ABOUT THE JAMES CONSTANT OF ABSOLUTE NORMED SPACES (II) (Nonlinear Analysis and Convex Analysis)

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ABOUT THE JAMES CONSTANT OF ABSOLUTE NORMED SPACES II

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ABSTRACT. In this note, we describe some recent results concerning James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

1. INTRODUCTION

A Banach space $X$ is called uniformly non-square if there is a $\delta > 0$ such that if $x, y \in S_X$ then $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$, where $S_X = \{x \in X : \|x\| = 1\}$. Gao and Lau [4] introduced the James constant of a Banach space $X$ as follows:

$$J(X) = \sup \left\{ \min \{\|x + y\|, \|x - y\|\} : x, y \in S_X \right\}.$$

We shall collect some properties about James constant:

1. For any Banach space $X$ we have $\sqrt{2} \leq J(X) \leq 2$.
2. If $X$ is a Hilbert space, then $J(X) = \sqrt{2}$.
3. $J(X) < 2$ if and only if $X$ is uniformly non-square.
4. If $1 \leq p \leq \infty$ and $\dim L_p \geq 2$, then

$$J(L_p) = \max \{2^{1/p}, 2^{1/p'}\}$$

where $1/p + 1/p' = 1$.

In this note, we describe some recent results concerning the James constant of absolute norms on $\mathbb{R}^2$ and the 2-dimensional Lorentz sequence spaces.

A norm $\| \cdot \|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all $x, y \in \mathbb{R}$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The $\ell_p$-norms $\| \cdot \|_p$ are
such examples:

\[ \| (x, y) \|_p = \begin{cases} \left( |x|^p + |y|^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{|x|, |y|\} & \text{if } p = \infty. \end{cases} \]

Let \( AN_2 \) be the family of all absolute normalized norms on \( \mathbb{R}^2 \). Bonsall and Duncan [2] showed that for any absolute normalized norm on \( \mathbb{R}^2 \) there corresponds a continuous convex function on \([0, 1]\) with some appropriate conditions as follows. Let \( \Psi_2 \) be the family of all continuous convex functions on \([0, 1]\) such that \( \psi(0) = \psi(1) = 1 \) and \( \max\{1 - t, t\} \leq \psi(t) \leq 1 \). Then \( AN_2 \) and \( \Psi_2 \) are in a one to one correspondence under the equation

(1) \[ \psi(t) = \| (1 - t, t) \| \quad (0 \leq t \leq 1). \]

Indeed, for any \( \| \cdot \| \in AN_2 \) we put \( \psi \) as (1). Then \( \psi \in \Psi_2 \). Also, for all \( \psi \in \Psi_2 \) we define

\[ \| (x, y) \|_{\psi} = \begin{cases} (|x| + |y|)\psi\left(\frac{|y|}{|x| + |y|}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases} \]

Then \( \| \cdot \|_{\psi} \in AN_2 \), and \( \| \cdot \|_{\psi} \) satisfies (1). From this result, we can consider many non-\( \ell_p \)-type norms easily. The functions which correspond with the \( \ell_p \)-norms \( \| \cdot \|_p \) on \( \mathbb{R}^2 \) are

\[ \psi_p(t) = \begin{cases} \left( (1 - t)^p + t^p \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1 - t, t\} & \text{if } p = \infty. \end{cases} \]

For \( 0 < \omega < 1 \) and \( 1 \leq q < \infty \), the 2-dimensional Lorentz sequence space \( d^{(2)}(\omega, q) \) is \( \mathbb{R}^2 \) with the norm

\[ \| x \|_{\omega, q} = (x_1^{*q} + \omega x_2^{*q})^{1/q}, \quad x = (x_1, x_2) \in \mathbb{R}^2, \]

where \((x_1^*, x_2^*)\) is the nonincreasing rearrangement of \((|x_1|, |x_2|)\); that is, \( x_1^* = \max\{|x_1|, |x_2|\} \) and \( x_2^* = \min\{|x_1|, |x_2|\} \).
Note here that the norm $\| \cdot \|_{\omega, q}$ of $d^{(2)}(\omega, q)$ is a symmetric absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by

$$\psi_{\omega, q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{1/q} & \text{if } 0 \leq t \leq 1/2, \\ (t^q + \omega (1-t)^q)^{1/q} & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

2. **James Constant of Absolute Normalized Norms on $\mathbb{R}^2$**

For a norm $\| \cdot \|$ on $\mathbb{R}^2$, we write $J(\| \cdot \|)$ for $J((\mathbb{R}^2, \| \cdot \|))$. Mitani and Saito [6] characterized the James constant of $(\mathbb{R}^2, \| \cdot \|_{\psi})$ in terms of $\psi$.

**Theorem 1** ([6]). Let $\psi \in \Psi_2$. If $\psi$ is symmetric with respect to $t = 1/2$, then

$$J(\| \cdot \|_{\psi}) = \max_{0 \leq t \leq 1/2} \frac{2-2t}{\psi(t)} \psi\left(\frac{1}{2-2t}\right).$$

**Example 2.** Let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. Then

$$(2) \quad J(\| \cdot \|_{p}) = \max\{2^{1/p}, 2^{1/p'}\}.$$

Indeed, we define a function $f$ on $[0, 1/2]$ as follows:

$$f(t) = \frac{2-2t}{\psi_p(t)} \psi_p\left(\frac{1}{2-2t}\right)$$

$$= \left(\frac{1 + (1-2t)^p}{(1-t)^p + t^p}\right)^{1/p}.$$

If $1 \leq p \leq 2$, then $f$ is the maximum at $t = 0$ and

$$J(\| \cdot \|_{\psi_p}) = f(0) = 2^{1/p}.$$

If $p \geq 2$, then $f$ is the maximum at $t = 1/2$ and

$$J(\| \cdot \|_{\psi_p}) = f(1/2) = 2^{1/p'}.$$

Thus we obtain (2).
**Example 3.** Let \(1/2 \leq \lambda \leq 1\). We define a function \(\varphi_\lambda\) as
\[
\varphi_\lambda(t) = \max\{1 - t, t, \lambda\}.
\]
Then it is obvious that \(\varphi_\lambda \in \Psi_2\). The corresponding absolute normalized norm \(\| \cdot \|_{\varphi_\lambda}\) is
\[
\| \cdot \|_{\varphi_\lambda} = \max\{\| \cdot \|_\infty, \lambda \| \cdot \|_1\}.
\]
Then
\[
J(\| \cdot \|_{\varphi_\lambda}) = \begin{cases} 1/\lambda & \text{if } 1/2 \leq \lambda \leq 1/\sqrt{2}, \\ 2\lambda & \text{if } 1/\sqrt{2} \leq \lambda \leq 1. \end{cases}
\]

3. **James Constant of 2-Dimensional Lorentz Sequence Spaces**

Kato and Maligranda [5] calculated \(d^{(2)}(\omega, q)\) in the case where \(q \geq 2\), that is, they proved that if \(0 < \omega < 1\) and \(q \geq 2\), then
\[
J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1 + \omega}\right)^{1/q}.
\]
However, from Theorem 1 we obtain the following.

**Lemma 4.** For \(0 < \omega < 1\) and \(1 \leq q < \infty\),
\[
J(d^{(2)}(\omega, q)) = J(\| \cdot \|_{\psi_{\omega,q}}) = \max_{0 \leq t \leq 1/2} \frac{2 - 2t}{\psi_{\omega,q}(t)} \psi_{\omega,q}\left(\frac{1}{2 - 2t}\right)
\]
holds.

By using this lemma, we calculate \(J(d^{(2)}(\omega, q))\) in the case where \(1 \leq q < 2\).

**Theorem 5** ([9], cf. [6, 12]). Let \(1 \leq q < 2\). (i) If \(0 < \omega \leq (\sqrt{2} - 1)^{2-q}\), then
\[
J(d^{(2)}(\omega, q)) = 2 \left(\frac{1}{1 + \omega}\right)^{1/q}.
\]
(ii) If \((\sqrt{2} - 1)^{2-q} < \omega < 1\), then there exists a unique solution \(s_0\) of the equation
\[
(1 + s_0)^{q-1}(1 - \omega s_0^{q-1}) = \omega(1 - s_0)^{q-1}(1 + \omega s_0^{q-1}), \quad 0 < s_0 < \omega^{1/(2-q)}.\]
(ii-a) If $(\sqrt{2} - 1)^{2-q} < \omega \leq \sqrt{2}^q - 1$, then

$$J(d^{(2)}(\omega, q)) = \max \left\{ \left( \frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}, 2 \left( \frac{1}{1 + \omega} \right)^{1/q} \right\}.$$  

(ii-b) If $\sqrt{2}^q - 1 < \omega < 1$, then

$$J(d^{(2)}(\omega, q)) = \left( \frac{2(1 + s_0)^{q-1}}{1 + \omega s_0^{q-1}} \right)^{1/q}.$$  

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