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Kyoto University
Octahedral Projection

Shyh-Nan Lee and Mau-Hsiang Shih

Department of Applied Mathematics, Chung Yuan Christian University, Chung Li 32023, Taiwan, Republic of China
Email: nan@math.cycu.edu.tw

Department of Mathematics, National Taiwan Normal University, 88 Sec. 4, Ting Chou Road, Taipei 116, Taiwan, Republic of China
Email: mhshih@math.ntnu.edu.tw

Abstract. The octahedral projection can be used to obtain the octahedral sub-division for a given simplicial subdivision of a simplex. By suitable labellings, we prove that our multiple balanced Sperner’s lemma, a generalized Sperner’s lemma of Shapley with the consideration of orientations, is equivalent to our combinatorial formula for multiple set-valued labellings. A multiple balanced KKM theorem can be derived from the multiple balanced Sperner’s lemma and can be used to prove the nonemptyness of the common core of coupled balanced games.

1. Preliminaries

The following (M1), (M2) and (M3) from matrix theory will be used later. All matrices are real here.

(M1) If $A$ is $p \times p$ and $B$ is $q \times q$, then

$$\det(I_p + kAB) = \det(I_q + kBA).$$

(1.1)

(M2) If $A$ is $p \times p$, $B$ is $q \times q$, $C$ is $p \times q$ and $D$ is $q \times p$ and if $A$, $B$ and $B + BDA^{-1}CB$ are nonsingular, then

$$(A + CBD)^{-1} = A^{-1} - A^{-1}CB(B + BDA^{-1}CB)^{-1}BDA.$$  (1.2)

(M3) If $A_{11}$ is $p \times p$, $A_{22}$ is $q \times q$, $A_{12}$ is $p \times q$ and $A_{21}$ is $q \times p$ and if

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

then

$$\det A = \det A_{22} \det(A_{11} - A_{12}A_{22}^{-1}A_{21}),$$

(1.3)

provided $A_{22}$ is nonsingular.

(M1) follows from the following two identities

$$\begin{pmatrix} I_p + kAB & kA \\ O & I_q \end{pmatrix} = \begin{pmatrix} I_p & kA \\ -B & I_q \end{pmatrix} \begin{pmatrix} I_p & O \\ B & I_q \end{pmatrix}$$
and
\[
\begin{pmatrix}
I_p & kA \\
O & I_q + kBA
\end{pmatrix}
= 
\begin{pmatrix}
I_p & O \\
B & I_q
\end{pmatrix}
\begin{pmatrix}
I_p & kA \\
-B & I_q
\end{pmatrix}.
\]
By multiplying the right side of (1.2) and the matrix \(A+CBD\) directly, (M2) follows. (M3) is a generalization of the expansion of \(2\times2\) determinants.

The \(p \times q\) matrix with all entries 1 will be denoted by \(1_{p\times q}\), thus
\[
1_{p\times q} = 1_{p\times 1}1_{1\times q} \quad \text{and} \quad 1_{1\times p}1_{p\times 1} = p.
\] (1.4)

Let \(n\) be a positive integer and \(\alpha\) be a real number. The following identities are direct consequences of (1.1), (1.2) and (1.4).

\[
\det(I_n - \alpha 1_{nxn}) = 1 - n\alpha,
\] (1.5)
\[
\det(-I_n - \alpha 1_{nxn}) = (-1)^n(1 + n\alpha),
\] (1.6)
\[
(I_n - \alpha 1_{nxn})^{-1} = I_n + \frac{\alpha}{1 - n\alpha}1_{nxn} \quad (\alpha \neq \frac{1}{n}).
\] (1.7)

2. Octahedral Projections

In this section, we shall let

\(a\) \(a_1, \ldots, a_n\) be the standard basis of the Euclidean \(n\)-space, \quad (2.1)
\(b\) \(a = \sum_{i=1}^{n} \alpha a_i = (\alpha, \ldots, \alpha)\) where \(\alpha\) is some real number, \quad (2.2)
\(c\) \(b_i = a_i - a\) for \(i = 1, \ldots, n\), \quad (2.3)
\(d\) \(b_i' = -a_i - a\) for \(i = 1, \ldots, n\). \quad (2.4)

Proposition 2.1.

\(a\) \(\det(b_1 \cdots b_n) = 1 - n\alpha\). \quad (2.5)
\(b\) \(\det(b_1' \cdots b_n') = (-1)^n(1 + n\alpha)\). \quad (2.6)
\(c\) \(\det(b_1' \cdots b_r' b_{r+1} \cdots b_{r+s}) = (-1)^r(1 + r\alpha - s\alpha)\), \(\text{where } r + s = n\). \quad (2.7)

Proof. \(a\) It follows from (1.5), (2.1), (2.2) and (2.3) that
\[
\det(b_1 \cdots b_n) = \det(I_n - \alpha 1_{nxn}) = 1 - n\alpha.
\]
\(b\) It follows from (1.6), (2.1), (2.2) and (2.4) that
\[
\det(b_1' \cdots b_n') = \det(-I_n - \alpha 1_{nxn}) = (-1)^n(1 + n\alpha).
\]
\(c\) If \(r = 0\) or \(s = 0\), then (2.7) becomes (2.5) or (2.6) respectively, so let \(r \geq 1\) and \(s \geq 1\). We first assume \(\alpha \neq \frac{1}{s}\), then by (1.7) with \(n = s\), we have
\[
(I_s - \alpha 1_{sxn})^{-1} = I_s + \frac{\alpha}{1 - s\alpha}1_{sxn}, \quad (2.8)
\]
also, by (1.5) with \( n = s \), we have

\[
\det(I_s - \alpha 1_{s \times s}) = 1 - sa. \tag{2.9}
\]

It follows from (1.1), (1.3), (1.4), (2.3), (2.4), (2.8), (2.9) and a computation that

\[
\det(b_1 \cdots b_r b_{r+1} \cdots b_{r+s})
= \det\left(
\begin{array}{cc}
-I_r - \alpha 1_{rxr} & -\alpha 1_{rxs} \\
-\alpha 1_{srx} & I_s - \alpha 1_{sxs}
\end{array}
\right)
= (-1)^r (1 - sa) \det(I_r + \frac{\alpha}{1 - sa} 1_{rxr})
= (-1)^r (1 - sa)(1 + \frac{r\alpha}{1 - sa})
= (-1)^r (1 - sa + r\alpha).
\]

Since the right side of (2.7) is a continuous function of \( \alpha \), (2.7) is also valid for \( \alpha = \frac{1}{s} \).

Corollary

If \( c_i = b_i \) or \( b_{i'} \) for \( i = 1, \cdots, n \) and if \( r \) of the vectors \( c_1, \cdots, c_n \) are of the form \( b_{i'} \) and \( n - r \) of the form \( b_i \), then

\[
\det(c_1 \cdots c_n) = (-1)^r (1 + r\alpha - sa). \tag{2.10}
\]

Consequently, they are linearly independent if

\[
\alpha \neq \frac{1}{n}, \frac{1}{n-2}, \cdots, \frac{1}{n-2}, \frac{1}{n}. \tag{2.11}
\]

Proof. If \( c_i = b_i \) and \( c_j = b_j \) for some \( i < j \), then the determinant does not change when interchanging the \( i \)th row and the \( j \)th row then interchanging the \( i \)th column and the \( j \)th column but \( c_i \) and \( c_j \) are replaced by \( b_{i'} \) and \( b_j \). Continue this process if necessary, we finally obtain

\[
d(c_1 \cdots c_n) = \det(b_1 \cdots b_r b_{r+1} \cdots b_{r+s})
\]

thus (2.10) follows from (2.7), consequently, \( c_1, \cdots, c_n \) are linearly independent if and only if

\[
1 + r\alpha - sa \neq 0
\]

or equivalently,

\[
(s - r)\alpha \neq 1.
\]

Since \( s + r = n \), we have

\[
s - r \in \{-n, -n + 2, \cdots, n - 2, n\}.
\]

For a given subset \( S \) of the Euclidean \( n \)-space, the convex hull and the affine hull of \( S \) will be denoted by \( \text{conv}S \) and \( \text{aff}S \) respectively. In particular,

\[
p = \sum_{i=1}^{n} x_i a_i \in \text{aff}\{a_1, \cdots, a_n\}
\]
if and only if

$$x_1 + \cdots + x_n = 1.$$  \hfill (2.12)

From (2.1), (2.2), (2.4) and (2.12) it follows that

$$a + tb_{i'} \in \text{aff}\{a_1, \cdots, a_n\}$$

if and only if

$$t = \frac{n\alpha - 1}{n\alpha + 1}, \alpha \neq -\frac{1}{n}.$$  

Recall that a **ray** emanating from $a$ is the set

$$\{a + tb \mid t \geq 0\}$$

where $b$ is a fixed nonzero vector. Note that in the following Proposition 2.2, $t$ is positive.

**Proposition 2.2.** Suppose that

$$\overline{a_j} = a + tb_j \quad \text{for } j = 1, \cdots, n$$  \hfill (2.13)

where

$$t = \frac{n\alpha - 1}{n\alpha + 1} > 0.$$  \hfill (2.14)

If

$$\overline{a_j} = \sum_{i=1}^{n} p_{ij} a_i \quad (1 \leq j \leq n),$$  \hfill (2.15)

$$a_j = \sum_{i=1}^{n} q_{ij} \overline{a_i} \quad (1 \leq j \leq n),$$  \hfill (2.16)

then

$$P = (p_{ij}) = \frac{2\alpha}{n\alpha + 1}1_{nxn} - \frac{n\alpha - 1}{n\alpha + 1}I_n,$$  \hfill (2.17)

$$Q = (q_{ij}) = \frac{2\alpha}{n\alpha - 1}1_{nxn} - \frac{n\alpha + 1}{n\alpha - 1}I_n.$$  \hfill (2.18)

**Proof.** Since

$$\overline{a_j} = a + tb_j = a + t(-a_j - a) = (1 - t)a + t(-a_j),$$

we have

$$\sum_{i=1}^{n} p_{ij} a_j = \sum_{i=1}^{n} \{(1 - t)\alpha - t\delta_{ij}\} a_i$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
which gives (2.17).
If we write
\[ P = \alpha(1 - t)\left\{ \frac{-t}{\alpha(1 - t)}I_n + 1_{n \times 1}1_{1 \times n} \right\} \]
then, by (1.2), a computation will show that (2.18) holds.

The relative interior of \( \text{conv}\{v_1, \cdots, v_m\} \) in \( \text{aff}\{v_1, \cdots, v_m\} \) is denoted by \( \text{Int}\{v_1, \cdots, v_m\} \) which is the set
\[ \{ \sum_{i=1}^{n} \lambda_i v_i | \sum_{i=1}^{n} \lambda_i = 1, \text{each } \lambda_i > 0 \} \]

**Corollary**

\( \text{conv}\{a_1, \cdots, a_n\} \subset \text{Int}\{\overline{a_1}, \cdots, \overline{a_n}\} \)

if and only if
\[ \frac{-1}{n-2} < \alpha < -\frac{1}{n} \] (2.19)
where \( n \geq 2 \) and \( -\frac{1}{n-2} \equiv -\infty \) if \( n = 2 \).

**Proof.** That \( a_i \) is an affine combination of \( \overline{a_1}, \cdots, \overline{a_n} \) follows from (2.16) and (2.17). We may write (2.18) as
\[ Q = \frac{n\alpha + 1}{n\alpha - 1} \left( \frac{2\alpha}{n\alpha + 1}1_{n \times n} - I_n \right). \]
Since
\[ \frac{n\alpha + 1}{n\alpha - 1} = \frac{1}{t} > 0, \]
we have \( q_{ij} > 0 \) for all \( i, j \) if and only if
\[ \frac{2\alpha}{n\alpha + 1} - 1 > 0 \]
which is equivalent to (2.19).

By the \((n - 1)\)-sphere \( S^{n-1} \) we mean the set of all points
\[ p = \sum_{i=1}^{n} x_i a_i \] (2.20)
satisfying
\[ \sum_{i=1}^{n} |x_i| = 1. \] (2.21)

We may write (2.20) as
\[ p = \sum_{x_i > 0} |x_i| a_i + \sum_{x_i < 0} |x_i| (-a_i) \] (2.22)
or, by (2.23) and (2.24),

\[
p - a = \sum_{x_i > 0} |x_i| b_i + \sum_{x_i < 0} |x_i| b_i'.
\]  

(2.23)

If

\[
s(p - a) = \sum_{x_i > 0} \mu_i b_i + \sum_{x_i < 0} \mu_i t b_i'
\]  

(2.24)

where

\[
\sum_{x_i \neq 0} \mu_i = 1
\]  

(2.25)

then, by (2.11), (2.13) and (2.19), the unknowns \( s \) and \( \mu_i \) can be solved. Geometrically, the point \( a + s(p - a) \) is the central projection of \( p \in S^{n-1} \) on \( \text{aff}\{a_1, \ldots, a_n\} \) relative to the center \( a \), this makes the following definition.

**Definition**

Let \( n \geq 2 \) and let

\[
t = \frac{n\alpha + 1}{n\alpha - 1} \quad \text{where} \quad -\frac{1}{n - 2} < \alpha < -\frac{1}{n}.
\]  

(2.26)

The mapping \( f : S^{n-1} \rightarrow \text{aff}\{a_1, \ldots, a_n\} \) defined by

\[
f(p) = \sum_{x_i > 0} \mu_i a_i + \sum_{x_i < 0} \mu_i \overline{a_i}
\]  

(2.27)

in which

\[
p = \sum_{i=1}^{n} x_i a_i \quad \text{with} \quad \sum_{i=1}^{n} |x_i| = 1
\]  

(2.28)

and

\[
\mu_i = s|x_i| \quad \text{if} \quad x_i > 0, \quad \mu_i = s|x_i|/t \quad \text{if} \quad x_i < 0
\]  

(2.29)

where

\[
s = 1/(\sum_{x_i > 0} |x_i| + \sum_{x_i < 0} |x_i|/t)
\]  

(2.30)

is called an **octahedral projection**. (In case \( n = 3 \), \( S^{n-1} \) is the surface of an octahedron, hence the name.)

**Remarks**

(a) Since \( x_i \) may be positive or negative or zero in (2.28), there are exactly \( 3^n - 1 \)

faces of the simplicial complex \( S^{n-1} \) consisting of \( C_2 \) \( r \)-faces for

\( r = 1, \ldots, n; \) the relative interiors of these faces form a partition of \( S^{n-1} \).

(b) It follows from (2.3), (2.4), (2.26) and the corollary of 2.1 that \( a + c_1, \ldots, a + c_n \)
are affinely independent, so by (2.27), (2.28), (2.29) and (2.30) that $f$ maps the open simplex

$$\text{Int}(\{a_i \mid x_i > 0\} \cup \{-a_i \mid x_i < 0\})$$

onto the open simplex

$$\text{Int}(\{a_i \mid x_i > 0\} \cup \{-a_i \mid x_i < 0\}).$$

(c) By (2.26), $\alpha < -\frac{1}{n}$, the ray

$$R : a + s(p - a), \ s \geq 0$$

will pierce the interior of the closed unit ball

$$B^n : \sum_{i=1}^{n} |x_i| \leq 1$$

if $p \in \text{Int}\{-a_1, \cdots, -a_n\}$, so that $R \cap S^{n-1}$ will have exactly two points for $S^{n-1}$ is the boundary of the convex body $B^n$.

(d) It follows from (2.26), (2.27), corollary of 2.2 and the previous remarks (a), (b) and (c) that $f$ maps the polyhedron $S^{n-1} \setminus \text{Int}\{-a_1, \cdots, -a_n\}$ onto the $(n - 1)$-simplex $\text{conv}\{\overline{a_1}, \cdots, \overline{a_n}\}$ bijectively and induces an octahedral subdivision of this image which consisting of the images of the $3^n - 2$ faces, all faces of $S^{n-1}$ but the face $\text{conv}\{-a_1, \cdots, -a_n\}$.

(e) Let $T$ be a simplicial subdivision of the $(n - 1)$-simplex $\text{conv}\{a_1, \cdots, a_n\}$. If $\sigma$ is an $(r - 1)$-simplex of $T$ with the vertices $v_1, \cdots, v_r$ and if the carrier of $\sigma$ is

$$\text{conv}\{a_i \mid i \in I\}$$

for some $I \subset \{1, \cdots, n\}$

then

$$\tilde{\sigma} = \text{conv}\{\{v_1, \cdots, v_r\} \cup \{\overline{a_j} \mid j \in J\}\},$$

where $J = \{1, \cdots, n\} \setminus I$, is an $(n - 1)$-simplex for

$$\{a_i \mid i \in I\} \cup \{\overline{a_j} \mid j \in J\}$$

is linearly independent. The set of all such $(n - 1)$-simplexes $\tilde{\sigma}$ together with their faces is then a simplicial complex $\tilde{T}$, the induced octahedral subdivision of $\text{conv}\{\overline{a_1}, \cdots, \overline{a_n}\}$ from $T$.

(f) (2.7) shows that

$$\det(\overline{a_1} \cdots \overline{a_n}) = (-t)^{n-1} \quad (2.34)$$

and that

$$\det(d_1 \cdots d_n) = (-t)^r |\frac{1 - (n - 2r)\alpha}{1 - n\alpha}| \quad (2.35)$$

where $t$ and $\alpha$ are given by (2.26), $1 \leq r \leq n - 1$, and where $d_i = \overline{a_i}$ or $a_i$ for $i = 1, \cdots, n$ and $r$ of $d_1, \cdots, d_n$ are of the from $\overline{a_i}$ and $n - r$ of the form $a_i$. 
(g) (2.33), (2.34) and (2.35) give the relation between the orientations of \( \sigma \) and \( \tilde{\sigma} \) in the coherently oriented \((n-1)\)-pseudomainfold \( \tilde{T} \), where the orientations are induced by the signs of their determinants.

By suitable labellings on the vertices of \( \overline{a_1}, \ldots, \overline{a_n} \), we have proven that our multiple balanced Sperner's lemma [2] is equivalent to our combinatorial formulas for multiple set-valued labellings [2]. A multiple balanced KKM theorem [2] can be derived from the multiple balanced Sperner's lemma and can be used to prove the nonemptiness of the common core of coupled balanced games [5]. Related results can be found in [1], [3] and [4].

References


