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Approximation Processes of Bernstein-type Operators

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Abstract

We give a generalization of the Bernstein polynomials on the closed unit interval of the real line, and consider the uniform convergence and the degree of approximation by the generalized Bernstein-type operators.

1. Introduction

Let $\mathbb{N}$ denote the set of all natural numbers. Let f be a real-valued continuous function on the closed unit interval $\mathbb{I} = [0,1]$ of the real line $\mathbb{R}$ and let $n \in \mathbb{N}$. Then nth Bernstein polynomial of f is defined by

$$B_n(f)(x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j(1-x)^{n-j} \qquad (x \in \mathbb{I}). \tag{1}$$

It is well known that the sequence $\{B_n(f)\}_{n\in\mathbb{N}}$ converges uniformly to f on $\mathbb{I}$ (cf. [3]).

Nowadays there are various generalizations of (1) and one of them is the following ([2], cf. [1], [9]):

$$C_n(f, s_n, x) = \frac{1}{s_n}\sum_{j=0}^{n}\sum_{k=0}^{s_n-1} f\left(\frac{j+k}{n+s_n-1}\right)\binom{n}{j} x^j(1-x)^{n-j}, \tag{2}$$

where $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of natural numbers. If $s_n = 1$ for all $n \in \mathbb{N}$, then $C_n(f, s_n, x) = B_n(f)(x)$.

In this paper, we further generalize (2) to the multidimensional case and consider its uniform convergence with rates in terms of the modulus of continuity of functions to be approximated. For the details, we refer to [8].

2. Convergence theorems

Let $1 \le p \le \infty$ be fixed and let $\mathbb{R}^r$ denote the metric linear space of all r-tuples of real numbers, equipped with the metric

$$d_p(x,y) = \begin{cases} \left(\sum_{i=1}^{r} |x_i - y_i|^p\right)^{1/p} & (1 \le p < \infty) \\ \max\{|x_i - y_i| : 1 \le i \le r\} & (p = \infty), \end{cases}$$

where

$$x = (x_1, x_2, \dots, x_r),\ y = (y_1, y_2, \dots, y_r) \in \mathbb{R}^r.$$

For $i = 1, 2, \dots, r$, e_i denotes the ith coordinate function on $\mathbb{R}^r$ defined by

$$e_i(x) = x_i$$

for all $x = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r$. Then we have

$$(d_p(x, y))^q \le C(p, q, r) \sum_{i=1}^{r} |e_i(x) - e_i(y)|^q \quad (x, y \in \mathbb{R}^r,\ q > 0),$$

where

$$C(p, q, r) = \begin{cases} r^{q/p} & (1 \le p < \infty,\ p \ne q) \\ 1 & (1 \le p < \infty,\ p = q) \\ 1 & (p = \infty). \end{cases}$$

A subset X of $\mathbb{R}^r$ is said to be locally closed if for each point $x \in X$, there exists an open neighborhood V_x such that $X \cap V_x$ is (relatively) closed in V_x. Note that X is locally closed if and only if there exist an open set O and closed set F such that $X = O \cap F$.

From now on let X be a locally closed subset of the first hyperquadrant

$$\mathbb{R}^r_+ = (x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \ge 0,\ 1 \le i \le r\}.$$

Let $B(X)$ denote the Banach space of all real-valued bounded functions on X with the supremum norm $\|\cdot\|_X$. Also, we denote by $C(X)$ the linear space of all real-valued continuous functions on X and set $BC(X) = B(X) \cap C(X)$. Let $b = (b_1, b_2, \dots, b_r) \in X$, where $b_i > 0$ for $i = 1, 2, \dots, r$. Let Y be a closed subset of $X \cap H_b$, where

$$H_b := \{(x_1, x_2, \dots, x_r) \in \mathbb{R}^r : x_i \le b_i, i = 1, 2, \dots, r\}.$$

Let $\{\nu_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be strictly monotone increasing sequences of positive integers and let $\{m_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be sequences of positive integers. Let $\{\gamma_{n,i}\}_{n \in \mathbb{N}}, i = 1, 2, \dots, r$, be sequences of positive real-valued functions defined on Y which satisfy

$$\gamma_n(j_1, j_2, \dots, j_r; k_1, k_2, \dots, k_r; x)$$
$$:= (\gamma_{n,1}(x)(j_1 + k_1), \gamma_{n,2}(x)(j_2 + k_2), \dots, \gamma_{n,r}(x)(j_r + k_r)) \in X$$

for all $x \in Y$ and all $n \in \mathbb{N}$, where $j_i = 0, 1, 2, \dots, \nu_{n,i}$ and $k_i = 0, 1, 2, \dots, m_{n,i} - 1$ $(i = 1, 2, \dots, r)$. For each $n \in \mathbb{N}, \alpha, \beta \in \mathbb{R}$, we define

$$p_{n,j}(\alpha, \beta) = \binom{n}{j} \beta^j (\alpha - \beta)^{n-j} \qquad (j = 0, 1, 2, \dots, n).$$

Let $n \in \mathbb{N}, f \in BC(X)$ and $x = (x_1, x_2, \ldots, x_r) \in Y$. Then we define

$$(3) \qquad B_{\nu_{n,1},\ldots,\nu_{n,r}}(f; m_{n,1}, \ldots, m_{n,r}; \gamma_{n,1}, \ldots, \gamma_{n,r}; b)(x)$$

$$= \prod_{i=1}^{r} \frac{1}{m_{n,i} b_i^{\nu_{n,i}}} \sum_{j_1=0}^{\nu_{n,1}} \sum_{k_1=0}^{m_{n,1}-1} \sum_{j_2=0}^{\nu_{n,2}} \sum_{k_2=0}^{m_{n,2}-1} \cdots \sum_{j_r=0}^{\nu_{n,r}} \sum_{k_r=0}^{m_{n,r}-1}$$

$$f(\gamma_n(j_1, j_2, \ldots, j_r; k_1, k_2, \ldots, k_r; x)) \prod_{i=1}^{r} p_{\nu_{n,i}, j_i}(b_i, x_i),$$

which forms a positive linear operator of $BC(X)$ into $B(Y)$.

Remark 1. Let $r = 1, b_1 = 1$ and $X = Y = \mathbb{I}$. We define

$$\nu_{n,1} = n, \quad m_{n,1} = s_n, \quad \gamma_{n,1}(x) = 1/(n + s_n - 1) \qquad (n \in \mathbb{N}, x \in Y).$$

Then (3) reduces to (2).

Remark 2. Let $X = Y = \mathbb{I}^r$ be the unit r-cube and $b = (1, 1, \ldots, 1)$. We define

$$m_{n,i} = 1, \quad \gamma_{n,i}(x) = 1/\nu_{n,i} \qquad (n \in \mathbb{N}, x \in Y, i = 1, 2, \ldots, r).$$

Thenn (3) reduces to the following r-dimensional Bernstein operators:

$$B_{\nu_{n,1},\nu_{n,2},\ldots,\nu_{n,r}}(f)(x) := \sum_{j_1=0}^{\nu_{n,1}} \sum_{j_2=0}^{\nu_{n,2}} \cdots \sum_{j_r=0}^{\nu_{n,r}} f\left(\frac{j_1}{\nu_{n,1}}, \frac{j_2}{\nu_{n,2}}, \ldots, \frac{j_r}{\nu_{n,r}}\right)$$

$$\times \prod_{i=1}^{r} \binom{\nu_{n,i}}{j_i} x_i^{j_i} (1 - x_i)^{\nu_{n,i} - j_i}$$

(cf. [3]).

Theorem 1 *If*

$$\lim_{n\to\infty} \|\gamma_{n,i}\|_Y = 0 \qquad (i = 1, 2, \ldots, r),$$

$$\lim_{n\to\infty} \|m_{n,i}\gamma_{n,i}\|_Y = 0 \qquad (i = 1, 2, \ldots, r)$$

and

$$\lim_{n\to\infty} \|\nu_{n,i}\gamma_{n,i} - b_i 1_X\|_Y = 0 \qquad (i = 1, 2, \ldots, r),$$

then for every $f \in BC(X)$,

$$\lim_{n\to\infty} \|B_{\nu_{n,1},\ldots,\nu_{n,r}}(f; m_{n,1}, \ldots, m_{n,r}; \gamma_{n,1}, \ldots, \gamma_{n,r}; b) - f\|_Y = 0.$$

Let $\{\varphi_{n,i}\}_{n\in\mathbb{N}}, i=1,2,\ldots,r$, be sequences of nonnegative real-valued functions defined on Y. We define

$$\gamma_{n,i}(x)=\frac{1}{\nu_{n,i}+m_{n,i}+\varphi_{n,i}(x)-1}$$

for all $n\in\mathbb{N}, x\in Y$ and for $i=1,2,\ldots,r$. Suppose that

$$\gamma_n(j_1,j_2,\ldots,j_r;k_1,k_2,\ldots,k_r;x)\in X$$

for all $x\in Y$ and all $n\in\mathbb{N}$, where $j_i=0,1,2,\ldots,\nu_{n,i}$ and $k_i=0,1,2,\ldots,m_{n,i}-1$ $(i=1,2,\ldots,r)$. Then from Theorem 1, we have the following:

Theorem 2 *If*

$$\lim_{n\to\infty}\frac{m_{n.i}}{\nu_{n,i}}=0 \qquad (i=1,2,\ldots,r)$$

and

$$\lim_{n\to\infty}\left\|b_i\Big(1+\frac{\varphi_{n,i}}{\nu_{n,i}}-1_X\right\|_Y=0 \qquad (i=1,2,\ldots,r),$$

then for every $f\in BC(X)$,

$$\lim_{n\to\infty}\|B_{\nu_{n,1},\ldots,\nu_{n,r}}(f;m_{n,1},\ldots,m_{n,r};\gamma_{n,1},\ldots,\gamma_{n,r};b)-f\|_Y=0.$$

Remark 3. Let $b=(1,1,\ldots,1)$ and we define

$$\varphi_{n,i}(x)=0 \qquad (n\in\mathbb{N},\ x\in Y,\ i=1,2,\ldots,r).$$

Then (3) reduces to the form

$$B_{\nu_{n,1},\nu_{n,2},\ldots,\nu_{n,r}}(f;m_{n,1},m_{n,2},\ldots,m_{m,r})(x) \tag{4}$$

$$:=\prod_{i=1}^{r}\frac{1}{m_{n,i}}\sum_{j_1=0}^{\nu_{n,1}}\sum_{k_1=0}^{m_{n,1}-1}\sum_{j_2=0}^{\nu_{n,2}}\sum_{k_2=0}^{m_{n,2}-1}\cdots\sum_{j_r=0}^{\nu_{n,r}}\sum_{k_r=0}^{m_{n,r}-1}$$

$$f\Big(\frac{j_1+k_1}{\nu_{n,1}+m_{n,1}-1},\frac{j_2+k_2}{\nu_{n,2}+m_{n,2}-1},\ldots,\frac{j_r+k_r}{\nu_{n,r}+m_{n,r}-1}\Big)$$

$$\times\prod_{i=1}^{r}\binom{\nu_{n,i}}{j_i}x_i^{j_i}(1-x_i)^{\nu_{n,i}-j_i}.$$

Furthermore, if

$$m_{n,i}=1 \qquad (n\in\mathbb{N}, i=1,2,\ldots,r),$$

then (4) becomes the following form:

$$B_{\nu_{n,1},\nu_{n,2},\ldots,\nu_{n,r}}(f)(x):=\sum_{j_1=0}^{\nu_{n,1}}\sum_{j_2=0}^{\nu_{n,2}}\cdots\sum_{j_r=0}^{\nu_{n,r}}f\big(\frac{j_1}{\nu_{n,1}},\frac{j_2}{\nu_{n,2}},\ldots,\frac{j_r}{\nu_{n,r}}\big)$$

$$\times \prod_{i=1} \binom{\nu_{n,i}}{j_i} x_i^{j_i}(1-x_i)^{\nu_{n,i}-j}$$
$$(n \in \mathbb{N}, f \in BC(X), x \in Y).$$

In particular, if $X = Y = \mathbb{I}^r$, we have $B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}} = B_{n,r}$ (cf. Remark 2).

Theorem 3 *The following statemets holds for all $f \in BC(X)$:*

(a) If

$$\lim_{n\to\infty} \frac{m_{n,i}}{\nu_{n,i}} = 0 \qquad (i = 1, 2, \dots, r),$$

then

$$\lim_{n\to\infty} \|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y = 0.$$

(b) $\lim_{n\to\infty} \|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f) - f\|_Y = 0.$

3. Rates of convergence

Let $f \in B(X)$ and $\delta \geq 0$. Then we define

$$\omega_p(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in X, d_p(x, y) \leq \delta\},$$

which is called the modulus of continuity of f. Obviously, $\omega_p(f, \cdot)$ is a monotone increasing function on $[0, \infty)$ and

$$\omega_p(f, \delta) = 0, \quad \omega_p(f, \delta) \leq 2\|f\| \qquad (\delta \geq 0).$$

Also, f is uniformly continuous on X if and only if

$$\lim_{\delta\to+0} \omega_p(f, \delta) = 0.$$

Now we here suppose that X is convex. Therefore, we have

$$\omega_p(f, \xi\delta) \leq (1 + \xi)\omega_p(f, \delta)$$

for all $\xi, \delta \geq 0$ and all $f \in B(X)$ (cf. [4, Lemma 3 (ii)], [5], [6, Lemma 1 (b)], [7, Lemma 2.4 (b)]). Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers.

Theorem 4 *For all $f \in BC(X)$ and all $n \in \mathbb{N}$,*

$$\|B_{\nu_{n,1},\dots,\nu_{n,r}}(f; m_{n,1}, \dots, m_{n,r}; \gamma_{n,1}, \dots, \gamma_{n,r}; b) - f\|_Y \leq \|1_X + \mu_n\|_Y \omega_p(f, \epsilon_n),$$

where for all $x \in Y$,

$$\mu_n(x) = \min\left\{C(p,r)\epsilon_n^{-2} \sum_{i=1}^r \mu_{n,i}(x),\ \sqrt{C(p,r)}\epsilon_n^{-1} \sqrt{\sum_{i=1}^r \mu_{n.i}(x)}\right\},$$

$$C(p,r)=\begin{cases} r^{2/p} & (1\le p<\infty, p\neq 2)\\ 1 & (p=2,\infty)\end{cases}$$

and

$$\mu_{n,i}(x)=\big((b_i-\nu_{n,i}\gamma_{n,i}(x))^2-\nu_{n,i}\gamma_{n,i}^2(x)\big)\big(\frac{x_i}{b_i}\big)^2$$
$$+\gamma_{n,i}(x)\big(\nu_{n,i}\gamma_{n,i}(x)-(m_{n,i}-1)(b_i-\nu_{n,i}\gamma_{n,i}(x))\big)\frac{x_i}{b_i}$$
$$+\frac{1}{3}(m_{n,i}-1)\big(m_{n,i}-\frac{1}{2}\big)\gamma_{n,i}^2(x).$$

Theorem 5 *Suppose that $\nu_{n,i}\gamma_{n,i}(x)\le b_i$ for all $n\in\mathbb{N}, x\in Y$ and for $i=1,2,\dots,r$. Then for all $f\in BC(X)$ and all $n\in\mathbb{N}$,*

$$B_{\nu_{n,1},\dots,\nu_{n,r}}(f;m_{n,1},\dots,m_{n,r};\gamma_{n,1},\dots,\gamma_{n,r};b)-f\|_Y\le\|1_X+\tau_n\|_Y\omega_p(f,\epsilon_n),$$

where for all $x\in Y$,

$$\tau_n(x)=\min\Big\{C(p,r)\epsilon_n^{-2}\sum_{i=1}^r\tau_{n,i}(x),\ \sqrt{C(p,r)}\epsilon_n^{-1}\sqrt{\sum_{i=1}^r\tau_{n,i}(x)}\Big\}$$

and

$$\tau_{n,i}(x)=\big(\frac{x_i}{b_i}\big)^2(b_i-\nu_{n,i}\gamma_{n,i}(x))^2+\frac{x_i}{b_i}\nu_{n,i}\gamma_{n,i}^2(x)+\ \frac{1}{3}(m_{n,i}-1)\big(m_{n,.i}-\frac{1}{2}\big)\gamma_{n,i}^2(x).$$

Theorem 6 *For all $f\in BC(X)$ and all $n\in\mathbb{N}$,*

$$\|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f;m_{n,1},m_{n,2},\dots,m_{n,r})-f\|_Y\le\|1_X+\eta_n\|_Y\omega_p(f,\epsilon_n),$$

where for all $x\in Y$,

$$\eta_n(x)=\min\Big\{C(p,r)\epsilon_n^{-2}\sum_{i=1}^r\eta_{n,i}(x),\ \sqrt{C(p,r)}\epsilon_n^{-1}\sqrt{\sum_{i=1}^r\eta_{n,i}(x)}\Big\}$$

and

$$\eta_{n,i}(x)=x_i^2\Big(\frac{m_{n,i}-1}{\nu_{n,i}+m_{n,i}-1}\Big)^2+\frac{\nu_{n,i}x_i}{(\nu_{n,i}+m_{n,i}-1)^2}$$
$$+\frac{1}{3}(m_{n,i}-1)\big(m_{n,i}-\frac{1}{2}\big)\frac{1}{(\nu_{n,i}+m_{n,i}-1)^2}.$$

Theorem 7 *For all $f \in BC(X)$ and all $n \in \mathbb{N}$,*

$$\|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y \tag{5}$$
$$\leq \Big(1 + \min\{M_{r,Y}C(p,r),\ \sqrt{M_{r,Y}C(p,r)}\}\Big)\omega_p(f, \theta_n),$$

where

$$M_{r,Y} = \max\{\|e_i^2 + \frac{1}{3}1_X\|_Y : i = 1, 2, \dots, r\}$$

and

$$\theta_n = \sqrt{\sum_{i=1}^{r}\Big(\big(\frac{m_{n,i}-1}{\nu_{n,i}+m_{n,i}-1}\big)^2 + \frac{1}{\nu_{n,i}}\Big)}.$$

Remark 5. From (5), we obtain the following estimate for all $f \in BC(X)$ and all $n \in \mathbb{N}$:

$$\|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f; m_{n,1}, m_{n,2}, \dots, m_{n,r}) - f\|_Y \tag{6}$$
$$\leq \Big(1 + 2\sqrt{\frac{C(p,r)}{3}}\Big)\omega_p(f, \zeta_n),$$

where

$$\zeta_n = \sum_{i=1}^{r}\Big(\frac{m_{n,i}-1}{\nu_{n,i}+m_{n,i}-1} + \frac{1}{\sqrt{\nu_{n,i}}}\Big) \leq \sum_{i=1}^{r}\Big(\frac{m_{n,i}-1}{nu_{n,i}} + \frac{1}{\sqrt{\nu_{n,i}}}\Big).$$

Therefore, the inequality (6) improves the estimate given in [2, Theorem 2] for $r = 1, b_1 = 1, X = Y = \mathbb{I}$ and $\nu_{n,1} = n$. Also, we can get the following estimate for all $f \in C(\mathbb{I}^r)$ and all $n \in \mathbb{N}$ (cf. [6], [7]):

$$\|B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,r}}(f) - f\|_{\mathbb{I}^r} \leq \xi_n(p,r)\omega_p\Big(f, \epsilon_n\sqrt{\sum_{i=1}^{r}\frac{1}{\nu_{n,i}}}\Big), \tag{7}$$

$$\xi_n(p,r) = 1 + \min\Big\{\frac{C(p,r)}{4\epsilon_n^2},\ \frac{\sqrt{C(p,r)}}{2\epsilon_n}\Big\}.$$

In particular, if $\nu_{n,i} = n$ for all $n \in \mathbb{N}, i = 1, 2, \dots, r$ and $B_n := B_{\nu_{n,1},\nu_{n,2},\dots,\nu_{n,i}}$, then (7) establishes the inequality

$$\|B_n(f) - f\|_{\mathbb{I}^r} \leq \Big(1 + \min\Big\{\frac{rC(p,r)}{4},\ \frac{\sqrt{rC(p,r)}}{2}\Big\}\Big)\omega_p(f, \frac{1}{\sqrt{n}})$$

for all $f \in C(\mathbb{I}^r)$ and all $n \in \mathbb{N}$.

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