A generalization of Hardy spaces on spaces of homogeneous type

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1. INTRODUCTION

This is an announcement of my recent work [10].

Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [1, 2] (see the next section for the definition). Using atoms, Coifman and Weiss [2] introduced the Hardy space $H^p(X)$. The purpose of this report is to generalize the definition of Hardy space $H^p(X)$ and prove that the generalized Hardy spaces have the same property as $H^p(X)$. Our definition includes a kind of Hardy spaces with variable exponent. The results are new even for the $\mathbb{R}^n$ case.

First we state definitions of Campanato and Hölder spaces. Let $1 \leq p < \infty$ and $\phi: X \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{loc}(X)$ and for a ball $B$, let $f_B = \mu(B)^{-1} \int_B f(x) \, d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p, \phi}(X)$ and the Hölder spaces $\Lambda_{\phi}(X)$ are defined to be the sets of all $f$ such that $\|f\|_{\mathcal{L}_{p, \phi}} < \infty$ and $\|f\|_{\Lambda_{\phi}} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p, \phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_{\phi}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, d(x, y)) + \phi(y, d(y, x))}.$$

Let $C$ be the space of all constant functions. Then $\mathcal{L}_{p, \phi}(X)/C$ and $\Lambda_{\phi}(X)/C$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p, \phi}}$ and $\|f\|_{\Lambda_{\phi}}$, respectively. Campanato spaces of these type were studied in [11, 7, 8, 12, 9]. See [9] for relations among these spaces. When $p = 1$, we denote $\mathcal{L}_{1, \phi}(X)$ by $\text{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1, \phi}(X) = \text{BMO}(X)$.

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For $\phi(x, r) = r^{\alpha(x)}$, $\alpha(x) > 0$, we denote $\Lambda_{\phi}(X)$ by $\text{Lip}_{\alpha(\cdot)}(X)$. Then

$$\|f\|_{\text{Lip}_{\alpha(\cdot)}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{d(x, y)^{\alpha(x)} + d(y, x)^{\alpha(y)}}.$$ 

If $\alpha(\cdot)$ satisfies a certain condition, then $\text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q', \phi}(X)$ for all $p \in [1, \infty)$. Using atoms, Coifman and Weiss [2] defined the Hardy space $H^p(X)$ as a subspace of the dual of $\text{Lip}_{\alpha}(X)$ and they proved that $\text{Lip}_{\alpha}(X)$ is the dual of $H^p(X)$. Their results are generalization of the case $X = \mathbb{R}^n$. In [2] $\text{Lip}_{\alpha}(X)$ was regarded as the space of functions modulo constants. Therefore, we denote by $(H^p(X))^* = \text{Lip}_{\alpha(X)}/C$ the fact above.

In this report, using $[\phi, q]$-atoms, we define a generalized Hardy space $H^p_{U}(X)$ as a subspace of the dual of $\mathcal{L}_{q', \phi}(X)/C$ and prove that $\mathcal{L}_{q', \phi}(X)/C$ is the dual of $H^p_{U}(X)$, i.e. $H^p_{U}(X)^* = \mathcal{L}_{q', \phi}(X)/C$, where $1 < q \leq \infty$, $1/q + 1/q' = 1$, $U$ is a concave strictly increasing function from $[0, \infty)$ to itself and $U(0) = 0$ (see the third section for the precise definition of $H^p_{U}(X)$). The definition of $H^p(X)$ in [2], $0 < p \leq 1$, is a special case of ours, since $\text{Lip}_{\alpha}(X)$ is a special case of $\mathcal{L}_{q', \phi}(X)$.

Coifman and Weiss [2] first defined $H^{p,q}(X)$, and then proved $H^{p,q}(X) = H^{p,\infty}(X)$, which was denoted by $H^p(X)$. We will prove that $H^p_{U}(X) = H^{p,\infty}_{U}(X)$ under a certain condition. In particular, for Hardy spaces with variable exponent $p(x)$, we use the condition that $p(x)$ is log-Hölder continuous (see Corollary 4.2).

The log-Hölder continuity was used to prove boundedness of the Hardy-Littlewood maximal operator on $L^{p(x)}$, Lebesgue spaces with variable exponent, as follows.

Let $G \subset \mathbb{R}^n$ be bounded. For a function $p : G \rightarrow [1, \infty)$, let

$$L^{p(x)}(G) = \left\{ f \in L^1(G) : \int_G (c |f(x)|)^{p(x)} \, dx < \infty \text{ for some } c > 0 \right\}.$$

For $f \in L^{p(x)}(G)$, let

$$\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_G \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.$$

Then $\| \cdot \|_{p(x)}$ is a norm and thereby $L^{p(x)}(G)$ is a Banach space. For a function $f$ on $G$, the Hardy-Littlewood maximal function of $f$ is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap G} |f(y)| \, dy,$$

where the supremum is taken over all balls $B$ containing $x$. By the definition we have

$$\|Mf\|_{\infty} \leq \|f\|_{\infty}.$$
We say that $p(x)$ is log-Hölder continuous if

$$|p(x) - p(y)| \leq \frac{c}{|\log |x - y||} \quad \text{for} \quad |x - y| \leq \frac{1}{2}.$$ 

**Theorem 1.1** (Diening [3]). If $p(x)$ is log-Hölder continuous, then the operator $M$ is bounded on $L^{p(x)}(G)$.

**Remark 1.1.** Let

$$p(x) = \begin{cases} 
4 & (-1 < x \leq 0) \\
2 & (0 < x < 1).
\end{cases}$$

If $f(x) = \begin{cases} 0 & (-1 < x \leq 0) \\
x^{-1/3} & (0 < x < 1),
\end{cases}$ then $Mf(x) \geq c|x|^{-1/3}$. In this case $f \in U^{(x)}(-1,1)x^{-1/3} (0 < x < 1)$, and $Mf \not\in U^{(x)}(-1,1)$.

\section*{2. Space of homogeneous type}

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. $X$ is a topological space endowed with a quasi-distance $d$ and a nonnegative measure $\mu$ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x),$$

\begin{equation}
(2.1) \quad d(x, y) \leq K_{1}(d(x, z) + d(z, y)),
\end{equation}

the balls ($d$-balls) $B(x, r) = B^{d}(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point $x$, $\mu$ is defined on a $\sigma$-algebra of subsets of $X$ which contains the balls, and

\begin{equation}
(2.2) \quad 0 < \mu(B(x, 2r)) \leq K_{2}\mu(B(x, r)) < \infty,
\end{equation}

If there are constants $\theta$ ($0 < \theta \leq 1$) and $K_{3} \geq 1$ such that

\begin{equation}
(2.3) \quad |d(x, z) - d(y, z)| \leq K_{3}(d(x, z) + d(y, z))^{1-\theta}d(x, y)^{\theta}, \quad x, y, z \in X,
\end{equation}

then the balls are open sets. Note that (2.1) for some $K_{1} \geq 1$ follows from (2.3) (Lemarié [4]). Conversely, from (2.1) it follows that there exist $\theta > 0$, $K_{3} \geq 1$ and a quasi-distance which is equivalent to the original $d$ such that (2.3) holds (Macías and Segovia [5]). Therefore we always assume (2.3) in this report.

It is known that, if $\mu(X) < +\infty$, then there is a constant $R_{0} > 0$ such that

\begin{equation}
(2.4) \quad X = B(x, R_{0}) \quad \text{for all } x \in X
\end{equation}

(see [12, Lemma 5.1]).
3. Definitions

Definition 3.1 ([φ, q]-atom (resp. (p(·), q)-atom)). Let φ : X × (0, ∞) → (0, ∞) and 1 < q ≤ ∞. A function a on X is called a [φ, q]-atom (resp. (p(·), q)-atom) if there exists a ball B such that

(i) \( \text{supp } a \subset B \),
(ii) \( \|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)} \)

(resp. \( \|a\|_q \leq \mu(B)^{1/q} \frac{1}{1/p(x)} \), where \( x \) is the center of \( B \)),

(iii) \( \int_X a(x) \, d\mu(x) = 0 \),

where \( \|a\|_q \) is the \( L^q \) norm of \( a \) and \( 1/q + 1/q' = 1 \). We denote by \( A[\phi, q] \) the set of all [φ, q]-atoms. (We denote by \( A(p(\cdot), q) \) the set of all (p(·), q)-atoms.)

We note that (p(·), q)-atoms are special cases of [φ, q]-atoms. If \( p(x) \equiv p \), then the (p(·), q)-atom is the usual (p, q)-atom. Let \( p_- = \inf p(x) \) and \( p_+ = \sup p(x) \).

Remark 3.1. Assume that \( \mu(B(x, r)) \sim r^Q \) \( (Q > 0) \) for \( x \in X \) and \( 0 < r < \infty \) \( (0 < r < R_0 \text{ if } \mu(X) < \infty) \). Let \( \alpha(x) = Q(1/p(x) - 1) \). If \( Q/(\theta + Q) \leq p_- \leq p_+ < 1 \), then \( 0 < \alpha_- \leq \alpha_+ \leq \theta \) and \( \text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q',\phi}(X) \) for all \( q' \in [1, \infty) \).

If \( a \) is a [φ, q]-atom and a ball \( B \) satisfies (i)–(iii), then

\[
\left| \int_X a(x)g(x) \, d\mu(x) \right| = \left| \int_B a(x)(g(x) - g_B) \, d\mu(x) \right| \\
\leq \|a\|_q \left( \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'} \\
\leq \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'} \\
\leq \|g\|_{\mathcal{L}_{q',\phi}}.
\]

That is, the mapping \( g \mapsto \int_X ag \, d\mu \) is a bounded linear functional on \( \mathcal{L}_{q',\phi}(X)/C \) with norm not exceeding 1.

Definition 3.2 (\( H^q_U[\phi,q](X) \)). Let \( \phi : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), \( 1 < q \leq \infty \) and \( 1/q + 1/q' = 1 \). Let \( U \) be a continuous, concave, increasing and bijective function from \( [0, +\infty) \) to itself. Assume that \( \mathcal{L}_{q',\phi}(X)/C \neq \{0\} \). We define the space \( H^q_U[\phi,q](X) \subset (\mathcal{L}_{q',\phi}(X)/C)^* \) as follows:
$f \in H_{U}^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_{j}\} \subset A[\phi,q]$ and positive numbers $\{\lambda_{j}\}$ such that

$$f = \sum_{j} \lambda_{j} a_{j} \text{ in } (L_{q',\phi}(X)/C)^{\ast} \text{ and } \sum_{j} U(\lambda_{j}) < \infty. \quad (3.2)$$

From $U(0) = 0$ and the concavity of $U$ it follows that

$$U(Cr) \leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty, \quad (3.3)$$

$$U(r+s) \leq U(r) + U(s), \quad 0 \leq r, s < \infty. \quad (3.4)$$

Then $H_{U}^{[\phi,q]}(X)$ is a linear space. (3.4) implies

$$\sum_{j} \lambda_{j} \leq U^{-1} \left( \sum_{j} U(\lambda_{j}) \right). \quad (3.5)$$

Therefore, if $\sum_{j} U(\lambda_{j}) < \infty$, then $\sum_{j} \lambda_{j} < \infty$ and $\sum_{j} \lambda_{j} a_{j}$ converges in $(L_{q',\phi}(X)/C)^{\ast}$.

In general, the expression (3.2) is not unique. We define

$$\|f\|_{H_{U}^{[\phi,q]}} = \inf \left\{ U^{-1} \left( \sum_{j} U(\lambda_{j}) \right) \right\},$$

where the infimum is taken over all expressions as in (3.2). We note that $\|f\|_{H_{U}^{[\phi,q]}}$ is not a norm in general. Let $d(f, g) = U(\|f - g\|_{H_{U}^{[\phi,q]}})$ for $f, g \in H_{U}^{[\phi,q]}(X)$. Then $d(f, g)$ is a metric and $H_{U}^{[\phi,q]}(X)$ is complete with respect to this metric. If $I(r) = r$, then $\|f\|_{H_{U}^{[\phi,q]}}$ is a norm and $H_{U}^{[\phi,q]}(X)$ is a Banach space.

In the case of $(p(\cdot), q)$-atoms instead of $[\phi,q]$-atoms, we denote $H_{U}^{[\phi,q]}(X)$ by $H_{U}^{p(\cdot),q}(X)$.

4. RESULTS

Theorem 4.1. If there exists a constant $C_{*} > 0$ such that

$$U(rs) \leq C_{*} U(r) U(s) \text{ for } 0 < r, s \leq 1, \quad (4.1)$$

$$U \left( \frac{\mu(B_{1}) \phi(B_{1})}{\mu(B_{2}) \phi(B_{2})} \right) \leq C_{*} \frac{\mu(B_{1})}{\mu(B_{2})} \text{ for all balls } B_{1} \text{ and } B_{2} \text{ with } B_{1} \subset B_{2}, \quad (4.2)$$

then

$$H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X),$$

with equivalent topologies.
Corollary 4.2. Let $Q > 0$. Assume that $\mu(X) < \infty$ and that $\mu(B(x,r)) \sim r^Q$ for all $x \in X$ and $0 < r < R_0$, where $R_0$ is the constant in (2.4). Let $U(r) = r^{p+}$ with $0 < p_- \leq p_+ \leq 1$, where $p_- = \inf p(x)$ and $p_+ = \sup p(x)$. If there exists a constant $C_0 > 0$ such that

\begin{equation}
|p(x) - p(y)| \leq \frac{C_0}{\log(1/d(x,y))} \quad \text{for} \quad d(x,y) < 1/2,
\end{equation}

then

$$H^{p(\cdot), q}_{U}(X) = H^{p(\cdot), \infty}_{U}(X),$$

with equivalent topologies.

In this case we denote $H^{p(\cdot), q}_{U}(X)$ by $H^{p(\cdot)}(X)$ simply, which is a kind of Hardy spaces with variable exponent.

Proof of Corollary 4.2. The inequality (4.1) holds clearly. We show (4.2).

For $B(x, r) \subset B(y, s)$,

$$\frac{U\left(\frac{\mu(B(x,r))}{\mu(B(y,s))}\right)}{\frac{\mu(B(x,r))}{\mu(B(y,s))}} \sim \left(\frac{r}{s}\right)^{Qp_+ + \frac{1}{p(y)} - \frac{1}{p(x)}} \frac{s^{Qp_+ + \frac{1}{p(y)} - \frac{1}{p(x)}}}{s^{Qp_+ + \frac{1}{p(y)} - \frac{1}{p(x)}}},$$

since $r/s \leq 1$. If $1/2 < s < R_0$, then

$$s^{Qp_+ + \frac{1}{p(y)} - \frac{1}{p(x)}} \leq R_0^{Qp_+/p_-}.$$ 

If $s \leq 1/2$, then $d(x,y) < s$ and

$$\log s^{Qp_+ + \frac{1}{p(y)} - \frac{1}{p(x)}} \leq Qp_+ \left(\frac{1}{p(y)} - \frac{1}{p(x)}\right) \log(1/s) \leq Qp_+ \left\|p(x) - p(y)\right\| \log(1/d(x,y)) \leq C_0 Qp_+ / p_-^2. \quad \square$$

Lemma 4.3. Let $E = H^{p(\cdot), q}_{U}(X)$. If

\begin{equation}
\sup_{0 < s \leq 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0),
\end{equation}

then

$$\|\ell\|_{E^*} = \sup \{ |\ell(f)| : \|f\|_E \leq 1 \}$$

is finite for all $\ell \in E^*$, and $\|\ell\|_{E^*}$ is a norm.

Remark 4.1. If (4.1) holds, then (4.4) holds. If (4.4) holds, then there exist constants $C > 0$ and $p > 0$ such that $U(r) \leq Cr^p$ for $r \in (0, 1]$. If $\alpha > 0$ and $U(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$, then $U$ does not satisfy (4.4).
Let $L^q_c(X)$ be the set of all $L^q$-functions with bounded support, and let
\[ L^q_c(X) = \left\{ f \in L^q(X) : \int_X f \, d\mu = 0 \right\}. \]
Then, for $1 < q \leq \infty$, $L^q_c(X)$ is dense in $H^{[\phi,q]}(X)$.

**Theorem 4.4.** If $U$ satisfies (4.4), then
\[ \left( H^{[\phi,q]}(X) \right)^* = L^{q',\phi}(X)/C. \]
More precisely, if $g \in L^{q',\phi}(X)/C$, then the mapping $\ell : f \mapsto \int_X f(g+c) \, d\mu$ for $f \in L^q_c(X)$, can be extended to a continuous linear functional on $H^{[\phi,q]}(X)$. Conversely, if $\ell$ is a continuous linear functional on $H^{[\phi,q]}(X)$, then there exists $g \in L^{q',\phi}(X)/C$ such that $\ell(f) = \int_X f(g+c) \, d\mu$ for $f \in L^q_c(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{L^{q',\phi}}$.

**Corollary 4.5.** Assume the conditions in Remark 3.1 and Corollary 4.2. Then
\[ (H^{p(\cdot)}(X))^* = \text{Lip}_{\alpha(\cdot)}(X)/C. \]

**References**


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