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A strong convergence theorem by hybrid method for a countable family of nonexpansive mappings and an equilibrium problem

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Abstract

In this paper, we introduce an iterative scheme by hybrid method for finding a common element of the set of fixed points of a countable family of nonexpansive mappings and the set of solutions of an equilibrium problem in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the above two sets under some parameters controlling conditions.

Keywords: Fixed point theorem; Nonexpansive mappings; Equilibrium problem; Common fixed points

1 Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. Let F be a bifunction from $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geqslant 0 \quad \text{for all} \quad y \in C. \tag{1.1}$$

The set of solution of (1.1) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem (see; [2, 4, 11, 18]). In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and they also proved a strong convergence theorem. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leqslant \|x - y\|,$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. If C is bounded closed convex and S is a nonexpansive mapping from C into itself, then $F(S)$ is nonempty (see; [8]). We write $x_n \to x$ ($x_n \rightharpoonup x$, resp.) if $\{x_n\}$ converges (weakly, resp.) to x.

In 1953, Mann [9] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n \tag{1.2}$$

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where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results is proved by Reich [14]. In an infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence [5]. Attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [12] proposed the following modification of Mann iteration method (1.2):

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n) S x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{1.3}$$

For finding an element of $EP(F) \cap F(S)$, Tada and Takahashi [20] introduced the following iterative scheme by the hybrid method in a Hilbert space: $x_0 = x \in H$ and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n) x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0,1]$ where $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Further, they proved $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} x_1$.

Recently, Takahashi et al. [17] proved a strong convergence theorem by the hybrid method for a family of nonexpansive mappings in Hilbert spaces: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$ and let

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leqslant \alpha_n \leqslant a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) = \varnothing$ and satisfy some appropriate conditions. Then, $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_0$.

In this paper, motivated and inspired by the above results, we introduce a new following iterative scheme:

$$\begin{cases} x_0 \in H, \quad \text{and} \quad C_0 = C, \\ u_n \in C \text{ such that } F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

for finding a common element of the set of fixed points of a countable family of nonexpansive mappings and the set of solutions of an equilibrium problem. Moreover, we show that $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{\bigcap_{n=1}^{\infty} F(S_n)\cap EP(F)} x_0$ by the hybrid method in mathematical programming.

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies the *Opial's condition* [13], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H, satisfies the *Kadec-Klee property* [6, 19], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [2]):

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [2]

Lemma 2.1. [2] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbf{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [3].

Lemma 2.2. [3] *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

1. *T_r is single- valued;*
2. *T_r is firmly nonexpansive,i.e.,for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
3. *$F(T_r) = EP(F)$;*
4. *$EP(F)$ is closed and convex.*

Let C be a subset of a Banach space E and let $\{S_n\}$ be a family of mappings from C into E. For a subset B of C, we say that $(\{S_n\}, B)$ satisfies condition $AKTT$ if

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty.$$

Aoyama et al. [1, Lemma 3.2], prove the following result which is very useful in our main result.

Lemma 2.3. *Let C be a nonempty closed subset of a Banach space E and let $\{S_n\}$ be a sequence of mappings from C into E. Let B be a subset of C with $(\{S_n\}, B)$ satisfies condition $AKTT$, then there exists a mapping $S : B \to E$ such that*

$$Sy = \lim_{n\to\infty} S_n y \quad \forall y \in B$$

and $\lim_{n\to\infty} \sup\{\|S_n z - Sz\| : z \in B\} = 0$.

3 Main result

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into H such that $\bigcap_{n=0}^{\infty} F(S_n) \cap EP(F) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in H, \quad \text{and } C_0 = C, \\ u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)S_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n = 0, 1, 2\ldots, \end{cases}$$

with the following restrictions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,
(ii) $r_n > 0$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\liminf_{n\to\infty} r_n > 0$.

Let $\sum_{n=0}^{\infty} \sup \{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ *for any bounded subset* B *of* C *and* S *be a mapping from* C *into* H *defined by* $Sz = \lim_{n\to\infty} S_n z$ *for all* $z \in C$ *and suppose that* $F(S) = \bigcap_{n=0}^{\infty} F(S_n)$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{F(S)\cap EP(F)}x_0$.

Proof. We first show by induction that $F(S) \cap EP(F) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. $F(S) \cap EP(F) \subset C = C_0$ is obvious. Suppose that $F(S) \cap EP(F) \subset C_k$ for some $k \in \mathbb{N} \cup \{0\}$. Then, we have, for $p \in F(S) \cap EP(F) \subset C_k$

$$\begin{aligned} \|y_k - p\| &= \|\alpha_k x_k + (1-\alpha_k)S_k u_k - p\| \leqslant \alpha_k \|x_k - p\| + (1-\alpha_k)\|S_k u_k - p\| \\ &= \alpha_k \|x_k - p\| + (1-\alpha_k)\|S_k T_{r_k} x_k - p\| \leqslant \|x_k - p\| \end{aligned}$$

and hence $p \in C_{k+1}$. This implies that $F(S) \cap EP(F) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N} \cup \{0\}$. It is obvious that $C_0 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N} \cup \{0\}$. For $z \in C_k$, we know that $\|y_k - z\| \leqslant \|x_k - z\|$ is equivalent to $\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z\rangle \geqslant 0$. So, C_{k+1} is closed and convex. Then, for any $n \in \mathbb{N} \cup \{0\}$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, we have $\langle x_0 - x_n, x_n - y\rangle \geqslant 0$ for all $y \in C_n$. In particular, we also have

$$\langle x_0 - x_n, x_n - p\rangle \geqslant 0 \quad \text{for all} \quad p \in F(S) \cap EP(F) \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}.$$

So, we have

$$0 \leqslant \langle x_0 - x_n, x_n - p\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|.$$

This implies that

$$\|x_0 - x_n\| \leqslant \|x_0 - p\| \quad \text{for all} \quad p \in F(S) \cap EP(F) \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Since $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geqslant 0. \tag{3.2}$$

So, we have

$$0 \leqslant \langle x_0 - x_n, x_n - x_{n+1}\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.$$

and hence

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|.$$

Since $\{\|x_n - x_0\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Next, we show that $\|x_n - x_{n+1}\| \to 0$. In fact, from (3.2) we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_0 + x_0 - x_{n+1}\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leqslant \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have that $\|x_n - x_{n+1}\| \to 0$. On the other hand $x_{n+1} \in C_{n+1} \subset C_n$ implies that $\|y_n - x_{n+1}\| \leqslant \|x_n - x_{n+1}\| \to 0$ and then

$$\|x_n - y_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0. \tag{3.3}$$

Further, since $\|y_n - x_n\| = (1-\alpha_n)\|S_n u_n - x_n\|$ and (i), we obtain

$$\lim_{n\to\infty}\|S_n u_n - x_n\| = 0. \tag{3.4}$$

For $p \in F(S) \cap EP(F)$, we have, from Lemma 2.2,

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \leqslant \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right\}, \end{aligned}$$

hence $\|u_n - p\|^2 \leqslant \|x_n - p\|^2 - \|x_n - u_n\|^2$. Therefore, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(S_n u_n - p)\|^2 \leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|S_n u_n - p\|^2 \\ &\leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \leqslant \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 - \|x_n - u_n\|^2\right\} \\ &= \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2, \end{aligned}$$

and then

$$\|x_n - u_n\|^2 \leqslant \frac{1}{1-\alpha_n}\left(\|x_n - p\|^2 - \|y_n - p\|^2\right) \leqslant \frac{1}{1-\alpha_n}\|x_n - y_n\|\left(\|x_n - p\| + \|y_n - p\|\right).$$

By (i) and (3.3), we obtain

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.5}$$

From (3.4) and (3.5), we obtain also

$$\|u_n - S_n u_n\| = \|u_n - x_n\| + \|x_n - S_n u_n\| \to 0. \tag{3.6}$$

And then

$$\|u_n - Su_n\| \leqslant \|u_n - S_n u_n\| + \|S_n u_n - Su_n\| \leqslant \|u_n - S_n u_n\| + \sup\{\|S_n z - Sz\| : z \in \{u_n\}\} \to 0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From (3.5), we obtain also that $u_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$. We shall show $w \in EP(F)$. By $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \text{for all } \ y \in C.$$

From the monotonicity of F, we get

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant F(y, u_n), \quad \text{for all } \ y \in C;$$

hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geqslant F(y, u_{n_i}), \quad \text{for all } \ y \in C.$$

From (ii), (3.5) and condition $(A4)$, we have $0 \geqslant F(y,w)$, for all $y \in C$. Let $y \in C$ and set $x_t = ty + (1-t)w$, for $t \in (0,1]$. Then, we have

$$0 = F(x_t, x_t) \leqslant tF(x_t, y) + (1-t)F(x_t, w) \leqslant tF(x_t, y).$$

or $F(x_t, y) \geqslant 0$. Letting $t \downarrow 0$ and using $(A3)$, we get

$$F(w,y) \geqslant 0 \quad \text{for all} \quad y \in C$$

and hence $w \in EP(F)$. We next show that $w \in F(S)$. Assume $w \notin F(S)$. Then, from the Opial's condition and (3.6), we have

$$\begin{aligned} \liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| \leqslant \liminf_{i\to\infty} \{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\} \\ &= \liminf_{i\to\infty} \left\{\|u_{n_i} - Su_{n_i}\| + \lim_{m\to\infty} \|S_m u_{n_i} - S_m w\|\right\} \leqslant \liminf_{i\to\infty} \|u_{n_i} - w\|. \end{aligned}$$

This is a contradiction. So, we get $w \in F(S)$. Therefore, we obtain $w \in F(S) \cap EP(F)$. Let $z = P_{F(S)\cap EP(F)} x_0$, by (3.1) we observe that

$$\|x_0 - z\| \leqslant \|x_0 - w\| \leqslant \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leqslant \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leqslant \|x_0 - z\|,$$

hence, $\lim_{n\to\infty} \|x_0 - x_{n_i}\| = \|x_0 - w\| = \|x_0 - z\|$. Since H is a Hilbert space, we obtain $x_{n_i} \to w = z$. Since $z = P_{F(S)\cap EP(F)} x_0$, we can conclude that $x_n \to P_{F(S)\cap EP(F)} x_0$. Moreover, from (3.5) we also have $u_n \to P_{F(S)\cap EP(F)} x_0$. ■

Setting $S_n = S$ in Theorem 3.1, we have the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let S be a nonexpansive mapping from C into H such that $F(S)\cap EP(F) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in H, \quad \text{and} \quad C_0 = C, \\ u_n \in C \text{ such that } \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

with the following restrictions:

(i) $0 \leqslant \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $r_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(S)\cap EP(F)} x_0$.

As direct consequences of corollary 3.2 , we can obtain two corollaries.

Corollary 3.3. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ such that $EP(F) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in H, \quad \text{and} \quad C_0 = C, \\ u_n \in C \text{ such that } \quad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|u_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

with $r_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{EP(F)}x_0$.

Proof. Putting $S = I$ and $\alpha_n = 0$ in Theorem 3.1. ∎

Corollary 3.4. *Let C be a nonempty closed convex subset of H and let S be a nonexpansive mapping from C into H such that $F(S) \neq \varnothing$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in H, \quad \text{and} \quad C_0 = C, \\ u_n \in C \text{ such that } \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) S u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n = 0, 1, 2 \ldots, \end{cases}$$

with $0 \leqslant \alpha_n < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F(S)}x_0$.

Proof. Putting $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1. ∎

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