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ABOUT THE JAMES CONSTANT OF ABSOLUTE NORMED SPACES II

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ABSTRACT. In this note, we describe some recent results concerning James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

1. INTRODUCTION

A Banach space X is called *uniformly non-square* if there is a $\delta > 0$ such that if $x, y \in S_X$ then $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$, where $S_X = \{x \in X : \|x\| = 1\}$. Gao and Lau [4] introduced the *James constant* of a Banach space X as follows:

$$J(X) = \sup\Big\{\min\left\{\|x+y\|, \|x-y\|\right\} : x, y \in S_X\Big\}.$$

We shall collect some properties about James constant:

(1) For any Banach space X we have $\sqrt{2} \leq J(X) \leq 2$.

(2) If X is a Hilbert space, then $J(X) = \sqrt{2}$.

(3) $J(X) < 2$ if and only if X is uniformly non-square.

(4) If $1 \leq p \leq \infty$ and $\dim L_p \geq 2$, then

$$J(L_p) = \max\{2^{1/p}, 2^{1/p'}\}$$

where $1/p + 1/p' = 1$.

In this note, we describe some recent results concerning the James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $x, y \in \mathbb{R}$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x,y)\|_p = \begin{cases} (|x|^p + |y|^p)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|x|, |y|\} & \text{if } p = \infty. \end{cases}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$. Bonsall and Duncan [2] showed that for any absolute normalized norm on $\mathbb{R}^2$ there corresponds a continuous convex function on $[0,1]$ with some appropriate conditions as follows. Let Ψ_2 be the family of all continuous convex functions on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$. Then AN_2 and Ψ_2 are in a one to one correspondence under the equation

$$\text{(1)} \qquad \psi(t) = \|(1-t,\, t)\| \quad (0 \le t \le 1).$$

Indeed, for any $\|\cdot\| \in AN_2$ we put ψ as (1). Then $\psi \in \Psi_2$. Also, for all $\psi \in \Psi_2$ we define

$$\|(x,y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if } (x,y) \ne (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1). From this result, we can consider many non-ℓ_p-type norms easily. The functions which correspond with the ℓ_p-norms $\|\cdot\|_p$ on $\mathbb{R}^2$ are

$$\psi_p(t) = \begin{cases} \{(1-t)^p + t^p\}^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty. \end{cases}$$

For $0 < \omega < 1$ and $1 \le q < \infty$, the *2-dimensional Lorentz sequence space* $d^{(2)}(\omega, q)$ is $\mathbb{R}^2$ with the norm

$$\|x\|_{\omega,q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

where (x_1^*, x_2^*) is the nonincreasing rearrangement of $(|x_1|, |x_2|)$; that is, $x_1^* = \max\{|x_1|, |x_2|\}$ and $x_2^* = \min\{|x_1|, |x_2|\}$.

Note here that the norm $\|\cdot\|_{\omega,q}$ of $d^{(2)}(\omega,q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega,q}(t)=\begin{cases}\left((1-t)^q+\omega t^q\right)^{1/q} & \text{if } 0\le t\le 1/2,\\ \left(t^q+\omega(1-t)^q\right)^{1/q} & \text{if } 1/2\le t\le 1.\end{cases}$$

2. James constant of absolute normalized norms on $\mathbb{R}^2$

For a norm $\|\cdot\|$ on $\mathbb{R}^2$, we write $J(\|\cdot\|)$ for $J((\mathbb{R}^2,\|\cdot\|))$. Mitani and Saito [6] chracterized the James constant of $(\mathbb{R}^2,\|\cdot\|_\psi)$ in terms of ψ.

Theorem 1 ([6]). *Let $\psi\in\Psi_2$. If ψ is symmetric with respect to $t=1/2$, then*

$$J(\|\cdot\|_\psi)=\max_{0\le t\le 1/2}\frac{2-2t}{\psi(t)}\psi\left(\frac{1}{2-2t}\right).$$

Example 2. Let $1\le p\le\infty$ and $1/p+1/p'=1$. Then

$$J(\|\cdot\|_p)=\max\{2^{1/p},2^{1/p'}\}. \tag{2}$$

Indeed, we define a function f on $[0,1/2]$ as follows:

$$\begin{aligned} f(t)&=\frac{2-2t}{\psi_p(t)}\psi_p\left(\frac{1}{2-2t}\right)\\ &=\left(\frac{1+(1-2t)^p}{(1-t)^p+t^p}\right)^{1/p}.\end{aligned}$$

If $1\le p\le 2$, then f is the maximum at $t=0$ and

$$J(\|\cdot\|_{\psi_p})=f(0)=2^{1/p}.$$

If $p\ge 2$, then f is the maximum at $t=1/2$ and

$$J(\|\cdot\|_{\psi_p})=f(1/2)=2^{1/p'}.$$

Thus we obtain (2).

Example 3. Let $1/2 \le \lambda \le 1$. We define a function φ_λ as

$$\varphi_\lambda(t) = \max\{1-t, t, \lambda\}.$$

Then it is obvious that $\varphi_\lambda \in \Psi_2$. The corresponding absolute normalized norm $\|\cdot\|_{\varphi_\lambda}$ is

$$\|\cdot\|_{\varphi_\lambda} = \max\{\|\cdot\|_\infty, \lambda\|\cdot\|_1\}.$$

Then

$$J(\|\cdot\|_{\varphi_\lambda}) = \begin{cases} 1/\lambda & \text{if } \ 1/2 \le \lambda \le 1/\sqrt{2}, \\ 2\lambda & \text{if } \ 1/\sqrt{2} \le \lambda \le 1. \end{cases}$$

3. James constant of 2-dimensional Lorentz sequence spaces

Kato and Maligranda [5] calculated $d^{(2)}(\omega, q)$ in the case where $q \ge 2$, that is, they proved that if $0 < \omega < 1$ and $q \ge 2$, then

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

However, from Theorem 1 we obtain the following.

Lemma 4. *For $0 < \omega < 1$ and $1 \le q < \infty$,*

$$J\big(d^{(2)}(\omega, q)\big)\big(= J(\|\cdot\|_{\psi_{\omega,q}})\big) = \max_{0 \le t \le 1/2} \frac{2-2t}{\psi_{\omega,q}(t)} \psi_{\omega,q}\left(\frac{1}{2-2t}\right)$$

holds.

By using this lemma, we calculate $J\big(d^{(2)}(\omega, q)\big)$ in the case where $1 \le q < 2$.

Theorem 5 ([9], cf. [6, 12]). *Let $1 \le q < 2$.* (i) *If $0 < \omega \le (\sqrt{2}-1)^{2-q}$, then*

$$J\big(d^{(2)}(\omega, q)\big) = 2\left(\frac{1}{1+\omega}\right)^{1/q}.$$

(ii) *If $(\sqrt{2}-1)^{2-q} < \omega < 1$, then there exists a unique solution s_0 of the equation*

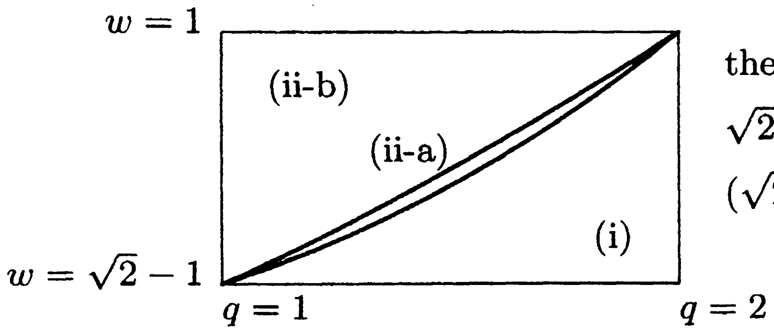
$$(1+s_0)^{q-1}(1-\omega s_0^{q-1}) = \omega(1-s_0)^{q-1}(1+\omega s_0^{q-1}), \quad 0 < s_0 < \omega^{1/(2-q)}.$$

(ii-a) *If* $(\sqrt{2}-1)^{2-q} < \omega \le \sqrt{2}^q - 1$, *then*

$$J\big(d^{(2)}(\omega,q)\big) = \max\left\{ \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q},\ 2\left(\frac{1}{1+\omega}\right)^{1/q} \right\}.$$

(ii-b) *If* $\sqrt{2}^q - 1 < \omega < 1$, *then*

$$J\big(d^{(2)}(\omega,q)\big) = \left(\frac{2(1+s_0)^{q-1}}{1+\omega s_0^{q-1}}\right)^{1/q}.$$



the graphs of $\sqrt{2}^q - 1$ (above) and $(\sqrt{2}-1)^{2-q}$ (below)

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