<table>
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<th>Title</th>
<th>Dynamical properties of equivariant holomorphic maps (Geometry of Transformation Groups and Related Topics)</th>
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<tr>
<td>Author(s)</td>
<td>Ueno, Kohei</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2008), 1612: 14-20</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-09</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140083">http://hdl.handle.net/2433/140083</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Dynamical properties of equivariant holomorphic maps

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Abstract
This paper is a resume of [10]. We consider complex dynamics of a holomorphic map from $P^k$ to $P^k$, which is $S_{k+2}$-equivariant and critically finite, for each $k \geq 1$. Here $S_{k+2}$ is the $k+2$-th symmetric group. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1 Introduction

For a finite group $G$ acting on $P^k$ as projective transformations, we say that a rational map $f$ on $P^k$ is $G$-equivariant if $f$ commutes with each element of $G$. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. P. Doyle and C. McMullen [3] introduced the notion of equivariant maps on $P^1$ to solve quintic equations. See also [11] for equivariant maps on $P^1$. In the study of extending P. Doyle and C. McMullen’s result to higher dimensions, S. Crass [2] found a good family of finite groups and equivariant maps for which one may say something about global dynamics. S. Crass [2] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Our results [10] give affirmative answers for the conjectures in [2].

In section 2 we shall explain an action of the symmetric group $S_{k+2}$ on $P^k$ and properties of our $S_{k+2}$-equivariant map. In section 3 and 4 we shall denote our results about the Fatou sets and hyperbolicity of our maps. We need the properties of our maps and Kobayashi metrics for the proofs.
2 \(S_{k+2}\)-equivariant maps on \(P^k\)

S. Crass [2] selected the symmetric group \(S_{k+2}\) as a finite group acting on \(P^k\) and found an \(S_{k+2}\)-equivariant map which is holomorphic and critically finite for each \(k \geq 1\). We denote by \(C = C(f)\) the critical set of \(f\) and say that \(f\) is critically finite if each irreducible component of \(C(f)\) is periodic or preperiodic. More precisely, \(S_{k+2}\)-equivariant map \(g_{k+3}\) defined in section 2.2 preserves each irreducible component of \(C(g_{k+3})\), which is a projective hyperplane. The complement of \(C(g_{k+3})\) is Kobayashi hyperbolic. Furthermore restrictions of \(g_{k+3}\) to invariant projective subspaces have the same properties as above. See section 2.3 for details.

2.1 \(S_{k+2}\) acts on \(P^k\)

An action of the \((k+2)\)-th symmetric group \(S_{k+2}\) on \(P^k\) is induced by the permutation action of \(S_{k+2}\) on \(C^{k+2}\) for each \(k \geq 1\). The transposition \((i, j)\) in \(S_{k+2}\) corresponds with the transposition "\(u_i \leftrightarrow u_j\)" on \(C^{k+2}_u\), which pointwise fixes the hyperplane \(\{u_i = u_j\} = \{u \in C^{k+2}_u \mid u_i = u_j\}\). Here \(C^{k+2}_u = C^{k+2} = \{u = (u_1, u_2, \ldots, u_{k+2}) \mid u_i \in C\ \text{for}\ i = 1, \ldots, k+2\}\).

The action of \(S_{k+2}\) preserves a hyperplane \(H\) in \(C^{k+2}_u\), which is identified with \(C^{k+1}_x\) by projection \(A : C^{k+2}_u \rightarrow C^{k+1}_x\),

\[
H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \overset{A}{\cong} C^{k+1}_x \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.
\]

Here \(C^{k+1}_x = C^{k+1}_x = \{x = (x_1, x_2, \ldots, x_{k+1}) \mid x_i \in C\ \text{for}\ i = 1, \ldots, k+1\}\).

Thus the permutation action of \(S_{k+2}\) on \(C^{k+2}_u\) induces an action of "\(S_{k+2}\)" on \(C^{k+1}_x\). Here "\(S_{k+2}\)" is generated by the permutation action \(S_{k+1}\) on \(C^{k+1}_x\) and a \((k+1, k+1)\)-matrix \(T\) which corresponds to the transposition \((1, k+2)\) in \(S_{k+2}\),

\[
T = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \ldots & 1 \end{pmatrix}.
\]

Hence the hyperplane \(\{u_i = u_j\}\) corresponds to \(\{x_i = x_j\}\) for \(1 \leq i < j \leq k+1\). The hyperplane \(\{u_i = u_{k+2}\}\) corresponds to \(\{x_i = 0\}\) for \(1 \leq i \leq k+1\). Each element in "\(S_{k+2}\)" which corresponds to some transposition in \(S_{k+2}\) pointwise fixes one of these hyperplanes in \(C^{k+1}_x\).
The action of "$S_{k+2}$" on $\mathbb{C}^{k+1}$ projects naturally to the action of "$S_{k+2}$" on $\mathbb{P}^k$. These hyperplanes on $\mathbb{C}^{k+1}$ projects naturally to projective hyperplanes on $\mathbb{P}^k$. Here $\mathbb{P}^k = \{ x = [x_1 : x_2 : \cdots : x_{k+1}] \mid (x_1, x_2, \cdots, x_{k+1}) \in \mathbb{C}^{k+1} \setminus \{0\} \}. Each element in the action of "$S_{k+2}$" on $\mathbb{P}^k$ which corresponds to some transposition in $S_{k+2}$ pointwise fixes one of these projective hyperplanes. We denote "$S_{k+2}$" also by $S_{k+2}$ and call these projective hyperplanes transposition hyperplanes.

2.2 Existence of our maps

One way to get $S_{k+2}$-equivariant maps on $\mathbb{P}^k$ which are critically finite is to make $S_{k+2}$-equivariant maps whose critical sets coincide with the union of the transposition hyperplanes.

**Theorem 1 ([2]).** For each $k \geq 1$, $g_{k+3}$ defined below is the unique $S_{k+2}$-equivariant holomorphic map of degree $k + 3$ which is doubly critical on each transposition hyperplane.

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}] : \mathbb{P}^k \rightarrow \mathbb{P}^k,$$

where $g_{k+3,i}(x) = x_i^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_i^s A_{k-s}$, $A_0 = 1$,

and $A_{k-s}$ is the elementary symmetric function of degree $k-s$ in $\mathbb{C}^{k+1}$.

Then the critical set of $g$ coincides with the union of the transposition hyperplanes. Since $g$ is $S_{k+2}$-equivariant and each transposition hyperplane is pointwise fixed by some element in $S_{k+2}$, $g$ preserves each transposition hyperplane. In particular $g$ is critically finite. Although Crass [2] used this explicit formula to prove Theorem 1, we shall only use properties of the $S_{k+2}$-equivariant maps described below.

2.3 Properties of our maps

Let us look at properties of the $S_{k+2}$-equivariant map $g$ on $\mathbb{P}^k$ for a fixed $k$, which is proved in [2] and shall be used to prove our results. Let $L^{k-1}$ denote one of the transposition hyperplanes, which is isomorphic to $\mathbb{P}^{k-1}$. Let $L^m$ denote one of the intersections of $(k-m)$ or more distinct transposition hyperplanes which is isomorphic to $\mathbb{P}^m$ for $m = 0, 1, \cdots, k-1$.

First, let us look at properties of $g$ itself. The critical set of $g$ consists of the union of the transposition hyperplanes. By $S_{k+2}$-equivariance, $g$ preserves each transposition hyperplane. Furthermore the complement of the critical set of $g$ is Kobayashi hyperbolic.
Next, let us look at properties of $g$ restricted to $L^m$ for $m = 1, 2, \cdots, k - 1$. Let us fix any $m$. Since $g$ preserves each $L^m$, we can also consider the dynamics of $g$ restricted to any $L^m$. Each restricted map has the same properties as above. Let us fix any $L^m$ and denote by $g\big|_{L^m}$ the restricted map of $g$ to the $L^m$. The critical set of $g\big|_{L^m}$ consists of the union of intersections of the $L^m$ and another $L^{k-1}$ which does not include the $L^m$. We denote it by $L^{m-1}$, which is an irreducible component of the critical set of $g\big|_{L^m}$. By $S_{k+2}$-equivariance, $g\big|_{L^m}$ preserves each irreducible component of the critical set of $g\big|_{L^m}$. Furthermore the complement of the critical set of $g\big|_{L^m}$ in $L^m$ is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of $g$. The set of superattracting points, where the derivative of $g$ vanishes for all directions, coincides with the set of $L^0$'s.

**Remark 1.** For every $k \geq 1$ and every $m$, $1 \leq m \leq k$, a restricted map of $g_{k+3}$ to any $L^m$ is not conjugate to $g_{m+3}$.

## 3 The Fatou sets of the $S_{k+2}$-equivariant maps

Let us recall theorems about critically finite holomorphic maps. Let $f$ be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. The Fatou set of $f$ is defined to be the maximal open subset where the iterates $\{f^n\}_{n \geq 0}$ is a normal family. The Julia set of $f$ is defined to be the complement of the Fatou set of $f$. Each connected component of the Fatou set is called a Fatou component. Let $U$ be a Fatou component of $f$. A holomorphic map $h$ is said to be a limit map on $U$ if there is a subsequence $\{f^n\}_{s \geq 0}$ which locally converges to $h$ on $U$. We say that a point $q$ is a Fatou limit point if there is a limit map $h$ on a Fatou component $U$ such that $q \in h(U)$. The set of all Fatou limit points is called the Fatou limit set. We define the $\omega$-limit set $E(f)$ of the critical points by

$$E(f) = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} f^n(C).$$

**Theorem 2.** ([9, Proposition 5.1]) If $f$ is a critically finite holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, then the Fatou limit set is contained in the $\omega$-limit set $E(f)$.

Let us recall the notion of Kobayashi metrics. Let $M$ be a complex manifold and $K_M(x, v)$ the Kobayashi quasimetric on $M$,

$$\inf \left\{ |a| : \varphi : D \to M : \text{holomorphic}, \varphi(0) = x, D\varphi \left( a \left( \frac{\partial}{\partial z} \right) \right) = v, a \in \mathbb{C} \right\}$$
for $x \in M$, $v \in T_x M$, $z \in D$, where $D$ is the unit disk in $C$. We say that $M$ is Kobayashi hyperbolic if $K_M$ becomes a metric.

Let us recall theorems about dynamics of critically finite holomorphic maps in low dimensions. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for $k = 1$ and 2.

**Theorem 3.** ([7, Corollary 14.5]) If $f$ is a critically finite holomorphic map from $P^1$ to $P^1$, then the only Fatou components of $f$ are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $P^1$.

**Theorem 4.** ([4, Theorem 7.7]) If $f$ is a critically finite holomorphic map from $P^2$ to $P^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of $f$ are attractive components of superattracting points.

We get our first result by using Theorem 2, Kobayashi metrics and the properties of our maps.

**Theorem 5.** For each $k \geq 1$, the Fatou set of the $S_{k+2}$-equivariant map $g$ consists of attractive basins of superattracting fixed points which are intersections of $k$ or more distinct transposition hyperplanes.

## 4 The $S_{k+2}$-equivariant maps satisfy Axiom A

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [5] for details. Let $f$ be a holomorphic map from $P^k$ to $P^k$ and $K$ a compact subset such that $f(K) = K$. Let $\hat{K}$ be the set of histories in $K$ and $\hat{f}$ the induced homeomorphism on $\hat{K}$. We say that $f$ is hyperbolic on $K$ if there exists a continuous decomposition $T_{\hat{K}} = E^u + E^s$ of the tangent bundle such that $D\hat{f}(E^{u/s}_{\hat{x}}) \subset E^{u/s}_{\hat{f}(\hat{x})}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\hat{f}^n(v)| \geq c\lambda^n |v| \text{ for all } v \in E^u$$

and

$$|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v| \text{ for all } v \in E^s.$$ 

Here $|\cdot|$ denotes the Fubini-Study metric on $P^k$. If a decomposition and inequalities above hold for $f$ and $K$, then it also holds for $\hat{f}$ and $\hat{K}$. In particular we say that $f$ is expanding on $K$ if $f$ is hyperbolic on $K$ with unstable dimension $k$. Let $\Omega$ be the non-wandering set of $f$, i.e., the set of points for any neighborhood $U$ of which there exists an integer $n$ such that
$f''(U)$ intersects with $U$. By definition, $\Omega$ is compact and $f(\Omega) = \Omega$. We say that $f$ satisfies Axiom A if $f$ is hyperbolic on $\Omega$ and periodic points are dense in $\Omega$.

Let us introduce a theorem which deals with repelling part of dynamics. Let $f$ be a holomorphic map from $P^k$ to $P^k$. We define the $k$-th Julia set $J_k$ of $f$ to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set $J_1$ coincides with the Julia set $J$. Let $K$ be a compact subset such that $f(K) = K$. We say that $K$ is a repeller if $f$ is expanding on $K$.

**Theorem 6.** ([6]) Let $f$ be a holomorphic map on $P^k$ of degree at least 2 such that the $\omega$-limit set $E(f)$ is pluripolar. Then any repeller for $f$ is contained in $J_k$. In particular,

$$J_k = \{\text{repelling periodic points of } f\}$$

If $f$ is critically finite, then $E(f)$ is pluripolar. Hence our maps satisfies the condition in the theorem above.

We get our second result by using Theorem 3, Kobayashi metrics and the properties of our maps.

**Theorem 7.** For each $k \geq 1$, the $S_{k+2}$-equivariant map $g$ satisfies Axiom A.

Since $g$ satisfies Axiom A, [1, Theorem 4.11] and [8] induces the following corollary.

**Corollary 1.** The Fatou set of the $S_{k+2}$-equivariant map $g$ has full measure in $P^k$ for each $k \geq 1$.

**Acknowledgments.** I would like to thank Professor S. Ushiki and Doctor K. Maegawa for their useful advice. Particularly in order to obtain our second result, K. Maegawa's suggestion to use Theorem 6 was helpful.

**References**


