nonempty bounded closed subsets of E and $KC(E)$ be the family of nonempty compact convex subsets of E. Let $H(.,.)$ denote the Hausdorff distance on $FB(X)$, i.e.,

$$H(A,B) := \max\{\sup_{a\in A}\operatorname{dist}(a,B), \sup_{b\in B}\operatorname{dist}(b,A)\}, \quad A,B \in FB(X),$$

where $\operatorname{dist}(a,B) := \inf\{\|a-b\| : b \in B\}$ is the distance from the point a to the subset B.

A multivalued mapping $T : E \to FB(E)$ is said to be *nonexpansive* if

$$H(Tx,Ty) \le \|x-y\| \text{ for all } x,y \in E.$$

We say that x is a *fixed point* of T if $x \in Tx$.

Let A be a nonempty bounded subset of X. The number $r(A) := \inf\{\sup_{y\in A}\|x-y\| : x \in A\}$ is called the *Chebyshev radius* of A. The number $\delta(A) := \sup\{\|x-y\| : x,y \in A\}$ is called the *diameter* of A. A Banach space X is said to have *normal structure* (respectively, *weak normal structure*) if $r(A) < \delta(A)$ for every bounded closed (respectively, weakly compact) convex subset A of X with $\delta(A) > 0$. X is said to have *uniform normal structure* (respectively, *weak uniform normal structure*) if

$$\gamma(X) := \inf\frac{\delta(A)}{r(A)} > 1, \tag{2.1}$$

where the infimum is taken over all bounded closed (respectively, weakly compact) convex subsets A of X with $\delta(A) > 0$. The *weakly convergent sequence coefficient* $WCS(X)$ [2] of X is the number

$$WCS(X) := \inf\{ \lim_{n,m\to\infty, n\neq m} \|x_n - x_m\|\}, \tag{2.2}$$

where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim_{n\to\infty}\|x_n\| = 1$ and $\lim_{n,m\to\infty,n\neq m}\|x_n - x_m\|$ exists. It is known that $1 \leq WCS(X) \leq 2$ and $WCS(X) > 1$ implies X has weak normal structure (see [2]).

For a Banach space X, the Jordan - von Neumann constant $C_{NJ}(X)$ of X, introduced by Clarkson [3], is defined by

$$C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + 2\|y\|^2} : x, y \in X \text{ not both zero}\}. \tag{2.3}$$

The James (or the uniform nonsquare) constant defined by Gao and Lau [11] by $J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}$, where B_X is the closed unit ball of X.
Dhompongsa et al.[6] extended this concept and defined a generalized James constant $J(a, X)$ for $a \in [0, \infty)$ as

$$J(a, X) = \sup\{\|x + y\| \wedge \|x - z\| : x, y, z \in B_X \text{ and } \|y - z\| \leq a\|x\|\}. \tag{2.4}$$

They proved in [6] that every space X with $J(X) < \frac{1+\sqrt{5}}{2}$ or $J(a, X) < \frac{3+a}{2}$ for some $a \in [0, 1]$ has uniform normal structure.

Let $\{x_n\}$ be a bounded sequence in X. We define the *asymptotic radius* and the *asymptotic center* of $\{x_n\}$ in E, respectively, by

$r(E, \{x_n\}) = \inf\{\limsup_{n\to\infty}\|x_n - x\| : x \in E\}$ and $A(E, \{x_n\}) = \{x \in E : \limsup_{n\to\infty}\|x_n - x\| = r(E, \{x_n\})\}$.

We call a sequence $\{x_n\}$ *regular relative to* E if $r(E, \{x_n\}) = r(E, \{y_n\})$ for all subsequences $\{y_n\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called *asymptotically uniform relative to* E if $A(E, \{x_n\}) = A(E, \{y_n\})$ for all subsequences $\{y_n\}$ of $\{x_n\}$.

Lemma 2.1. Let $\{x_n\}$ and E be as above. Then

(i) (Gobel [12], Lim [17]) there always exists a subsequence of $\{x_n\}$ which is regular relative to E,

(ii) (Kirk [15]) if E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform relative to E.

If C is a bounded subset of X, the *Chebyshev radius* of C relative to E is defined by $r_E(C) = \inf\{r_x(C) : x \in E\}$, where $r_x(C) = \sup\{\|x - y\| : y \in C\}$.

The Domínguez - Lorenzo condition introduced in [5] is defined as follows :

Definition 2.2. [5, Definition 3.1] A Banach space X is said to satisfy the *Domínguez - Lorenzo ((DL)-) condition* if there exists $\lambda \in [0, 1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular relative to E,

$$r_E(A(E, \{x_n\})) \leq \lambda r(E, \{x_n\}).$$

Finally, we give a brief formulation of an ultrapower of a Banach space X. Let $\mathcal{U}$ be a nontrivial ultrafilter on the set of positive integers $\mathbb{Z}^+$. Let $l_\infty(X)$ be the space of all bounded sequences in X, that is, $l_\infty(X) = \{\{x_n\} \subset X : \sup \|x_n\| < \infty\}$ and consider the closed subspace $\mathcal{N}$ of $l_\infty(X)$: $\mathcal{N} = \{\{x_n\} \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. Let $\widetilde{X}$ be the quotient space $l_\infty(X)/\mathcal{N}$ and call it an ultrapower of X. For each $x = \{x_n\} \in l_\infty(X)$, let $\widetilde{x}$ stand for the equivalence class of x. Then the quotient norm $\|\widetilde{x}\|$ of $\widetilde{x}$ is $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$. Note that for every sequence $\{a_n\}$ of real numbers, $\liminf_n a_n \le \lim_{\mathcal{U}} a_n \le \limsup_n a_n$. We denote for each subset E of X, $\dot{E}$ the set $\{\tilde{x} : x = \{x_n\}, x_n = x_1 \in E \text{ for all n}\}$, and denote for each $v \in X$, $\dot{v}$ the equivalence class of the sequence $\{v_n\}$ where $v_n = v$ for all n. Thus $\dot{E} = \{\dot{v} : v \in E\}$. For more details on the subject, we refer to [1] and [19].

3 Results

For a sequence $\{x_n\}$, let $\text{sep}\{x_n\} = \inf_{n \neq m} \|x_n - x_m\|$.

Definition 3.1. A Banach space X is said to have the *Uniform Kadec-Klee (UKK) property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x_n \in B_X$, $x_n \xrightarrow{w} x$ and $\text{sep}\{x_n\} \ge \varepsilon$ imply $\|x\| \le 1 - \delta$.

Definition 3.2. A Banach space X is said to be *nearly uniformly convex* (NUC) if for any $\varepsilon > 0$, there exists $\eta < 1$ such that $x_n \in B_X$ and $\text{sep}\{x_n\} \ge \varepsilon$ imply $\text{co}\{x_n\} \cap \eta B_X \neq \emptyset$.

A space X is NUC if and only if it has the UKK property and is reflexive (see [13]). It is a consequence of Domínguez and Gavira [8, Corollary 2] that UKK property implies the (DL)-condition. Here we give a direct proof.

Theorem 3.3. Every space X which has the UKK property satisfies the (DL)-condition.

Proof. Let E be a weakly compact convex subset of X, $\{x_n\} \subset E$ a sequence in E which is regular relative to E. Take a subsequence $\{y_n\}$ of $\{x_n\}$ such that $y_n \xrightarrow{w} z \in E$ and $\lim_{n \neq m} \|y_n - y_m\|$ exists. Let $r = r(E, \{x_n\})$ and $A = A(E, \{x_n\})$. Let $0 < \rho < 1$ and take $\delta > 0$ corresponding to ρ from the definition of the UKK property. Let $\eta > 0$ and $\varepsilon > 0$ so that $\frac{r-\eta}{r+\varepsilon} > \rho$. Let $x \in A$ and choose n_0 so that $\|y_n - x\| < r + \varepsilon$ for all $n \ge n_0$.
Since

$$r \le \limsup_n \|y_n - z\| \le \limsup_n \limsup_m \|y_n - y_m\| = \lim_{n \neq m} \|y_n - y_m\|,$$

we can choose $n_1 \ge n_0$ so that

$$\|y_n - y_m\| \ge r - \eta \text{ for all } n, m \ge n_1 \text{ with } n \neq m.$$

For convenience, assume $n_0 = n_1 = 1$. Now $\frac{y_n - x}{r+\varepsilon} \in B_X$, $\frac{y_n - x}{r+\varepsilon} \xrightarrow{w} \frac{z-x}{r+\varepsilon}$, and $\text{sep}\{\frac{y_n - x}{r+\varepsilon}\} \ge \frac{r-\eta}{r+\varepsilon} > \rho$. Thus $\frac{\|z-x\|}{r+\varepsilon} \le 1 - \delta$ and this implies

$$r_E(A(E, \{x_n\})) \le \|z - x\| \le (1 - \delta)(r + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrarily small, we obtain

$$r_E(A(E, \{x_n\})) \le (1 - \delta) r(E, \{x_n\})$$

and the (DL)-condition holds. □

The proof given above is based on the proof of [9, Theorem 3.4].

In [7] it is proved that every space X with $C_{NJ}(X) \leq 1 + \frac{WCS^2(X)}{4}$ has property(D). Here we obtain its analogue in terms of $\gamma(X)$ in (2.1). Observe that, under the present condition, we obtain a stronger result.

Theorem 3.4. Let X be a Banach space. If $C_{NJ}(X) \leq 1 + \frac{\gamma^2(X)}{4}$, then X satisfies the (DL) - condition.

Proof. Let E be a weakly compact convex subset of X and let $\{x_n\} \subset E$ be a bounded sequence which is regular relative to E. Let $A = A(E, \{x_n\})$. Put $\lambda = \frac{2}{\gamma(X)}\sqrt{C_N J(X) - 1} < 1$. Let $u, v \in A$. Thus $\frac{u+v}{2} \in A$ since A is convex. Consider $\widetilde{x} = \widetilde{(x_n)}$ in an ultrapower $\widetilde{X}$ of X with respect to some non-trivial ultrafilter $\mathcal{U}$ on $\mathbb{Z}^+$. Note that $\|\widetilde{x} - \dot{a}\| \leq r(E, \{x_n\}) := r$ for all $a \in A$. From the definition of the Jordan-von Neumann constant (2.3) and the fact that $C_{\mathrm{NJ}}(\widetilde{X}) = C_{\mathrm{NJ}}(X)$, we obtain the following estimates:

$$\begin{aligned}
\|\dot{u} - \dot{v}\|^2 = \|(\dot{u} - \tilde{x}) - (\dot{v} - \tilde{x})\|^2 &\leq 4r^2 C_{\mathrm{NJ}}(X) - \|\dot{u} + \dot{v} - 2\widetilde{x}\|^2 \\
&= 4r^2 C_{\mathrm{NJ}}(X) - 4\|\frac{\dot{u} + \dot{v}}{2} - \widetilde{x}\|^2 \\
&= 4r^2 C_{\mathrm{NJ}}(X) - 4r^2.
\end{aligned}$$

Thus, in terms of $\gamma(X)$, we have

$$\gamma(X) r_E(A) \leq \delta(A) \leq 2r\sqrt{C_{\mathrm{NJ}}(X) - 1} = 2\sqrt{C_{\mathrm{NJ}}(X) - 1}\, r(E, \{x_n\}).$$

Therefore,

$$r_E(A(E, \{x_n\})) \leq \lambda r(E, \{x_n\})$$

as desired. □

In [5] it is shown that every Banach space X which has property WORTH and $J(X) < 2$ satisfies the (DL)-condition. In [4], we have the following results.

Theorem 3.5. Let a Banach space X satisfy the non-strict Opial condition and let E be a weakly compact convex subset of X. Assume that $\{x_n\}$ is a sequence in E which is regular relative to E. Then

$$r_E(A(E, \{x_n\})) \leq \frac{J(1, X)}{2} r(E, \{x_n\}).$$

Corollary 3.6. Let X be a Banach space with $J(1, X) < 2$ and satisfies *non-strict Opial condition.* Then X satisfies the (DL) - condition.

Example 3.7. *There exists a space X satisfying $J(1, X) < 2$ and the non-strictly Opial but does not have WORTH:*

Let $1 < p < 2$, $X_1 = \mathbb{R}^2$ with norm $\|x\| = \|(x_1, x_2) = \|x\|_1$ or $\|x\|_p$ according as $x_1 x_2 \geq 0$ or $x_1 x_2 \leq 0$. Then let our space X be $\ell_2(X_1)$.

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