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Octahedral Projection

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Abstract. The octahedral projection can be used to obtain the octahedral subdivision for a given simplicial subdivision of a simplex. By suitable labellings, we prove that our multiple balanced Sperner's lemma, a generalized Sperner's lemma of Shapley with the consideration of orientations, is equivalent to our combinatiorial formula for multiple set-valued labellings. A multiple balanced KKM theorem can be derived from the multiple balanced Sperner's lemma and can be used to prove the nonemptyness of the common core of coupled balanced games.

1. Preliminaries

The following (M1), (M2) and (M3) from matrix theory will be used later. All matrices are real here.

(M1) If A is $p \times p$ and B is $q \times q$, then

$$\det(I_p + kAB) = \det(I_q + kBA). \tag{1.1}$$

(M2) If A is $p \times p$, B is $q \times q$, C is $p \times q$ and D is $q \times p$ and if A, B and $B + BDA^{-1}CB$ are nonsigular, then

$$(A + CBD)^{-1} = A^{-1} - A^{-1}CB(B + BDA^{-1}CB)^{-1}BDA. \tag{1.2}$$

(M3) If A_{11} is $p \times p$, A_{22} is $q \times q$, A_{12} is $p \times q$ and A_{21} is $q \times p$ and if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

then

$$\det A = \det A_{22} \det(A_{11} - A_{12}{A_{22}}^{-1}A_{21}), \tag{1.3}$$

provided A_{22} is nonsigular.

(M1) follows from the following two identities

$$\begin{pmatrix} I_p + kAB & kA \\ O & I_q \end{pmatrix} = \begin{pmatrix} I_p & kA \\ -B & I_q \end{pmatrix} \begin{pmatrix} I_p & O \\ B & I_q \end{pmatrix}$$

and

$$\begin{pmatrix} I_p & kA \\ O & I_q + kBA \end{pmatrix} = \begin{pmatrix} I_p & O \\ B & I_q \end{pmatrix} \begin{pmatrix} I_p & kA \\ -B & I_q \end{pmatrix}.$$

By multiplying the right side of (1.2) and the matrix $A+CBD$ directly, (M2) follows. (M3) is a generalization of the expansion of 2×2 determinants.

The $p \times q$ matrix with all entries 1 will be denoted by $\mathbf{1}_{p\times q}$, thus

$$\mathbf{1}_{p\times q} = \mathbf{1}_{p\times 1}\mathbf{1}_{1\times q} \text{ and } \mathbf{1}_{1\times p}\mathbf{1}_{p\times 1} = p. \tag{1.4}$$

Let n be a positive integer and α be a real number. The following identities are direct consequences of (1.1), (1.2) and (1.4).

$$\det(I_n - \alpha\mathbf{1}_{n\times n}) = 1 - n\alpha, \tag{1.5}$$

$$\det(-I_n - \alpha\mathbf{1}_{n\times n}) = (-1)^n(1 + n\alpha), \tag{1.6}$$

$$(I_n - \alpha\mathbf{1}_{n\times n})^{-1} = I_n + \frac{\alpha}{1 - n\alpha}\mathbf{1}_{n\times n} \quad (\alpha \neq \frac{1}{n}). \tag{1.7}$$

2. Octahedral Projections

In this section, we shall let

(a) $a_1, \cdots, a_n$ be the standard basis of the Euclidean n-space, (2.1)

(b) $a = \sum_{i=1}^{n} \alpha a_i = (\alpha, \cdots, \alpha)$ where α is some real number, (2.2)

(c) $b_i = a_i - a$ for $i = 1, \cdots, n$, (2.3)

(d) $b_i' = -a_i - a$ for $i = 1, \cdots, n$. (2.4)

Proposition 2.1.

(a) $\det(b_1 \cdots b_n) = 1 - n\alpha$. (2.5)

(b) $\det(b_1' \cdots b_n') = (-1)^n(1 + n\alpha)$. (2.6)

(c) $\det(b_1' \cdots b_r' b_{r+1} \cdots b_{r+s}) = (-1)^r(1 + r\alpha - s\alpha)$, where $r + s = n$. (2.7)

Proof. (a) It follows from (1.5), (2.1), (2.2) and (2.3) that

$$\det(b_1 \cdots b_n) = \det(I_n - \alpha\mathbf{1}_{n\times n}) = 1 - n\alpha.$$

(b) It follows from (1.6), (2.1), (2.2) and (2.4) that

$$\det(b_1' \cdots b_n') = \det(-I_n - \alpha\mathbf{1}_{n\times n}) = (-1)^n(1 + n\alpha).$$

(c) If $r = 0$ or $s = 0$, then (2.7) becomes (2.5) or (2.6) respectively, so let $r \geq 1$ and $s \geq 1$. We first assume $\alpha \neq \frac{1}{s}$, then by (1.7) with $n = s$, we have

$$(I_s - \alpha\mathbf{1}_{s\times s})^{-1} = I_s + \frac{\alpha}{1 - s\alpha}\mathbf{1}_{s\times s}, \tag{2.8}$$

also, by (1.5) with $n = s$, we have

$$\det(I_s - \alpha \mathbf{1}_{s\times s}) = 1 - s\alpha. \tag{2.9}$$

It follows from (1.1), (1.3), (1.4), (2.3), (2.4), (2.8), (2.9) and a computation that

$$\begin{aligned}
& \det(b_1{}' \cdots b_r{}' b_{r+1} \cdots b_{r+s}) \\
= \; & \det \begin{pmatrix} -I_r - \alpha \mathbf{1}_{r\times r} & -\alpha \mathbf{1}_{r\times s} \\ -\alpha \mathbf{1}_{s\times r} & I_s - \alpha \mathbf{1}_{s\times s} \end{pmatrix} \\
= \; & (-1)^r (1 - s\alpha) \det(I_r + \frac{\alpha}{1 - s\alpha} \mathbf{1}_{r\times r}) \\
= \; & (-1)^r (1 - s\alpha)(1 + \frac{r\alpha}{1 - s\alpha}) \\
= \; & (-1)^r (1 - s\alpha + r\alpha).
\end{aligned}$$

Since the right side of (2.7) is a continuous function of α, (2.7) is also valied for $\alpha = \frac{1}{s}$.

Corollary

If $c_i = b_i$ or $b_i{}'$ for $i = 1, \cdots, n$ and if r of the vectors $c_1, \cdots, c_n$ are of the form $b_i{}'$ and $n - r$ of the form b_i, then

$$\det(c_1 \cdots c_n) = (-1)^r (1 + r\alpha - s\alpha). \tag{2.10}$$

Consequently, they are linearly independent if

$$\alpha \neq \frac{-1}{n}, \frac{-1}{n-2}, \cdots, \frac{1}{n-2}, \frac{1}{n}. \tag{2.11}$$

Proof. If $c_i = b_i$ and $c_j = b_j{}'$ for some $i < j$, then the determinant does not change when interchanging the ith row and the jth row then interchanging the ith column and the jth column but c_i and c_j are replaced by $b_i{}'$ and b_j. Continue this process if necessary, we finally obtain

$$d(c_1 \cdots c_n) = \det(b_1{}' \cdots b_r{}' b_{r+1} \cdots b_{r+s})$$

thus (2.10) follows from (2.7), consequently, $c_1, \cdots, c_n$ are linearly independent if and only if

$$1 + r\alpha - s\alpha \neq 0$$

or equivalently,

$$(s - r)\alpha \neq 1.$$

Since $s + r = n$, we have

$$s - r \in \{-n, \; -n+2, \; \cdots, \; n-2, \; n\}.$$

For a given subset S of the Euclidean n-space, the convex hull and the affine hull of S will be denoted by *convS* and *affS* respectively. In particular,

$$p = \sum_{i=1}^{n} x_i a_i \in aff\{a_1, \cdots, a_n\}$$

if and only if

$$x_1 + \cdots + x_n = 1. \tag{2.12}$$

From (2.1), (2.2), (2.4) and (2.12) it follows that

$$a + tb_i' \in aff\{a_1, \cdots, a_n\}$$

if and only if

$$t = \frac{n\alpha - 1}{n\alpha + 1}, \ \alpha \neq -\frac{1}{n}.$$

Recall that a *ray* emanting from a is the set

$$\{a + tb \mid t \geq 0\}$$

where b is a fixed nonzero vector. Note that in the following Proposition 2.2, t is positive.

Proposition 2.2. Suppose that

$$\overline{a_j} = a + tb_j' \quad \text{for } j = 1, \cdots, n \tag{2.13}$$

where

$$t = \frac{n\alpha - 1}{n\alpha + 1} > 0. \tag{2.14}$$

If

$$\overline{a_j} = \sum_{i=1}^{n} p_{ij} a_i \qquad (1 \leq j \leq n), \tag{2.15}$$

$$a_j = \sum_{i=1}^{n} q_{ij} \overline{a_i} \qquad (1 \leq j \leq n), \tag{2.16}$$

then

$$P = (p_{ij}) = \frac{2\alpha}{n\alpha + 1} \mathbf{1}_{n \times n} - \frac{n\alpha - 1}{n\alpha + 1} I_n, \tag{2.17}$$

$$Q = (q_{ij}) = \frac{2\alpha}{n\alpha - 1} \mathbf{1}_{n \times n} - \frac{n\alpha + 1}{n\alpha - 1} I_n. \tag{2.18}$$

Proof. Since

$$\overline{a_j} = a + tb_j' = a + t(-a_j - a) = (1 - t)a + t(-a_j),$$

we have

$$\sum_{i=1}^{n} p_{ij} a_j = \sum_{i=1}^{n} \{(1 - t)\alpha - t\delta_{ij}\} a_i$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

which gives (2.17).

If we write

$$P = \alpha(1-t)\{\frac{-t}{\alpha(1-t)}I_n + \mathbf{1}_{n\times 1}\ 1\ \mathbf{1}_{1\times n}\}$$

then, by (1.2), a computation will show that (2.18) holds.

The *relative interior* of $conv\{v_1, \cdots, v_m\}$ in $aff\{v_1, \cdots, v_m\}$ is denoted by $Int\{v_1, \cdots, v_m\}$ which is the set

$$\{\sum_{i=1}^{n} \lambda_i v_i \mid \sum_{i=1}^{n} \lambda_i = 1, \text{ each } \lambda_i > 0\}.$$

Corollary

$$conv\{a_1, \cdots, a_n\} \subset Int\{\overline{a_1}, \cdots, \overline{a_n}\}$$

if and only if

$$-\frac{1}{n-2} < \alpha < -\frac{1}{n} \tag{2.19}$$

where $n \geq 2$ and $-\frac{1}{n-2} \equiv -\infty$ if $n = 2$.

Proof. That a_i is an affine combination of $\overline{a_1}, \cdots, \overline{a_n}$ follows from (2.16) and (2.17). We may write (2.18) as

$$Q = \frac{n\alpha+1}{n\alpha-1}(\frac{2\alpha}{n\alpha+1}\mathbf{1}_{n\times n} - I_n).$$

Since

$$\frac{n\alpha+1}{n\alpha-1} = \frac{1}{t} > 0,$$

we have $q_{ij} > 0$ for all i, j if and only if

$$\frac{2\alpha}{n\alpha+1} - 1 > 0$$

which is equivalent to (2.19).

By the $(n-1)$-*sphere* S^{n-1} we mean the set of all points

$$p = \sum_{i=1}^{n} x_i a_i \tag{2.20}$$

satisfying

$$\sum_{i=1}^{n} |x_i| = 1. \tag{2.21}$$

We may write (2.20) as

$$p = \sum_{x_i>0} |x_i| a_i + \sum_{x_i<0} |x_i|(-a_i) \tag{2.22}$$

or,by (2.23) and (2.24),

$$p - a = \sum_{x_i>0} |x_i| b_i + \sum_{x_i<0} |x_i| b_i' . \tag{2.23}$$

If

$$s(p - a) = \sum_{x_i>0} \mu_i b_i + \sum_{x_i<0} \mu_i t b_i' \tag{2.24}$$

where

$$\sum_{x_i \neq 0} \mu_i = 1 \tag{2.25}$$

then, by (2.11), (2.13) and (2.19), the unknows s and μ_i can be solved. Geometrically, the point $a+s(p-a)$ is the central projection of $p \in S^{n-1}$ on $aff\{a_1, \cdots, a_n\}$ relative to the center a, this makes the following definition.

Definition

Let $n \geq 2$ and let

$$t = \frac{n\alpha + 1}{n\alpha - 1} \quad \text{where} \quad -\frac{1}{n-2} < \alpha < -\frac{1}{n}. \tag{2.26}$$

The mapping $f : S^{n-1} \to aff\{a_1, \cdots, a_n\}$ defined by

$$f(p) = \sum_{x_i>0} \mu_i a_i + \sum_{x_i<0} \mu_i \overline{a_i} \tag{2.27}$$

in which

$$p = \sum_{i=1}^{n} x_i a_i \quad \text{with} \quad \sum_{i=1}^{n} |x_i| = 1 \tag{2.28}$$

and

$$\mu_i = s|x_i| \ \text{ if } x_i > 0, \quad \mu_i = s|x_i|/t \text{ if } x_i < 0 \tag{2.29}$$

where

$$s = 1/(\sum_{x_i>0} |x_i| + \sum_{x_i<0} |x_i|/t) \tag{2.30}$$

is called an *octahedral projection.* (In case $n = 3$, S^{n-1} is the surface of an octahedron, hence the name.)

Remarks

(a) Since x_i may be positive or negative or zero in (2.28), there are exactly $3^n - 1$ faces of the simplicial complex S^{n-1} consisting of $C_2^n 2^r$ $(r-1)$-faces for $r = 1, \cdots, n$; the relative interiors of these faces form a partition of S^{n-1}.

(b) It follows from (2.3), (2.4), (2.26) and the corollary of 2.1 that $a+c_1, \cdots, a+c_n$

are affinely independent, so by (2.27), (2.28), (2.29) and (2.30) that f maps the open simplex

$$Int(\{a_i|x_i > 0\} \cup \{-a_i|x_i < 0\})$$

onto the open simplex

$$Int(\{a_i|x_i > 0\} \cup \{\overline{a_i}|x_i < 0\}).$$

(c) By (2.26), $\alpha < -\frac{1}{n}$, the ray

$$R : a + s(p - a), \quad s \geq 0$$

will pierce the interior of the closed unit ball

$$B^n : \sum_{i=1}^{n} |x_i| \leq 1$$

if $p \in Int\{-a_1, \cdots, -a_n\}$, so that $R \cap S^{n-1}$ will have exactly two points for S^{n-1} is the boundary of the convex body B^n.

(d) It follows from (2.26), (2.27), corollary of 2.2 and the previous remarks (a), (b) and (c) that f maps the polyhedron $S^{n-1} \setminus Int\{-a_1, \cdots, -a_n\}$ onto the $(n-1)$-simplex $conv\{\overline{a_1}, \cdots, \overline{a_n}\}$ bijectively and induces an *octahedral subdivision* of this image which consisting of the images of the $3^n - 2$ faces, all faces of S^{n-1} but the face $conv\{-a_1, \cdots, -a_n\}$.

(e) Let T be a simplicial subdivision of the $(n-1)$-simplex $conv\{a_1, \cdots, a_n\}$. If σ is an $(r-1)$-simplex of T with the vertices $v_1, \cdots, v_r$ and if the carrier of σ is

$$conv\{a_i|i \in I\} \text{ for some } I \subset \{1, \cdots, n\}$$

then

$$\tilde{\sigma} = conv(\{v_1, \cdots, v_r\} \cup \{\overline{a_j}|j \in J\}),$$

where $J = \{1, \cdots, n\} \backslash I$, is an $(n-1)$-simplex for

$$\{a_i|i \in I\} \cup \{\overline{a_j}|j \in J\}$$

is linearly independent. The set of all such $(n-1)$-simplexes $\tilde{\sigma}$ together with their faces is then a simplicial complex $\tilde{T}$, the *induced octahedral subdivision* of $conv\{\overline{a_1}, \cdots, \overline{a_n}\}$ from T.

(f) (2.7) shows that

$$\det(\overline{a_1} \cdots \overline{a_n}) = (-t)^{n-1} \tag{2.34}$$

and that

$$\det(d_1 \cdots d_n) = (-t)^r \left|\frac{1-(n-2r)\alpha}{1-n\alpha}\right| \tag{2.35}$$

where t and α are given by (2.26), $1 \leq r \leq n-1$, and where $d_i = \overline{a_i}$ or a_i for $i = 1, \cdots, n$ and r of $d_1, \cdots, d_n$ are of the from $\overline{a_i}$ and $n - r$ of the form a_i.

(g) (2.33), (2.34) and (2.35) give the relation between the orientations of σ and $\tilde{\sigma}$ in the coherently oriented $(n-1)$-pseudomainfold $\tilde{T}$, where the orientations are induced by the signs of their determinants.

By suitable labellings on the vertices of $\overline{a_1}, \cdots, \overline{a_n}$, we have proven that our multiple balanced Sperner's lemma [2] is equivalent to our combinatorial formulas for multiple set-valued labellings [2]. A multiple balanced KKM theorem [2] can be derived from the multiple balanced Sperner's lemma and can be used to prove the nonemptyness of the common core of coupled balanced games [5]. Related results can be found in [1], [3] and [4].

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