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Mean ergodic theorems for nonlinear nonexpansive semigroups in Banach spaces

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Abstract

In this article, we deal with weak and strong convergence theorems for abstract semigroups of nonlinear operators in Banach spaces. We first discuss nonlinear mean ergodic theorems for nonexpansive semigroups in a uniformly convex Banach space whose norm is Fréchet differentiable. Next, we consider nonlinear ergodic theorems in the case when a Banach space is general and the domains of the nonexpansive semigroups are compact. Further, we deal with weak and strong convergence theorems of Halpern's type and Mann's type for nonexpansive semigroups in Banach spaces.

1 Introduction

Let C be a closed and convex subset of a real Banach space. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1975, Baillon [5] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means $S_n(x)$ converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, we have that P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction *"an ergodic retraction"*. In 1981, Takahashi [26, 28] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [20] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [10]. In 1999, Lau, Shioji and Takahashi [15] extended Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach space. By using Rodé's method, Kido and Takahashi [12] also proved a mean ergodic theorem for noncommutative semigroups of linear bounded operators in Banach spaces. On the other hand, Edelstein [9] studied a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset in a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space, let T be a nonexpansive mapping of C into itself and let $\xi \in C$. Then, for each point x of the closure of convex hull of the ω-limit set $\omega(\xi)$

of ξ, the Cesàro means converge to a fixed point of T, where the ω-limit set $\omega(\xi)$ of ξ is the set of cluster points of the sequence $\{T^n\xi : n = 1, 2, \dots\}$. Atsushiba and Takahashi [4] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means converge to a fixed point of T. This result was extended to commutative semigroups of nonexpansive mappings by Atsushiba, Lau and Takahashi [1]. Suzuki and Takahashi [24] constructed a nonexpansive mapping of a compact and convex subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means converge to a point of C, but the limit point is not a fixed point of T.

In this article, we deal with weak and strong convergence theorems for abstract semigroups of nonlinear operators in Banach spaces. We first discuss nonlinear mean ergodic theorems for nonexpansive semigroups in a uniformly convex Banach space whose norm is Fréchet differentiable. Next, we consider nonlinear ergodic theorems in the case when a Banach space is general and the domains of the nonexpansive semigroups are compact. Further, we deal with weak and strong convergence theorems of Halpern's type and Mann's type for nonexpansive semigroups in Banach spaces.

2 Preliminaries

Let C be a nonempty closed convex subset of a Banach space E and let T be a mapping of C into C. Let D be a subset of C and let P be a mapping of C onto D. Then P is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into C is said to be a retraction if $P^2 = P$. A subset D of C is said to be a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction of C onto D. Let E be a Banach space. Then, for every ε with $0 \leq \varepsilon \leq 2$, the modulus $\delta(\varepsilon)$ of convexity of E is defined by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}.$$

A Banach space E is said to be uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. E is also said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A uniformly convex Banach space is strictly convex and reflexive. Let E be a Banach space and let E^* be its dual, that is, the space of all continuous linear functionals x^* on E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$. With each $x \in E$, we associate the set $J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$. Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \emptyset$ for any $x \in E$. Then the multi-valued operator $J : E \to E^*$ is called the duality mapping of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. When this is the case, the norm of E is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each x in U, this limit is attained uniformly for y in U. The space E is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. It is well known that if E is smooth, then the duality mapping J is single valued. It is also known that if E has a Fréchet differentiable norm, then J is norm to norm continuous; see [30, 31] for more details.

Let S be a semitopological semigroup, i.e., a semigroup with Hausdorff topology such that for each $s \in S$, the mappings $t \mapsto ts$ and $t \mapsto st$ of S into itself are continuous. Let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* is called a mean on X if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on X if and only if

$$\inf\{f(s) : s \in S\} \leq \mu(f) \leq \sup\{f(s) : s \in S\}$$

for every $f \in X$. For a mean μ on X and $f \in X$, sometimes we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $\ell_s f$ and $r_s f$ of $B(S)$ given by $(\ell_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing constants which is invariant under ℓ_s, $s \in S$ (resp. r_s, $s \in S$). Then a mean μ on X is said to be left invariant (resp. right invariant) if $\mu(f) = \mu(\ell_s f)$ (resp. $\mu(f) = \mu(r_s f)$) for all $f \in X$ and $s \in S$. An invariant mean is a left and right invariant mean. X is said to be amenable if there exists an invariant mean on X. We know from [8] that if S is commutative, then $B(S)$ is amenable. S is called left reversible if any two right ideals in S have nonvoid intersection, i.e., $aS \cup bS \neq \emptyset$ for $a, b \in S$. The class of left reversible semigroups includes all groups and commutative semigroups. Let S be a semitopological semigroup and let C be a nonempty subset of a Banach space E. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following: (i) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$; (ii) for each $x \in C$, the mapping $s \mapsto T_s x$ is continuous; (iii) for each $s \in S$, T_s is a nonexpansive mapping of C into itself. In the case of $S = \mathbb{R}_+ = [0, \infty)$ and $T_0 = I$, we denote a nonexpansive semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C by $\{S(t) : t \in \mathbb{R}_+\}$ and call it a one-parameter nonexpansive semigroup on C. For a nonexpansive semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$. We also denote by $C(S)$ the Banach space of all bounded continuous functions on S.

3 Mean Ergodic Theorems

Let S be a semitopological semigroup and let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on $C(S)$. Then $\{\mu_\alpha \in A\}$ is said to be asymptotically invariant if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

If C is a closed convex subset of a reflexive Banach space E and $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Then we know that for each $x \in C$ and $y^* \in E^*$, the real valued function $t \mapsto \langle T_t x, y^* \rangle$ is in $C(S)$. So, we can define the value $\mu_t \langle T_t x, y^* \rangle$ of μ at this function. So, by the Riesz theorem, there exists an $x_0 \in E$ such that $\mu_t \langle T_t x, y^* \rangle = \langle x_0, y^* \rangle$ for every $y^* \in E^*$. We write such an x_0 by $T_\mu x$ or $\int T_t x d\mu(t)$; see [26, 10] for more details. The following is the first mean ergodic theorem for a noncommutative nonexpansive semigroup on C in a Hilbert space.

Theorem 3.1 (Takahashi [26]). *Let H be a Hilbert space and let C be a closed convex subset of H. Let $C(S)$ be amenable and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_t P = P$ for every $t \in S$ and $Px \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$.*

The following is Rodé's theorem [20] which extends Baillon's theorem to a noncommutative nonexpansive semigroup on C in a Hilbert space.

Theorem 3.2 (Rodé [20]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. If $\{\mu_\alpha\}$ is an asymptotically invariant net of means on $C(S)$, then for each $x \in C$, $T_{\mu_\alpha}x$ converges weakly to an element of $F(\mathcal{S})$.*

Next, we state a nonlinear ergodic theorem for a nonexpansive semigroup in a Banach space. Before stating it, we give a definition. A net $\{\mu_\alpha\}$ of continuous linear functionals on $C(S)$ is called strongly regular if it satisfies the following conditions: (i) $\sup_\alpha \|\mu_\alpha\| < +\infty$; (ii) $\lim_\alpha \mu_\alpha(1) = 1$; (iii) $\lim_\alpha \|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for every $s \in S$.

Theorem 3.3 (Hirano, Kido and Takahashi [10]). *Let S be a commutative semitopological semigroup and let E be a unifromly convex Banach space with a Fréchet differentiable norm. Let C be a nonempty closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S})$ is nonempty. Then there exists a unique nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Further, if $\{\mu_\alpha\}$ is a strongly regular net of continuous linear functionals on $C(S)$, then for each $x \in C$, $T_{\mu_\alpha}T_tx$ converges weakly to Px uniformly in $t \in S$.*

In 1999, Lau, Shioji and Takahashi [15] extended Theorem 3.3 to an amenable semigroup of nonexpansive mappings on a uniformly convex Banach space.

Theorem 3.4 (Lau, Shioji and Takahashi [15]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let S be a semitopological semigroup. Let C be a closed convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C with $F(\mathcal{S}) \neq \emptyset$. Suppose that $C(S)$ has an invariant mean. Then there exists a unique nonexpansive retraction P from C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for each $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. Further, if $\{\mu_\alpha\}$ is an asymptotically invariant net of means on $C(S)$, then for each $x \in C$, $\{T_{\mu_\alpha}x\}$ converges weakly to Px.*

Atsushiba and Takahashi [4] proved a nonlinear ergodic theorem for a one-parameter nonexpansive semigroup in a strictly convex Banach space which is connected with Dafermos and Slemrod [7].

Theorem 3.5 (Atsushiba and Takahashi [4]). *Let E be a strictly convex Banach space and let C be a nonempty compact convex subset of E. Let $\mathcal{S} = \{S(t) : 0 \leqq t < \infty\}$ be a one-parameter nonexpansive semigroup on C and let $x \in C$. Then, $(1/t)\int_0^t S(\tau + h)xd\tau$ converges strongly to a common fixed point of $S(t)$, $t \in [0, \infty)$ unifomly in $h \in [0, \infty)$.*

Further, Atsushiba, Lau and Takahashi [1] obtained the following theorem which generalizes Theorem 3.5.

Theorem 3.6 (Atsushiba, Lau and Takahashi [1]). *Let E be a strictly convex Banach space, let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C, where S is commutative. Let $\{\lambda_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on $C(S)$ and let $x \in C$. Then, $\int T_{h+t}xd\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of T_t, $t \in S$ uniformly in $h \in S$.*

Remark Suzuki and Takahashi [24] constructed a nonexpansive mapping T of a compact convex subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means converge, but the limit point is not a fixed point of T. Suzuki [23] showed that there exists a one-parameter semigroup $\{T(t) : t \in \mathbb{R}_+\}$ of nonexpansive mappings of a closed and convex subset C in a Banach space into itself such that

$$\lim_{t\to\infty} \|1/t \int_0^t T(t)x\, dt - x\| = 0$$

for some $x \in C$ which is not a common fixed point of $\{T(t) : t \in \mathbb{R}_+\}$. Very recently, Miyake and Takahashi [19] extended Theorem 3.3 to that of noncommutative nonexpansive semigroups in general Banach spaces.

Theorem 3.7 (Miyake and Takahashi [19]). *Let E be a Banach space, let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C, where S is a semitopological semigroup. Let $\{\lambda_\alpha : \alpha \in A\}$ be an asymptotically invariant net of means on $C(S)$ and let $x \in C$. Then, $\int T_{h+t}xd\lambda_\alpha(t)$ converges strongly to a point y_0 uniformly in $h \in S$.*

Using Theorem 3.7, they also obtained the following mean ergodic theorem.

Theorem 3.8 (Miyake and Takahashi [19]). *Let E be a strictly convex Banach space, let C be a nonempty compact convex subset of E and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C, where S is a semitopological semigroup. Let $\{\lambda_\alpha : \alpha \in A\}$ be an asymptotically invariant net of means on $C(S)$ and let $x \in C$. Then, $\int T_{h+t}xd\lambda_\alpha(t)$ converges strongly to a common fixed point y_0 of T_t, $t \in S$ uniformly in $h \in S$.*

4 Strong Convergence Theorems of Halpern's Type

Shimizu and Takahashi [21] introduced the first iterative scheme for finding a common fixed point of a family of nonexpansive mappings and proved the following theorem:

Theorem 4.1 (Shimizu and Takahashi [21]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C such that $\bigcap_{t\in\mathbb{R}_+} F(S(t)) \neq \emptyset$. Then, for each $x \in C$, $\{x_n\}$ generated by $x_1 = x$ and*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)x_n\, ds, \quad n = 1, 2, \ldots,$$

converges strongly to an element Px of $\bigcap_{t\in\mathbb{R}_+} F(S(t))$ as $t_n \to \infty$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Shioji and Takahashi [22] extended Theorem 4.1 to that of a Banach space.

Theorem 4.2 (Shioji and Takahashi [22]). *Let E be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ and $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^*\mu_n\| = 0$ for every $s \in S$. Suppose that $x, y_1 \in C$ and $\{y_n\}$ is given by*

$$y_{n+1} = \beta_n x + (1-\beta_n)T_{\mu_n}y_n, \quad n = 1, 2, \ldots,$$

where $\{\beta_n\}$ is in $[0,1]$. If $\{\beta_n\}$ is chosen so that $\lim_{n\to\infty} \beta_n = 0$ and $\Sigma_{n=1}^{\infty}\beta_n = \infty$, then $\{y_n\}$ converges strongly to an element of $F(\mathcal{S})$.

Lau, Miyake and Takahashi [13] also proved such a strong theorem of Halpern's type in the case when a Banach space is smooth and the domains of the nonexpansive semigroups are compact.

Theorem 4.3 (Lau, Miyake and Takahashi [13]). *Let E be a strictly convex and smooth Banach space and let C be a compact convex subset of E. Let S be a left reversible semigroup and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C. Let $C(S)$ be amenable and let $\{\mu_n\}$ be a strongly left regular sequence of means on $C(S)$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T_{\mu_n}x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$. Then $\{y_n\}$ converges strongly to an element Px of $F(\mathcal{S})$, where P denotes the sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

As direct consequences of Theorem 4.3, we have the following corollaries.

Corollary 4.4. *Let E be a strictly convex and smooth Banach space and let C be a compact and convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a one-parameter semigroup of nonexpansive mappings of C into itself. Then, for each $x_1 = x \in C$, define*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ and $\{t_n\} \subset 0, \infty]$ satisfies $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty}\frac{t_n}{t_{n+1}} = 1$. Then, $\{x_n\}$ converges to a common fixed point of $\mathcal{S}$.

Corollary 4.5. *Let E be a strictly convex smooth Banach space and let C be a compact and convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a one-parameter semigroup of nonexpansive mappings of C into itself. Then, for each $x_1 = x \in C$, define*

$$x_{n+1} = \alpha_n x + (1-\alpha_n)r_n\int_0^{\infty} \exp(-r_n s)T(s)x_n ds, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty]$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty} r_n = 0$. Then, $\{x_n\}$ converges to a common fixed point of $\mathcal{S}$.

5 Weak Convergence Theorems of Mann's Type

Motivated by Shimizu and Takahashi [21], Atsushiba and Takahashi [3] also obtained the following weak convergence theorem.

Theorem 5.1 (Atsushiba and Takahashi [3]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C such that $\bigcap_{t\in\mathbb{R}_+} F(S(t))$ is nonempty. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s)x_n\, ds, \quad n = 1, 2, \ldots,$$

where $t_n \to \infty$ as $n \to \infty$ and $\{\alpha_n\} \subset [0,1]$ satisfies $0 < \alpha_n \le a < 1$. Then $\{x_n\}$ converges weakly to a common fixed point of $\bigcap_{t\in\mathbb{R}_+} F(S(t))$.

Atsushiba, Shioji and Takahashi [2] extended Theorem 5.1 to that of a Banach space.

Theorem 5.2 (Atsushiba, Shioji and Takahashi [2]). *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$ ande let $\{\mu_n\}$ be a sequence of means on $C(S)$ such that $\|\mu_n - \ell_s^* \mu_n\| = 0$ for every $s \in S$. Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $0 \leq \alpha_n \leq a < 1$. Then $\{x_n\}$ converges weakly to an element $x_0 \in F(\mathcal{S})$.

Suzuki and Takahashi [25] also proved such a theorem of Mann's type when a Banach space is general and the domains of one-parameter nonexpansive semigroups are compact.

Theorem 5.3 (Suzuki and Takahashi [25]). *Let C be a compact convex subset of a Banach space E and let $\mathcal{S} = \{S(t) : t \in \mathbb{R}_+\}$ be a one-parameter nonexpansive semigroup on C. Let $x_1 \in C$ and define a sequence $\{x_n\}$ in C by*

$$x_{n+1} = \frac{\alpha_n}{t_n} \int_0^{t_n} S(s) x_n \, ds + (1 - \alpha_n) x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ and $\{t_n\} \subset (0, \infty)$ satisfy the following conditions:

$$0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1, \qquad \lim_{n \to \infty} t_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{t_{n+1}}{t_n} = 1.$$

Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Miyake and Takahashi [16] extended Theorem 5.3 to commutative nonexpansive semigroups in a Banach space.

Theorem 5.4 (Miyake and Takahashi [16]). *Let C be a compact convex subset of a Banach space E and let $\mathcal{S} = \{T_t : t \in S\}$ be a commutative nonexpansive semigroup on C. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on $C(S)$ such that $\lim_{n \to \infty} \|\mu_n - \mu_{n+1}\| = 0$. Let $\{x_n\}$ be the sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1 - \alpha_n) x_n, \quad n = 1, 2, \ldots,$$

where $\{\alpha_n\} \subset [0,1]$ satisfies $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S}$.

Problem Let C be a compact convex subset of a Banach space E and let $\mathcal{S} = \{T_t : t \in S\}$ be a noncommutative nonexpansive semigroup on C. Let $\{\mu_n\}$ be an asymptotically invariant sequence of means on $C(S)$ such that $\lim_{n \to \infty} \|\mu_n - \mu_{n+1}\| = 0$. Suppose that $\{\alpha_n\} \subset [0,1]$ satisfies $0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1$ and $\{x_n\}$ is the sequence defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n T_{\mu_n} x_n + (1 - \alpha_n) x_n, \quad n = 1, 2, \ldots.$$

Then, does $\{x_n\}$ converge strongly to a common fixed point of $\mathcal{S}$?

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