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SMITH PROBLEM FOR A FINITE OLIVER GROUP WITH NON-TRIVIAL CENTER

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1. Introduction

The Smith problem is that two tangential representations are isomorphic or not for a smooth action on a homotopy sphere with exactly two fixed points. Two real G-modules U and V are called *Smith equivalent* if there exists a smooth action of G on a sphere Σ such that $S^G = \{x, y\}$ for two points x and y at which $T_x(\Sigma) \cong U$ and $T_y(\Sigma) \cong V$ as real G-modules. We will consider a subset $Sm(G)$ of the real representation ring $RO(G)$ of G consisting of the differences $U-V$ of real G-modules U and V which are Smith equivalent. We also define a subset $CSm(G)$ of $RO(G)$ consisting of the differences $U - V \in Sm(G)$ of real G-modules U and V such that for the sphere Σ appearing in the notion of Smith equivalence of U and V satisfies that Σ^P is connected for every $P \in \mathcal{P}(G)$. Moreover, we assume that $0 \in CSm(G)$ as definition.

In many groups, Smith equivalent modules are not isomorphic. In this paper we discuss the Smith problem for an Oliver group with non-trivial center. Throughout this paper we assume a group is finite.

2. Topological viewpoint

We denote by $\mathcal{P}(G)$ the family of subgroups of G consisting of the trivial subgroup of G and all subgroups of G of prime power order, and by $\mathcal{L}(G)$ the family of large subgroups of G. Here, by a *large subgroup* of G we mean a subgroup $H \leq G$ such that $O^p(G) \leq H$ for some prime p, where $O^p(G)$ is the smallest normal subgroup of G such that $|G/O^p(G)| = p^k$ for some integer $k \geq 0$. A real G-module V is called *$\mathcal{L}(G)$-free* if $\dim V^H = 0$ for each $H \in \mathcal{L}(G)$, which amounts to saying that $\dim V^{O^p(G)} = 0$ for each prime p dividing $|G|$. Following [PSo], we denote by $LO(G)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of two real $\mathcal{L}(G)$-free G-modules U and V such that $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for every $P \in \mathcal{P}(G)$.

For two subgroups $P < H$ of G with $P \in \mathcal{P}(G)$, and a smooth G-manifold X or a real G-module X, we consider the number

$$d_X(P, H) = \dim X^P - 2\dim X^H$$

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where *dim* means the dimension of the G-CW complex. Furthermore we define by $\dim Z = \dim X - \dim Y$ for a virtual real G-module $Z = X - Y$ of $RO(G)$. A smooth G-manifold X satisfies the *gap condition* (GC) if $d_X(P, H) > 0$ for every pair (P, H) of subgroups $P < H$ of G with $P \in \mathcal{P}(G)$.

The following theorem goes back to [PSo], the Realization Theorem.

Theorem 2.1 ([PSo]). *Let G be a finite Oliver gap group. Then $LO(G) \subseteq CSm(G)$.*

We impose a number of restrictions on a smooth G-manifold, in particular, a real G-module X. The restrictions are collected in the following conditions, where we consider series $P < H \leq G$ of subgroups P and H of G *always* with $P \in \mathcal{P}(G)$. We say that a smooth G-manifold X satisfies the *weak gap condition* (WGC) if the conditions (WGC1)–(WGC4) all hold (cf. [LM], [MP]), and we say that X satisfies the *semi-weak gap condition* (SWGC) if the conditions (WGC1) and (WGC2) both hold.

(WGC1) $d_X(P, H) \geq 0$ for every $P < H \leq G$, $P \in \mathcal{P}(G)$.

(WGC2) If $d_X(P, H) = 0$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, then $[H : P] = 2$, $\dim X^H > \dim X^K + 1$ for every $H < K \leq G$, and X^H is connected.

(WGC3) If $d_X(P, H) = 0$ for some $P < H \leq G$, $P \in \mathcal{P}(G)$, and $[H : P] = 2$, then X^H can be oriented in such a way that the map $g\colon X^H \to X^H$ is orientation preserving for any $g \in N_G(H)$.

(WGC4) If $d_X(P, H) = d_X(P, H') = 0$ for some $P < H$, $P < H'$, $P \in \mathcal{P}(G)$, then the smallest subgroup $\langle H, H' \rangle$ of G containing H and H' is not a large subgroup of G.

Now, for a finite group G, we define subgroups $VLO(G)$, $WLO(G)$ and $MLO(G)$ of the free abelian group $LO(G)$ as follows.

$VLO(G) = \{U - V \in LO(G) \mid U \oplus W$ and $V \oplus W$ both satisfy the gap condition for some real $\mathcal{L}(G)$-free G-module $W\}$

$WLO(G) = \{U - V \in LO(G) \mid U \oplus W$ and $V \oplus W$ both satisfy the weak gap condition for some real $\mathcal{L}(G)$-free G-module $W\}$

$MLO(G) = \{U - V \in LO(G) \mid U \oplus W$ and $V \oplus W$ both satisfy the semi-weak gap condition for some real $\mathcal{L}(G)$-free G-module $W\}$

Note that if $\mathcal{P}(G) \cap \mathcal{L}(G) = \varnothing$ then for an $\mathcal{L}(G)$-free real G-modules U and V there is a real $\mathcal{L}(G)$-free G-module W such that both $U \oplus W$ and $V \oplus W$ satisfy (WGC2), and if G is an Oliver group then for an $\mathcal{L}(G)$-free real G-modules U and V there is a real $\mathcal{L}(G)$-free G-module W such that both $U \oplus W$ and $V \oplus W$ satisfy (WGC2) and (WGC4).

In general, $VLO(G) \subseteq WLO(G) \subseteq MLO(G) \subseteq LO(G)$ by definitions. But if G is a gap group, then for every $U - V \in LO(G)$, there exists a real $\mathcal{L}(G)$-free G-module W satisfying the gap condition, such that $U \oplus W$ and $V \oplus W$ also satisfy the gap condition, and thus $U - V \in VLO(G)$, and hence

$$VLO(G) = WLO(G) = MLO(G) = LO(G).$$

Therefore, the following theorem extends the result in Theorem 2.1 by using Theorem in [MP].

Theorem 2.2. *Let G be a finite Oliver group. Then* $WLO(G) \subseteq CSm(G)$.

3. Algebraic viewpoint

We denote by $PO(G)$ the subgroup of $RO(G)$ of G consisting of the differences $U - V$ of representations U and V such that $\dim U^G = \dim V^G$ and $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for any subgroup P of G of prime power order. We note that in [PSo], $PO(G)$ is denoted by $IO(G, G)$. Similarly, we denote by $\overline{PO}(G)$ the subgroup of $RO(G)$ of G consisting of the differences $U - V$ of representations U and V such that $\dim U^G = \dim V^G$ and $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for any subgroup P of G of odd prime power order and order 2, 4. By a theorem of Sanchez [Sa], the difference of two Smith equivalent representations lies in $\overline{PO}(G)$ and the difference of two $\mathcal{P}$-matched Smith equivalent representations lies in $PO(G)$.

We define the Laitinen number a_G as the number of real conjugacy classes in G represented by elements of G not of prime power order. The rank of $PO(G)$ is equal to the maximum of 0 and $a_G - 1$. Moreover the rank of $\overline{PO}(G)$ is equal to the rank of $PO(G)$ plus the number of all real conjugacy classes represented by 2-elements of order ≥ 8. Now, let H be a normal subgroup of G. We denote by $PO(G, H)$ the subgroup of $RO(G)$ consisting of the differences $U - V$ of representations U and V such that $U^H \cong V^H$ as representations over G/H, and $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for any subgroup P of prime power order. Again, we note that in [PSo], $PO(G, H)$ is denoted by $IO(G, H)$. It holds that $PO(G) = PO(G, G)$. Let $b_{G/H}$ be the number of all real conjugacy classes in G/H which are images from real conjugacy classes of G represented by elements not of prime power order by the surjection $G \to G/H$. Then the rank of $PO(G, H)$ is equal to $a_G - b_{G/H}$ (see [PSo]).

Proposition 3.1 (cf. [PSo]). *It holds that*

$$PO(G, G^{nil}) \leq LO(G) \leq PO(G) \leq \overline{PO}(G) \leq RO(G).$$

Note that $G^{nil} = \cap_p O^p(G)$. Also it is known that

$$LO(G) \subseteq CSm(G) \subseteq Sm(G)$$

if G is an Oliver gap group.

4. Upper restriction

Let S be a set of primes dividing $|G|$ and 1, and let denote by $G^{\cap S}$ the normal subgroup of G defined as

$$G^{\cap S} = \bigcap_{L \trianglelefteq G; [G:L] \in S} L.$$

Theorem 4.1 ([M07a, KMK]). *Let G be a finite Oliver group. We set* $S = \{2,3\}$ *if a Sylow 2-subgroup of G is normal and set* $S = \{2\}$ *otherwise. Then it holds that*

$$CSm(G) \subseteq PO(G, G^{\cap S}) \quad and \quad Sm(G) \subseteq \overline{PO}(G, G^{\cap S}).$$

In addition if G is a gap group and $G^{nil} = G^{\cap S}$, *then it holds that*

$$LO(G) = CSm(G) = PO(G, G^{nil}).$$

Here G^{nil} *is the minimal subgroup among normal subgroups N of G such that G/N is nilpotent.*

In particular, $a_G = b_{G/G^{\cap S}}$ yields that $CSm(G) = 0$.

Proposition 4.2 (cf. [PSu08]). $G/G^{\cap S}$ *is an elementary abelian group.*

5. Known results

In this section we summarize several known results ([Ju, M07a, M07b, PSo, PSu07, Su]). First we treat a non-solvable group. Pawałowski and Solomon [PSo] showed that $0 \neq PO(G, G^{nil}) \subseteq CSm(G)$ if G is a non-solvable gap group with $a_G \geq 2$, Pawałowski and Sumi [PSu07]showed that $0 \neq LO(G) \cap CSm(G)$ if G is a non-solvable group with $a_G \geq 2$, except $Aut(A_6)$, $P\Sigma L(2, 27)$, and Morimoto [M07a, M07b] showed that $Sm(Aut(A_6)) = 0$ and $CSm(P\Sigma L(2, 27)) \neq 0$. Combining these results we can state that

Theorem 5.1. *For a finite non-solvable group G,* $Sm(G) \neq 0$ *if and only if* $a_G \leq 1$ *or* $G \cong Aut(A_6)$.

We say that an element not of prime power order is an NPP element. Morimoto showed the following theorem to get $CSm(P\Sigma L(2, 27)) \neq 0$.

Theorem 5.2 (Morimoto). *Let G be an Oliver gap group. Suppose that* $O^2(G)$ *has a dihedral subgroup* D_{2pq} *of order 2pq with distinct primes p and q and G has two real conjugacy classes of NPP elements contained in* $O^2(G)$. *Then* $CSm(G) \neq 0$.

To show $LO(G) \cap CSm(G) \neq 0$ for a non-solvable group with $LO(G) \neq 0$, Pawałowski and Sumi introduced a basic pair (cf. [PSu07, Su]). Let $f \colon G \to G/G^{nil}$ be a natural homomorphism. For two NPP elements x and y of an finite Oliver group G, we call (x, y) a basic pair, if $f(x) = f(y)$, x is not real conjugate to y, and one of the following claims is satisfied:

(1) x and y are elements of some gap subgroup of G.
(2) $|x|$ is even and the involution of $\langle x \rangle$ is conjugate to the involution of $\langle y \rangle$ in G.

We denote by $\pi(G)$ the set of all primes dividing the order of G. Note that $\langle x \rangle G^{nil} = \langle y \rangle G^{nil}$ as $f(x) = f(y)$. Recall that if $|x|$ is even, then for the involution c of $\langle x \rangle$, $c \in O^2(G)$ or $|\pi(O^2(C_G(c)))| \geq 2$, then $\langle x \rangle O^2(G)$ is a gap group.

Theorem 5.3 ([PSu07]). *If an Oliver group has a basic pair, it holds* $LO(G) \cap CSm(G) \neq 0$.

Recall that $LO(G/G^{nil}) \subseteq LO(G)$. Furthermore we have

Proposition 5.4. *$2LO(G/G^{nil}) \subseteq WLO(G)$ and in particular $LO(G/G^{nil}) \neq 0$ implies $CSm(G) \neq 0$.*

Then $LO(G) \cap CSm(G) = 0$ implies $LO(G/G^{nil}) = 0$. Thus the following proposition is important.

Proposition 5.5 ([PSu07]). *Let H be a nilpotent group with $LO(H) = 0$. Then H is isomorphic to one of the following groups:*

(1) *a p-group for a prime p,*
(2) *$C_2 \times P$ for an odd prime p and a p-group P, or*
(3) *$P \times C_3$ for a 2-group P such that any element is self-conjugate.*

Lemma 5.6. *If $a_G \geq 2$ and $LO(G) = 0$ it holds $|\pi(G/G^{nil})| = 1, 2$.*

Proof. If $|\pi(G/G^{nil})| \geq 3$, then G/G^{nil} is a gap group with $LO(G/G^{nil}) \neq 0$, a contrary. If $|\pi(G/G^{nil})| = 0$, then G is perfect and thus rank $LO(G) = a_G - 1 > 0$, a contrary. □

Theorem 5.7. *If $LO(G) \cap CSm(G) = 0$, then G has no element x with $|\pi(\langle x \rangle)| \geq 3$.*

Proof. We assume that x is an element of G of order pqr such that p, q, r are distinct primes. It is clear that $a_G \geq 4$. We may assume that $x^{pq} \in G^{nil}$ by Lemma 5.6. Then $(x^{pq}x^{qr}x^{pr}, x^{qr}x^{pr})$ is a basic pair, a contrary. □

Thus $|\pi(\langle c \rangle)| \leq 2$ for each non-trivial element $c \in Z(G)$.

6. Induced modules and $PO(G)$

Let G be a finite group and NPP(G) be the set of all elements of G not of prime power order. Note that NPP(G) does not contain the identity element. For the real representation ring $RO(G)$, the real vector space $RO(G) \otimes \mathbb{R}$ is identified with the vector space consisting of all maps from the set of real conjugacy classes of G to the real number field $\mathbb{R}$. We denote by $1^G_{(g)^G_\pm}$ the map defined by $1^G_{(g)^G_\pm}((g)^G_\pm) = 1$ and $1^G_{(g)^G_\pm}((a)^G_\pm) = 0$ if a is not real conjugate to g. Then

$$RO(G) \otimes \mathbb{R} \cong \langle 1^G_{(g)^G_\pm} \mid (g)^G_\pm \subseteq G \rangle$$

and

$$RO(G)_{\mathcal{P}(G)} \otimes \mathbb{R} \cong \langle 1^G_{(g)^G_\pm} \mid g \in \text{NPP}(G) \rangle.$$

Let K be a subgroup of G. The induced map $\text{Ind}^G_K 1^K_{(k)^K_\pm}$ has a non-zero value at $(g)^G_\pm$ only if g is real conjugate to k in G, i.e. $(g)^G_\pm = (k)^G_\pm$, since

$$\text{Ind}^G_K 1^K_{(k)^K_\pm}((a)^G_\pm) = \sum_{\substack{bK \in G/K \\ b^{-1}ab \in K}} 1^K_{(k)^K_\pm}((b^{-1}ab)^K_\pm).$$

We denote by $RO(G)_{\mathcal{P}(G)}$ the subset of $RO(G)$ consisting the differences $U - V$ of real representations U and V such that $\mathrm{Res}^G_P(U) \cong \mathrm{Res}^G_P(V)$ for $P \in \mathcal{P}(G)$. It is clear that

$$PO(G) = \mathrm{Ker}(\mathrm{Fix}^G : RO(G)_{\mathcal{P}(G)} \to \mathbb{R}).$$

We have the following commutative diagram.

$$\begin{array}{ccccc} RO(K)_{\mathcal{P}(K)} \otimes \mathbb{R} & \longrightarrow & (\mathrm{Ind}^G_K RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{R} & \underset{\subseteq}{\longrightarrow} & RO(G)_{\mathcal{P}(G)} \otimes \mathbb{R} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \langle 1^K_{(k)^K_\pm} \mid k \in \mathrm{NPP}(K) \rangle & \longrightarrow & \langle 1^G_{(k)^G_\pm} \mid k \in \mathrm{NPP}(K) \rangle & \underset{\subseteq}{\longrightarrow} & \langle 1^G_{(g)^G_\pm} \mid g \in \mathrm{NPP}(G) \rangle \end{array}$$

It holds that

$$(\mathrm{Ind}^G_K RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{R} = (\mathrm{Ind}^G_K RO(K))_{\mathcal{P}(G)} \otimes \mathbb{R}$$

and then that

$$(\mathrm{Ind}^G_K RO(K)_{\mathcal{P}(K)}) \otimes \mathbb{Q} = (\mathrm{Ind}^G_K RO(K))_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

Since an element of $RO(G)$ is a linear combination with rational coefficients of induced modules of $RO(C)$ for cyclic subgroups C of G, we obtain that

$$\sum_{\substack{(C)^G \\ C \le G}} (\mathrm{Ind}^G_C RO(C)_{\mathcal{P}(C)}) \otimes \mathbb{Q} = RO(G)_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

Furthermore, noting $\mathrm{Ind}^G_C RO(C)_{\mathcal{P}(C)} = 0$ for $C \in \mathcal{P}(G)$, it holds that

$$\sum_{\substack{(\langle g \rangle)^G \\ g \in \mathrm{NPP}(G)}} (\mathrm{Ind}^G_{\langle g \rangle} RO(\langle g \rangle)_{\mathcal{P}(\langle g \rangle)}) \otimes \mathbb{Q} = RO(G)_{\mathcal{P}(G)} \otimes \mathbb{Q}.$$

If g has order $2p$ for an odd prime p, then $RO(\langle g \rangle)_{\mathcal{P}(\langle g \rangle)}) \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[\langle x^p \rangle]) \otimes (2\mathbb{R} - \eta)$$

for all real irreducible modules η over $\langle g^2 \rangle$ and $PO(\langle g \rangle) \otimes \mathbb{Q}$ is spanned by

$$(2\mathbb{R} - \mathbb{R}[\langle x^p \rangle]) \otimes (\eta - \eta')$$

for all non-trivial real irreducible modules η, η' over $\langle g^2 \rangle$. Hence we can investigate $LO(G)$ for a finite non-gap group G with $G/O^2(G)$ an elementary abelian 2-group. Letting C_2^n be an elementary abelian 2-group of order 2^n, we obtain the following results.

Theorem 6.1. *Let $G := K \times C_2^n$, $n \ge 2$ be an Oliver group such that $K/O^2(K)$ is an elementary abelian 2-group. Then it holds $MLO(G) \subseteq CSm(G) \subseteq LO(G)$. Furthermore if G is a gap group, it holds the equality $CSm(G) = LO(G)$.*

We will discuss in the case when G is a non-gap group in Theorem 6.1.

Proposition 6.2. *Let G be an Oliver non-gap group such that $[G : O^2(G)] = 2$. The following two claims are equivalent.*

(1) *$MLO(G) = LO(G)$.*

(2) *If two involutions x and y of G outside of $O^2(G)$ are not conjugate then $C_G(x)$ or $C_G(y)$ is a 2-group.*

The author does not know a group G with $MLO(G) \neq LO(G)$.

7. NON-TRIVIAL CENTRAL

In this section we consider whether $CSm(G) = 0$ or not for an Oliver group G with $a_G \geq 2$. In the section 5 we know completely it for a non-solvable group G. From now on we assume that G is an Oliver solvable group with $LO(G) \cap CSm(G) = 0$ and $a_G \geq 2$. Recall that $PO(G, G^{nil}) \neq 0$ implies $a_G \geq 2$.

Lemma 7.1. *If $Z(G) \neq \{1\}$ then $|\pi(G^{nil})| = 2$.*

Proof. Since $LO(G/G^{nil}) = 0$, G/G^{nil} is isomorphic to P, $C_2 \times P$, or $C_3 \times P_2$, where P is a p-group and P_2 is a 2-group. Then for some subgroup K of G, the sequence $G^{nil} \trianglelefteq K \trianglelefteq G$ such that $|\pi(G/K)| = 1$ and K/G^{nil} is cyclic. Thus $|\pi(G^{nil})| \geq 2$. We assume that $|\pi(G^{nil})| \geq 3$. Take distinct primes p, q, r in $\pi(G^{nil})$. Let $c \in Z(G)$ be an element of prime order. We may assume that $|c| \neq q, r$. Take elements x_q and x_r of G^{nil} of order q and r respectively. Then cx_q and cx_r are NPP elements of distinct order. Therefore (cx_q, cx_r) is a basic pair. □

Lemma 7.2. *$Z(G)$ has no NPP element.*

Proof. We suppose that $Z(G)$ has an NPP element c of order pq where p and q are primes. Then $|\pi(G)| = 2$ and $\pi(G) = \pi(\langle c\rangle) = \{p, q\}$ by Theorem 5.7. First we show that G^{nil} is not a subgroup of $\langle c\rangle$. Suppose $G^{nil} \leq \langle c\rangle$. Let $f\colon G \to G/\langle c\rangle$ be a canonical epimorphism. Note that $\pi(G/\langle c\rangle) = \{p, q\}$. Since $f(G)$ is nilpotent, $O^q(f(G))$ is a Sylow p-subgroup of $f(G)$ and a Sylow p-subgroup $O^q(G)_p$ of $O^q(G)$ is normal and its quotient $O^q(G)/O^q(G)_p$ is cyclic. This is a contrary against G is Oliver.

$$
\begin{array}{ccccc}
\langle c\rangle & \longrightarrow & G & \xrightarrow[f]{} & G/\langle c\rangle \\
\uparrow & & \uparrow & & \uparrow \\
\langle c\rangle \cap O^q(G) & \longrightarrow & O^q(G) & \longrightarrow & O^q(G/\langle c\rangle) \\
\uparrow & & \uparrow & & \uparrow = \\
\langle c\rangle \cap O^q(G)_p & \longrightarrow & O^q(G)_p & \longrightarrow & O^q(G/\langle c\rangle)
\end{array}
$$

Thus we can take an element x of G^{nil} which is not in $\langle c\rangle$. Since f sends two NPP elements xc and c to elements of distinct order, xc and c are not real conjugate. It is clear that they are sent to the same element by $G \to G/G^{nil}$. Then (xc, c) is a basic pair, which is a contrary. Thus $Z(G)$ has no NPP element. □

The following can be straightforward checked.

Lemma 7.3. *Let $c \in Z(G)$ be an element of order a prime p. If G^{nil} has an element x of order q^2 for some prime $q \neq p$, then G has a basic pair (cx, cx^q).*

We define the DressLength(G) as the minimal length n of sequences

$$G = G_0 > G_1 > G_2 > \cdots > G_n = \{1\}$$

such that $O^{p_j}(G_{j-1}) = G_j$ with some prime p_j for each j. In convenient, we assume DressLength(G) $= \infty$ if there is no sequence as above. For example, DressLength(G) $= \infty$ for a non-solvable group. It is easy to see that DressLength(G) ≥ 3 if G is an Oliver group and that DressLength(G) ≥ 3 if G is a gap group.

Now we recall classical results. A finite group is called a CP group if it has no NPP elements.

Lemma 7.4 (Higman, cf. [PSo, Lemma 2.5]). *Let H be a finite solvable CP group. Then one of the following conclusions holds:*

(1) *H is a p-group for some prime p; or*
(2) *$H = K \rtimes C$ is a Frobenius group with kernel K and complement C, where K is a p-group and C is a q-group of q-rank 1 for two distinct primes p and q; or*
(3) *$H = K \rtimes C \rtimes A$ is a 3-step group, in the sense that $K \rtimes C$ is a Frobenius group as in the conclusion (2) with C cyclic, and $C \rtimes A$ is a Frobenius group with kernel C and complement A, a cyclic p-group.*

Proposition 7.5 ([Hu, Proposition 22.3 and Remark on p.193]). Aut(C_{2^a}) $= C_2 \times C_{2^{a-2}}$ *where $x \mapsto x^5$ is a generator of $C_{2^{a-2}}$ and $x \mapsto x^{-1}$ is a generator of C_2.* Aut(C_{p^a}) $=$ *$C_{p^{a-1}(p-1)}$ for an odd prime p.*

With these results we use a Frattini subgroup and a Fitting subgroup and then we obtain the following results.

Theorem 7.6. *Let G be an Oliver solvable group with $a_G \geq 2$ and $Z(G) \neq \{1\}$. If $CSm(G) = 0$, then it holds the following.*

(1) *$Z(G)$ has no NPP element.*
(2) *If $Z(G)$ is a p-group, an element of G^{nil} not of p power order has prime order.*
(3) *$|\pi(G)| = 2$.*
(4) DressLength(G) $= 3, 4$.

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