数理解析研究所講究録
第 1611 巻 2008 年 34-41

Discrete fixed point theorems and their applications to the game theory [1]

九州大学・大学院数理学府 佐藤潤一 (Jun-ichi Sato)
Graduate School of Mathematics, Kyushu University
九州大学・大学院数理学研究院 川崎英文 (Hidefumi Kawasaki)
Faculty of Mathematics, Kyushu University

Abstract

In this paper, we present discrete fixed point theorems. They are based on monotonicity of the mapping. We apply them to a non-cooperative n-person game and give an existence theorem of a Nash equilibrium of pure strategies. As a special case, we consider bimatrix games.

1 Introduction

Fixed point theorems are powerful tools not only in mathematics but also in economics. Existence theorem of Nash equilibrium is one of the most important applications of fixed point theorems. The aims of this paper are to provide discrete fixed point theorems and to apply them to a non-cooperative n-person game.

There are three types of discrete fixed point theorems. In 1955, Tarski [9] proved a lattice-theoretical fixed point theorem. It asserts that any increasing mapping defined on a complete lattice has a fixed point. Recently, Shih and Dong [4] proved that a mapping from the n-dimensional hypercube $\{0,1\}^n$ to itself has the property that all the Boolean eigenvalues of the discrete Jacobian matrix of each element of the hypercube are zero, then it has a unique fixed point. Their result is an answer to a combinatorial version of the Jacobian conjecture. Further, Richard [8] extended Shih-Dong's fixed point theorem to the product of n finite intervals of integers of cardinality ≥ 2. On the other hand, Iimura-Murota-Tamura [2] gave a fixed point on an integrally convex set by using Brower's fixed point theorem, and Yang [10] obtained some extensions, see Section 4 for details. Our discrete fixed point theorems are of the first type, and based on the following two ideas. (a) The base set V is finite. (b) The mapping $f : V \to V$ reduces the area of candidates for fixed points. Our theorems need no convexity assumption.

This paper is organized as follows. In Section 2, we give discrete fixed point theorems. In Section 3, we apply our fixed point theorems to a class of non-cooperative games and obtain some existence theorems of a Nash equilibrium of pure strategies. In Section 4, we compare our discrete fixed point theorems to conventional ones.

Throughout this paper, $(V, \preceqq)$ is a partially ordered set and $f : V \to V$ is a nonempty set-valued mapping. The symbol $x \preceq y$ means $x \preceqq y$ and $x \neq y$. For any $x \in \mathbb{Z}^n$, x_i denotes the i-th component of x. We denote the component-wise order by $\leqq$. Further, $x \leq y$ means $x \leqq y$ and $x \neq y$.

2 Discrete fixed point theorems

Theorem 2.1 *Assume that there exist $x^0 \in V$ and $x^1 \in f(x^0)$ such that $x^0 \preceqq x^1$ and $\{x \in V;\ x^0 \preceqq x\}$ is finite. Further assume that for any $x \in V$ and $y \in f(x)$,*

$$(1) \qquad x \preceq y \ \Rightarrow\ \exists z \in f(y) \ s.t.\ y \preceqq z.$$

Then, f has a fixed point x^*, that is, $x^* \in f(x^*)$.

We can easily weaken the assumptions of Theorem 2.1 as follows.

Theorem 2.2 *Assume that there exists a sequence $\{x^m\}_{m \geq 0}$ in V such that $x^m \preceqq x^{m+1} \in f(x^m)$ for any $m \geq 0$ and $\{x \in V;\ x^0 \preceqq x\}$ is finite. Then, f has a fixed point $x^* \in f(x^*)$.*

Theorem 2.3 below shows a way to find x^0 and x^1 such that $x^0 \preceqq x^1 \in f(x^0)$ in the case where $\preceqq$ is the component-wise order $\leqq$ or $\geqq$ and V is a finite set.

Definition 2.1 *For any $k \in \{1, \ldots, n\}$ and $x \in \mathbb{Z}^n$, we denote $K := \{1, \ldots, k\}$, $x_K := (x_i)_{i \in K}$, $K+1 := \{1, \ldots, k+1\}$, and $x_{K+1} := (x_i)_{i \in K+1}$. We call $x \in V$ a fixed point of f w.r.t. $\mathbb{Z}^k$ if $x_K \in f_K(x) := \{y_K\, ;\, y \in f(x)\}$.*

Theorem 2.3 *Let V be a finite set in $\mathbb{Z}^n$, and assume that for any $k \in \{1, \ldots, n\}$, $x \in V$ and $y \in f(x)$, $x_K \leq$ (resp. $\geq$) $y_K \Rightarrow \exists z \in f(y)$ s.t. $y_K \leqq$ (resp. $\geqq$) z_K. Then, f has a fixed point $x^* \in f(x^*)$.*

3 Nash equilibrium of pure strategies

As an application of our discrete fixed point theorems, we shall present a class of non-cooperative games that have a Nash equilibrium of pure strategies. We consider the following non-cooperative n-person game $G = (N, \{S_i\}_{i\in N}, \{p_i\}_{i\in N})$: $N := \{1, \ldots, n\}$ is the set of all players. For any $i \in N$, S_i denotes the set of player i's strategies. Its element is denoted by s_i. We assume that each S_i is a finite subset of $\mathbb{Z}$. $p_i : S := \prod_{j=1}^{n} S_j \to \mathbb{R}$ denotes the payoff function of player i.

Furthermore, we use the following notation in this section. $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. S is equipped with a signed component-wise order, that is, N is divided into two subsets (possibly empty) N_+ and N_-, $\varepsilon_j = \pm 1$ are allocated to $j \in N_+$ and $j \in N_-$, respectively, and $s \precsim t$ is defined by $\varepsilon_j s_j \leqq \varepsilon_j t_j$ for any $j \in N$. $S_{-i} = \prod_{j\neq i} S_j$ is also equipped with the signed component-wise order. For any given $s_{-i} \in S_{-i}$, we denote by $f_i(s_{-i})$ the set of best responses of player i, that is, $f_i(s_{-i}) := \{s_i \in S_i\,;\, p_i(s_i, s_{-i}) = \max_{t_i\in S_i} p_i(t_i, s_{-i})\}$. $f(s) = f_1(s_{-1}) \times \cdots \times f_n(s_{-n})$ for any $s = (s_1, \ldots, s_n)$.

An n-tuple $s^* \in S$ is called a Nash equilibrium if $s^* \in f(s^*)$. As is well-known, Nash's theorem asserts that G has a Nash equilibrium if we allow mixed strategies. However pure strategies are not enough to guarantee a Nash equilibrium. We have to impose an additional assumption to get a Nash equilibrium of pure strategies. Our assumption is monotonicity.

Definition 3.1 *We say a game G monotone if, for any $i \in N$, $s^0_{-i}, s^1_{-i} \in S_{-i}$ with $s^0_{-i} \precsim s^1_{-i}$ and for any $t^1_i \in f_i(s^0_{-i})$, there exists $t^2_i \in f_i(s^1_{-i})$ such that $\varepsilon_i t^1_i \leqq \varepsilon_i t^2_i$.*

Theorem 3.1 *Any monotone game G has a Nash equilibrium of pure strategies.*

As a special case, let us consider the following bimatrix game. $A = (a_{ij})$ is a payoff matrix of player 1 (P1), that is, $p_1(i,j) = a_{ij}$. $B = (b_{ij})$ is a payoff matrix of player 2 (P2), that is, $p_2(i,j) = b_{ij}$. $S_1 := \{1, \ldots, m_1\}$ is the set of pure strategies of P1, where $m_1 \in \mathbb{N}$. $S_2 := \{1, \ldots, m_2\}$ is the set of pure strategies of P2, where $m_2 \in \mathbb{N}$. For any $j \in S_2$, $I(j) := \{i^* \in S_1\,;\, a_{i^*j} = \max_{i\in S_1} a_{ij}\}$ is the set of best responses of P1. For any $i \in S_1$, $J(i) := \{j^* \in S_2\,;\, b_{ij^*} = \max_{j\in S_2} b_{ij}\}$ is the set of best responses of P2. $f(i,j) := I(j) \times J(i)$ denotes the set of best responses of $(i,j) \in S_1 \times S_2$. A pair (i^*, j^*) is a Nash equilibrium of pure strategies if $(i^*, j^*) \in f(i^*, j^*)$.

Then Definition 3.1 reduces to Definition 3.2 below.

Definition 3.2 *We say payoff matrix A monotone if for any $j^0, j^1 \in S_2$ such that $\varepsilon_2 j^0 < \varepsilon_2 j^1$ and for any $i^1 \in I(j^0)$, there exists $i^2 \in I(j^1)$ such that $\varepsilon_1 i^1 \leqq \varepsilon_1 i^2$. Also, we say payoff matrix B monotone if for any i^0, i^1 such that $\varepsilon_1 i^0 < \varepsilon_1 i^1$ and for any $j^1 \in J(i^0)$, there exists $j^2 \in J(i^1)$ such that $\varepsilon_2 j^1 \leqq \varepsilon_2 j^2$. When both A and B are monotone, we say the bimatrix game monotone.*

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.1 *Any monotone bimatrix game has a Nash equilibrium of pure strategies.*

Example 3.1 The following matrices are monotone for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, where framed numbers correspond to best responses, and circled numbers correspond to the Nash equilibrium.

$$A = \left(\begin{array}{c|c|c|c} 5 & \boxed{7} & 1 & \boxed{9} \\ \boxed{8} & 2 & 3 & 5 \\ 4 & \boxed{7} & \boxed{4} & 8 \\ 7 & 6 & 2 & \boxed{9} \end{array}\right), \quad B = \left(\begin{array}{cccc} \boxed{7} & 2 & 6 & 3 \\ \hline 3 & \boxed{9} & 5 & 4 \\ \hline \boxed{8} & 6 & \boxed{8} & 5 \\ \hline 1 & \boxed{3} & \boxed{3} & 2 \end{array}\right).$$

Moreover, since $(i, j) = (3, 3)$ belongs to the set of best responses to itself, $(3, 3)$ is a Nash equilibrium of pure strategies.

Remark 3.1 Suppose that both players test whether each payoff matrix is monotone, respectively. If they answer "yes", then the matrix game has a Nash equilibrium of pure strategies. They don't need to answer the set of their best responses. This is an advantage of Theorem 3.1.

Example 3.2 The following matrices are monotone for $(\varepsilon_1, \varepsilon_2) = (1, -1)$.

$$A = \left(\begin{array}{c|c|c} 4 & 2 & 1 \\ 5 & \boxed{7} & \boxed{4} \\ \boxed{8} & 6 & 3 \end{array}\right), \quad B = \left(\begin{array}{ccc} 2 & 3 & \boxed{9} \\ \hline 4 & 5 & \boxed{6} \\ \hline 7 & \boxed{8} & 6 \end{array}\right)$$

Therefore, the matrix game has a Nash equilibrium of pure strategies $(2, 3)$.

We can weaken monotonicity so that we can deal with Example 3.3 below.

Definition 3.3 *We say payoff matrix A sequentially monotone if there exists a sequence of best responses $j^k \in J(k)$ such that $\varepsilon_2 j^k \leqq \varepsilon_2 j^{k+1}$ for any $k = 1, \ldots, m_1 - 1$. We say payoff matrix B sequentially monotone if there exists a sequence of best responses $i^k \in I(k)$ such that $\varepsilon_1 i^k \leqq \varepsilon_1 i^{k+1}$ for any $k = 1, \ldots, m_2 - 1$. When both A and B are sequentially monotone, we say the bimatrix game sequentially monotone.*

It is clear that any monotone bimatrix game is sequentially monotone.

Corollary 3.2 *Any sequentially monotone bimatrix game has a Nash equilibrium of pure strategies.*

Example 3.3 Although matrix A below is not monotone for $(\varepsilon_1, \varepsilon_2) = (1, 1)$, it is sequentially monotone. In fact, asterisked numbers give a sequence of best responses in Definition 3.3.

$$A = \left(\begin{array}{c|c|c|c} 5 & 2 & 1 & \boxed{9} \\ \boxed{8^*} & \boxed{7^*} & \boxed{4^*} & 5 \\ 4 & \boxed{7} & 3 & 8 \\ \boxed{8} & 6 & 2 & \boxed{9^*} \end{array}\right), \quad B = \left(\begin{array}{cccc} \boxed{7^*} & 2 & 6 & 3 \\ \hline 3 & 5 & \boxed{9^*} & 4 \\ \hline \boxed{8} & 6 & \boxed{8^*} & 5 \\ \hline 1 & \boxed{3} & \boxed{3^*} & 2 \end{array}\right).$$

4 Concluding remarks

In this section, we compare our results to others. Throughout this section, V is a subset of $\mathbb{Z}^n$ and $f : V \to V$ is a nonempty set-valued mapping. $N(y) := \{z \in \mathbb{Z}^n\,;\ \lfloor y \rfloor \leqq z \leqq \lceil y \rceil\}$ for all $y \in \mathbb{R}^n$, where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are rounding up and rounding down to the nearest integer, respectively. It is called the integral neighbourhood. $\|y\|_2 := (\sum_{i=1}^n y_i^2)^{1/2}$ and $\|y\|_\infty := \max\{|y_i|\,;\ i \in N\}$ for $y \in \mathbb{R}^n$. For $x^1, x^2 \in \mathbb{Z}^n$, $x^1 \simeq x^2$ is defined by $\|x^1 - x^2\|_\infty \leqq 1$. We say V integrally convex if $y \in \operatorname{co} V$ implies $y \in \operatorname{co}(V \cap N(y))$, where $\operatorname{co} V$ denotes the convex hull of V, see e.g. [2][3]. For each $x \in V$, $\pi_f(x)$ denotes the projection of x onto $\operatorname{co} f(x)$, that is, $\|\pi_f(x) - x\|_2 = \min_{y \in \operatorname{co} f(x)} \|y - x\|_2$. We say f direction preserving if for any $x, y \in V$ with $x \simeq y$

$$(2) \qquad x_i < (\pi_f(x))_i \ \Rightarrow\ y_i \leqq (\pi_f(y))_i \quad \forall i = 1, 2, \ldots, n.$$

Theorem 4.1 ([2, Theorem 2]) *Let V be a nonempty finite integrally convex set. If f is a nonempty- and discretely convex-valued direction preserving set-valued mapping, then f has a fixed point.*

For the sake of simplicity, we consider the case where f is single-valued. Then, since $\pi_f(x) = f(x)$, (2) reduces to

$$x_i < f_i(x) \Rightarrow y_i \leqq f_i(y) \quad \forall i = 1, 2, \ldots, n \quad \forall y \simeq x. \tag{3}$$

On the other hand, (1) in Theorem 2.1 reduces to

$$x \leq f(x) \Rightarrow f(x) \leqq f(f(x)). \tag{4}$$

For example, when V consists of sixteen points as in Figures 1 and 2, it is integrally convex. In order to apply Theorem 4.1, we have to test (3) for eight solid points in Figure 1. On the other hands, when we apply our results, it suffices to test (4) only for one point in Figure 2.

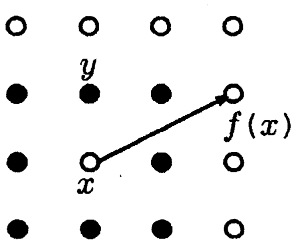

Figure 1:

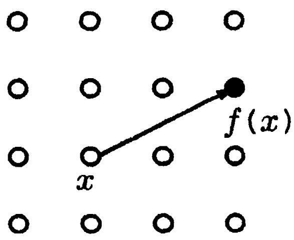

Figure 2:

Another advantage is that we don't need any convexity assumption on V. Yang [10] extended Theorem 4.1 by introducing a local gross direction preserving correspondence, which is weaker than direction preserving correspondence. However, his theorems also need information on eight points and convexity assumption, see [10, Definition 4.6, Theorem 3.12] for details.

Tarski gave an excellent fixed point theorem on a complete lattice, that is, every subset of the lattice has a least upper bound and a greatest lower bound.

Theorem 4.2 ([9, Theorem 1]) *Let f be an increasing mapping on a complete lattice to itself, that is, $x \trianglelefteq y$ implies $f(y) \trianglelefteq f(y)$. Then f has a fixed point.*

Its advantage is that V can be infinite. However, when V is finite, the assumption on the lattice seems restrictive. For example, when V is equipped with the component-wise order and has a hole as in Figure 3, it is not a complete lattice. Indeed, since gray points are upper bounds of set U in Figure 3, U has no least upper bound. Further, since f is not a set-valued

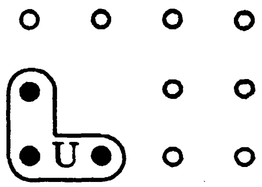

Figure 3: This set is not a complete lattice since U has no least upper bound.

mapping, even if one applies Theorem 4.2 to non-cooperative games, Examples 3.1, 3.2, and 3.3 are outside of his scope.

Shih-Dong's fixed point theorem is a remarkable result. Since the base set of their theorem is the Boolean algebra $\{0,1\}^n$, there seems no direct relationship between their fixed point theorem and ours. Finally, we close this paper with introducing Shih-Lee's formulation [5][6]. They provided Lefschetz type fixed points theorem for simplex mapping on some simplex not using homology theory, see [5, Theorem 2]. Moreover, they apply similar idea of [5] to prove existence of discretized economic equilibrium, see [6, Theorem 5].

Acknowledgment

The authors would like to thank Prof. M.-H. Shih and Prof. T. Tanino for their valuable suggestions.

References

1. T. Iimura, A discrete fixed point theorem and its applications, *J. Math. Econom.*, **39** (2003), 725–742.

2. T. Iimura, K. Murota and A. Tamura, Discrete fixed point theorem reconsidered, *J. Math. Econom.*, **41** (2005), 1030–1036.

3. K. Murota, *Discrete Convex Analysis*, SIAM, Philadelphia, 2003.

4. M.-H. Shih and J.-L. Dong, A combinatorial analogue of the Jacobian problem in automata networks, *Advances in Appl. Math.*, **34** (2005), 30–46.

5. M.-H. Shih and S.-N. Lee, A combinatorial Lefschetz fixed-point formula, *J. Combin. Theory Ser. A*, **61** (1992), 123–129.

6. M.-H. Shih and S.-N. Lee, Combinatorial formulae for multiple set-valued labellings, *Math. Ann.*, **296** (1993), 35–61.

7. G. Owen, *Game theory*, Third ed., Academic Press, 1995.

8. A. Richard, An extension of the Shih-Dong's combinatorial fixed point theorem, to appear in *Advances in Appl. Math.*.

9. A. Tarski, A lattice-theoretical fixpoint theorem and its applications, *Pacific J. Math.*, **5** (1955), 285–309.

10. Z. Yang, Discrete fixed point analysis and its applications, FBA Working Paper No. 210, Yokohama National University, Yokohama, 2004.