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One-Point Solutions Obtained from Best Approximation Problems for Cooperative Games

Tetsuzo Tanino
Division of Electrical, Electronic and Information Engineering
Graduate School of Engineering, Osaka University

1 Introduction

A main topic in the cooperative game theory focuses on allocation schemes of profits among the players. These allocation schemes are called solutions for games. They are classified into two cateories: set-valued solutions and one-point solutions. A set-valued solution associates a set of profit vectors with each cooperative game. On the other hand, a one-point solution associates a profit vector with each game. Since each profit vector is an n dimensional vector, where n is the number of players, and therefore can be identified with an additive cooperative game in which the worth of each coalition is given as the simple sum of individual worths of players in the coalition. Hence we can regard a one-point solution for cooperative games on a finite set of players as a mapping from the set of cooperative games to the set of additive games (see Mondere and Samet [5]).

Some authors have studied one-point solutions in the settings of best approximation problems (Charnes et al. [2], Ruiz et al. [6, 7], Kultti and Salonen [4] and so on). Among them, Kultti and Salonen formulated a minimum norm problem to find the best approximation in the set of efficient addive games for a given cooperative game. They intorduced norms from inner products in the space of cooperative games, and proved that any efficient linear solution that has the inessential game property can be obtained as a solution to the minimum norm problem with an appropriate inner product. However, they did not provide any information on how to choose the inner product to obtain a special class of solutions such as random order values and, in particular, the Shapley value.

On the other hand, it is known well that each cooperative game can be completely characterized by its dividends. In particular all the dividends of coalitions with more than one players are zero for any additive game. Therefore, in this paper, we formulate best approximation problems in the dividend space, which enables us to deal with simpler problems. A norm in the dividend space is derived from an inner product and we will make clear how to choose the inner product to obtain a Harsanyi payoff vector, a random order value and in particular the Shapley value.

The paper is organized in the following way. In Section 2 we review cooperative games and their divideds. Section 3 is devoted to concise introduction of one-point solutions for games. In Section 4 we formulate minimum norm problems in the dividend space and show how to choose the inner product to obtain a Harsanyi payoff vector, a random order value and in particular the Shapley value.

2 Cooperative games and Harsanyi dividends

Let N be a finite set of n elements, i.e., $N = \{1, 2, \dots, n\}$. Elements of N are called players. Any subset S of N is called a coalition. A cooperative game (transferable utility game) on N is a set funcion $v : 2^N \to \mathbf{R}$ with $v(\emptyset) = 0$. The function v is usually called a characteristic function and each value $v(S)$ is called the worth of the coalition S. Since N is fixed throughout this paper, the set of all cooperative games on N is simply denoted by $\mathcal{G}$. In the following, we use abbreviated notations such as $v(\{i\}) = v(i)$, $v(\{i,j\}) = v(i,j)$, $S \cup \{i\} = S \cup i$, $S \setminus \{i\} = S \setminus i$ and so on. We also distinguish two inclusive relations $S \subset T$ and $S \subseteq T$. The former means that $S \subseteq T$ and $S \neq T$.

The sum of two games $v, w \in \mathcal{G}$ is defined by $(v+w)(S) = v(S) + w(S)$ for all $S \subseteq N$, and the scalar multiplication of $v \in \mathcal{G}$ by a scalar $\alpha \in \mathbf{R}$ is defined by $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. Thus the space $\mathcal{G}$ of all games on N is a vector space and its dimension is clearly $2^n - 1$, since each game is specified by the worths $v(S)$ for all $S \subseteq N$ with $S \neq \emptyset$. As a basis in $\mathcal{G}$ we may consider unanimity games u_T defined by

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise,} \end{cases}$$

for any $T \subseteq N$ with $T \neq \emptyset$. Then each game $v \in \mathcal{G}$ is a linear combination of unanimity games,

$$v = \sum_{T \subseteq N, T \neq \emptyset} d^v(T) u_T.$$

The coefficient $d^v(v)$ is given by

$$d^v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$$

and called the (Harsanyi) dividend of T for the game v. For convenience' sake, we may put $d^v(\emptyset) = 0$ so that $v = \sum_{T \subseteq N} d^v(T) u_T$. In combinatorics, $d^v(\cdot)$ viewed as a set function on $2^N \setminus \{\emptyset\}$ is called the Möbius transform of v. The dividends satisfy the following recursive formula:

$$d^v(T) = \begin{cases} 0, & \text{if } T = \emptyset, \\ v(T) - \sum_{S \subset T} d^v(S), & \text{if } T \neq \emptyset. \end{cases}$$

It is obvious that $d^{v+w}(T) = d^v(T) + d^w(T)$, $d^{\alpha v}(T) = \alpha d^v(T)$. We should also note that

$$v(S) = \sum_{T \subseteq S} d^v(T), \quad \forall S \subseteq N.$$

Therefore if we regard both v and d^v as $2^n - 1$ dimensional vector such as $(v(1), \dots, v(n), v(1,2), \dots, v(N))^{\mathsf{T}}$ and $(d^v(1), \dots, d^v(n), d^v(1,2), \dots, d^v(N))^{\mathsf{T}}$ respectively, they are related in terms of a matrix D as $v = Dd^v$. Here the (S,T) element of D is 1 if $S \supseteq T$ and 0 otherwise.

Definition 1 *A game $v \in \mathcal{G}$ is said to be additive if $v(S) + v(T) = v(S \cup T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$. The set of all additive cooperative games on N is denoted by $\mathcal{A}$.*

We should note that an additive game is completely specified by n worths $v(1), v(2), \ldots,$ and $v(n)$, and therefore the set $\mathcal{A}$ of all additive games is a subspace of $\mathcal{G}$ and $\dim \mathcal{A} = n$.

Proposition 1 *A cooperative game $v \in \mathcal{G}$ is additive if and only if*

$$d^v(T) = \begin{cases} v(i) & \text{if } T = \{i\},\ i \in N, \\ 0 & \text{otherwise} \end{cases}$$

(Proof) First suppose that v is additive. From the definition $d^v(i) = v(i)$ for $i \in N$. Now we prove that $d^v(T) = 0$ for $T \subseteq N$ with $|T| > 1$ by induction with respect to $|T|$. For $T = \{i, j\}$,

$$d^v(i,j) = v(i,j) - d^v(i) - d^v(j) = v(i) + v(j) - v(i) - v(j) = 0.$$

Assume that $d^v(T) = 0$ for any T with $2 \leq |T| \leq k$ and take T with $|T| = k + 1$. Then

$$d^v(T) = v(T) - \sum_{S \subset T} d^v(S) = v(T) - \sum_{i \in T} d^v(i) = \sum_{i \in T} v(i) - \sum_{i \in T} v(i) = 0.$$

Conversely suppose that $d^v(T) = 0$ for any T with $|T| > 1$. Then

$$v(S) = \sum_{T \subseteq S} d^v(T) = \sum_{i \in S} v(i),$$

for any $S \subseteq N$. Hence v is additive. □

In cooperative games the concept of dummy players is important.

Definition 2 *A player $i \in N$ is said to be a dummy player in a game $v \in \mathcal{G}$ if*

$$v(S \cup i) - v(S) = v(i), \quad \forall S \subseteq N \setminus i.$$

We characterize a dummy player by the dividends. We should note that

$$v(S \cup i) - v(S) = \sum_{T \subseteq S \cup i} d^v(T) - \sum_{T \subseteq S} d^v(T) = \sum_{T \subseteq S} d^v(T \cup i)$$

for any $S \subseteq N \setminus i$.

Proposition 2 *Given a game $v \in \mathcal{G}$, a player $i \in N$ is a dummy player in v if and only if*

$$d^v(S \cup i) = 0 \ \ \forall S \subseteq N \setminus i,\ S \neq \emptyset \ (\iff d^v(T) = 0 \ \forall T \subseteq N,\ T \ni i,\ |T| > 1).$$

(Proof) "If part" is obvious. We suppose that $i \in N$ is a dummy player in v and prove "only if part" by induction. Let $|S| = 1$, i.e., $S = \{j\}$ with $j \neq i \in N$.

$$v(i,j) - v(j) = d^v(i) + d^v(i,j) = v(i) + d^v(i,j).$$

Hence $d^v(S \cup i) = d^v(i,j) = 0$. Next suppose that the relation holds for $|S| < k$ and let $|S| = k$. Then

$$\begin{aligned} v(S \cup i) - v(S) &= \sum_{T \subseteq S} d^v(T \cup i) \\ &= d^v(i) + \sum_{T \subset S, T \neq \emptyset} d^v(T \cup i) + d^v(S \cup i). \end{aligned}$$

Since $d^v(i) = v(i)$ and $d^v(T \cup i) = 0$ for all nonempty $T \subset S$ from the assumption of induction,

$$v(i) = v(S \cup i) - v(S) = v(i) + d^v(S \cup i).$$

Therefore $d^v(S \cup i) = 0$ as was to be proved. □

3 One-point solutions as minimum norm solutions for cooperative games

In a game $v \in \mathcal{G}$, the main issue is the distribution of the worth $v(N)$ among the palyers. A one-point solution of a game is specified by a function $\phi : \mathcal{G} \to \mathbf{R}^n$, which asscoaites a payoff vector $\phi(v) = (\phi_i(v))_{i \in N}$ called the value with each game $v \in \mathcal{G}$. Another kind of solution is given by a set-valued solution such as the core. Since this function ϕ is usually assumed to be linear with respect to v, the value is a linear combination of the values for unanimity games, i.e.,

$$\phi(v) = \sum_{T \subseteq N} d^v(T)\phi(u_T).$$

Typical examples of values are the Shapley and Banzhaf values given by

$$\varphi_i(u_T) = \begin{cases} \frac{1}{|T|}, & \text{if } i \in T \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta_i'(u_T) = \begin{cases} \frac{1}{2^{|T \setminus i|}}, & \text{if } i \in T \\ 0, & \text{otherwise} \end{cases}$$

respectively. Thus

$$\varphi_i(v) = \sum_{T \subseteq N, T \ni i} \frac{d^v(T)}{|T|}, \quad \text{and} \quad \beta_i'(v) = \sum_{T \subseteq N, T \ni i} \frac{d^v(T)}{2^{|T \setminus i|}}$$

respectively.

More general value is given in terms of the sharing system $p = (p_i^T)_{T \subseteq N, i \in T}$ satisfying $p \geq 0$, which means that any component $p_i^T \geq 0$, and $\sum_{i \in T} p_i^T = 1$ for each nonempty $T \subseteq N$. The set of all sharing systems satisfying the above relations is denoted by P, i.e.,

$$P = \{(p_i^T)_{T \subseteq N, i \in T} \mid p \geq 0, \sum_{i \in T} p_i^T = 1, \forall T \subseteq N, T \neq \emptyset\}.$$

The payoff vector $\phi^p(v) \in \mathbf{R}^n$, $p \in P$, given by

$$\phi_i^p(v) = \sum_{T \subseteq N, T \ni i} p_i^T d^v(T), \quad i \in N$$

is called a Harsanyi payoff vector [3] or Möbius value (in a restricted case) [1]. It is obvious that $\sum_{i \in N} \phi_i^p(v) = v(N)$, i.e., the Harsanyi payoff vector is efficient. Strictly speaking, the Shapley value is a Harsanyi payoff vector, but the Banzhaf value is not. We should also note that the Harsanyi payoff vector satisfies the following dummy palyer axiom: If $i \in N$ is a dummy player in v, then $\phi_i(v) = v(i)$.

Another type of one-point solutions can be obtained by marginal contributions of players. Let π be a permutation on N, which assigns rank number $\pi(i) \in \{1, 2, \dots, n\} = N$ to player $i \in N$. Let

$$\pi^i = \{j \in N \mid \pi(j) \leq \pi(i)\}.$$

The marginal contribution vector $m^\pi(v) \in \mathbf{R}^n$ of v and π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus i), \quad i \in N.$$

The Weber set, denoted by $W(v)$, is the convex hull of all marginal contribution vectors $m^\pi(v)$. Each element of $W(v)$ is called a random order value. Derks et al. characterized random order values by Harsanyi payoff vectors.

Proposition 3 *[3] If we define the sharing system p^π for a permutaion π on N by*

$$(p^\pi)_i^T = \begin{cases} 1 & \text{if } i \in T \text{ and } T \subseteq \pi^i, \\ 0 & \text{otherwise,} \end{cases}$$

then $\phi^{p^\pi}(v) = m^\pi(v)$.

Proposition 4 *[3] A Harsanyi payoff vector ϕ^p is a random order value if and only if the sharing system p belongs to the following set P^*, i.e.,*

$$p \in P^* = \{p \in P \mid \sum_{S \supseteq T} (-1)^{|S|-|T|} p_i^S \geq 0, \ \forall T \subseteq N, \ T \ni i\}.$$

We may identify a value which is a n dimensional vector with an additive game (see e.g. Monderer and Samet [5]). Thus a one-point solution is a function f from $\mathcal{G}$ to $\mathcal{A}$, since dim $\mathcal{A} = n$. Of course, given a game $\bar{v} \in \mathcal{G}$, we consider that $\phi_i(\bar{v}) = v^*(i)$ for some $v^* \in \mathcal{A}$. In other words, the value $\phi : \mathcal{G} \to \mathbf{R}^n$ is identified with the function $f : \mathcal{G} \to \mathcal{A}$ with $\phi_i(v) = f(v)(i)$.

Kultti and Salonen proposed the efficient minimum norm solutions [4], which is a generalization of the results by Ruiz et al. [6, 7]. Namely they considered the following optimization problem (minimum norm problem) for each cooperative game $\bar{v} \in \mathcal{G}$.

$$\begin{array}{ll} \text{minimize} & \langle v - \bar{v}, v - \bar{v} \rangle \\ \text{subject to} & v \in \mathcal{A}, \ v(N) = \bar{v}(N). \end{array} \tag{1}$$

Here $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{G}$. In this case f is a function $f : \bar{v} \mapsto v^*$, where v^* is the unique optimal solution to the above problem, i.e., the minimum norm solution.

They discussed some properties of solutions.

- *Efficiency.* $f(v)(N) = v(N)\ \forall v \in \mathcal{G}$.
- *Linearity.* $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w),\ \ \forall \alpha, \beta \in \mathbf{R},\ v, w \in \mathcal{G}$.
- *Inessential game property.* If $v \in \mathcal{A}$, then $f(v) = v$.

They have obtained the following two main results.

Theorem 1 *([4]) For each $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ to the above minimum norm prolem exists uniquely. The function $f : \mathcal{G} \to \mathcal{A}$ is efficient, linear and has the inessential game property.*

Theorem 2 *([4]) Let $f : \mathcal{G} \to \mathcal{A}$ be any efficient linear solution that has the inessential game property. Then there is an inner product such that $f(\bar{v})$ solves the above minimum norm problem for all $\bar{v} \in \mathcal{G}$.*

4 Minimum norm solutions in the dividend space

In the minimum norm problem

$$\begin{array}{ll} \text{minimize} & \langle v - \bar{v}, v - \bar{v} \rangle \\ \text{subject to} & v \in \mathcal{A},\ v(N) = \bar{v}(N), \end{array}$$

we may eliminate the variables $v(S)$ with $|S| > 1$ by the equality constraints $v(S) = \sum_{i \in S} v(i)$ for any $S \subseteq N$, $|S| > 1$ for $v \in \mathcal{A}$. Then the remaining essential variables are only $v(1), v(2), \ldots,$ and $v(n)$.

Since we can obtain the dividends $\{d^v(S) | S \subseteq N, S \neq \emptyset\}$ of v by the linear transformation from the worths $\{v(S) | S \subseteq N, S \neq \emptyset\}$ and vice versa, the above problem can be rewritten as the optimization problem with respect to the dividends as in the following.

$$\begin{array}{ll} \text{minimize} & \langle d - \bar{d}, d - \bar{d} \rangle \\ \text{subject to} & d(S) = 0 \text{ if } |S| > 1,\ \sum\limits_{i \in N} d(i) = \bar{v}(N) = \sum\limits_{S \subseteq N} \bar{d}(S). \end{array} \qquad (2)$$

Here $\bar{d} = d^{\bar{v}}$ is the $2^n - 1$ dimensional dividend vector of $\bar{v}$, and $\langle \cdot, \cdot \rangle$ is an appropriate inner product in the dividend space. Of course, the essential variables in the above optimization problem are $d(1), d(2), \ldots,$ and $d(n)$. If we denote the optimal solution of the above problem by d^*, then for each game $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v}) \in \mathcal{A}$ can be obtained by $f(\bar{v})(i) = d^*(i)$ for all $i \in N$.

We may describe an inner product $\langle \cdot, \cdot \rangle$ in terms of $(2^n - 1) \times (2^n - 1)$ positive definite symmetric matrix Q whose (S, T) element is q_{ST}. Thus

$$\langle d, d' \rangle = \sum_{S, T \subseteq N, S, T \neq \emptyset} q_{ST} d(S) d'(T).$$

Since $v = Dd^v$, the minimum norm problem (2) in the dividend space with the inner product specified by the positive definite matrix Q is obviously equivalent to the minimum norm problem (1) in the game space with the inner product specified by the positive definite matrix $D^{-T}QD^{-1}$. However the number of constraints is actually only one in the problem (2) and selection of the matrix is much easier as is shown below.

Noting that $d(S) = 0$ for $|S| > 1$, but that $\bar{d}(S) \neq 0$ generally, the above optimization problem can be essentially rewritten (by deleting the constant term) as

$$\begin{array}{ll} \text{minimize} & \sum_{i\in N}\sum_{j\in N} q_{ij}d(i)d(j) - 2\sum_{i\in N}\sum_{S\subseteq N, S\neq\emptyset} q_{iS}d(i)\bar{d}(S) \\ \text{subject to} & \sum_{i\in N} d(i) = \sum_{S\subseteq N} \bar{d}(S), \end{array} \tag{3}$$

where $q_{ij} = q_{\{i\}\{j\}}$ and $q_{iS} = q_{\{i\}S}$.

The theorems by Kultti and Salonen are obviously valid in this case.

Theorem 3 *For each $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ obtained through the above minimum norm prolem (3) with $\bar{d} = d^{\bar{v}}$ exists uniquely. The function $f : \mathcal{G} \to \mathcal{A}$ is efficient, linear and has the inessential game property.*

Theorem 4 *Let $f : \mathcal{G} \to \mathcal{A}$ be any efficient linear solution that has the inessential game property. Then there is an inner product such that $f(\bar{v})$ can be obtained by the solution to the above minimum norm problem with $\bar{d} = d^{\bar{v}}$ for all $\bar{v} \in \mathcal{G}$.*

Now we consider special cases of the inner products.

Lemma 1 *Given a sharing system $p = (p_i^T)_{T\subseteq N, i\in T} \in P$, let*

$$q_{iS} = \begin{cases} p_i^S & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

and, when $|S| > 1$ and $|T| > 1$,

$$q_{ST} = \begin{cases} \text{sufficiently large} & \text{if } S = T, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix Q is positive definite.

Theorem 5 *Given a game $\bar{v} \in \mathcal{G}$, the solution $f(\bar{v})$ obtained from the minimum norm problem (3) in the dividend space with $\bar{d} = d^{\bar{v}}$ and the inner product specified by the matrix Q defined in the above lemma coincides with the Harsanyi payoff vector $\phi^p(\bar{v})$.*

(Proof) Let us consider the unanimity game u_T for each nonempty $T \subseteq N$. Then $d^{u_T}(T) = 1$ and $d^{u_T}(S) = 0$ for $S \neq T$. It is straightforward to show that the minimum norm solution to the problem with u_T and the inner product by the matrix induced from the sharing system p is exactly $d^*(i) = p_i^T$ for $i \in T$ and $d^*(i) = 0$ for $i \notin T$. Thus

$$f(\bar{v})(i) = \sum_{T\subseteq N} d^{\bar{v}}(T) f(u_T)(i) = \sum_{T\subseteq N, T\ni i} p_i^T d^{\bar{v}}(T)$$

and therefore we obtain the Harsanyi payoff vector $\phi^p(\bar{v})$. □

Corollary 1 *If the sharing system p is in P^*, then the solution $f(\bar{v})$ obtained as the minimum norm solution in the dividend space is a random order value, i.e., $f(\bar{v}) \in W(\bar{v})$.*

Corollary 2 *If the sharing system is given by $p = p^{\pi}$ for a permutaion π on N, then the solution obtained as the minimum norm solution in the dividend space is the marginal contribution vector $m^{\pi}(\bar{v})$.*

Corollary 3 *If the sharing system is given by $p_i^T = \dfrac{1}{|T|}$ for all $i \in T$, the solution obtained as the minimum norm solution in the dividend space coincides with the Shapley value.*

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