A generalization of Hardy spaces on spaces of homogeneous type

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1. INTRODUCTION

This is an announcement of my recent work [10].

Let $X = (X, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss [1, 2] (see the next section for the definition). Using atoms, Coifman and Weiss [2] introduced the Hardy space $H^p(X)$. The purpose of this report is to generalize the definition of Hardy space $H^p(X)$ and prove that the generalized Hardy spaces have the same property as $H^p(X)$. Our definition includes a kind of Hardy spaces with variable exponent. The results are new even for the $\mathbb{R}^n$ case.

First we state definitions of Campanato and Hölder spaces. Let $1 \leq p < \infty$ and $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{\text{loc}}(X)$ and for a ball $B$, let $f_B = \mu(B)^{-1} \int_B f(x) \, d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p, \phi}(X)$ and the Hölder spaces $\Lambda_{\phi}(X)$ are defined to be the sets of all $f$ such that $\|f\|_{\mathcal{L}_{p, \phi}} < \infty$ and $\|f\|_{\Lambda_{\phi}} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p, \phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_{\phi}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, d(x, y)) + \phi(y, d(y, x))}.$$

Let $C$ be the space of all constant functions. Then $\mathcal{L}_{p, \phi}(X)/C$ and $\Lambda_{\phi}(X)/C$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p, \phi}}$ and $\|f\|_{\Lambda_{\phi}}$, respectively. Campanato spaces of these type were studied in [11, 7, 8, 12, 9]. See [9] for relations among these spaces. When $p = 1$, we denote $\mathcal{L}_{1, \phi}(X)$ by $\text{BMO}_{\phi}(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1, \phi}(X) = \text{BMO}(X)$.

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For $\phi(x, r) = r^{\alpha(x)}$, $\alpha(x) > 0$, we denote $\Lambda_{\phi}(X)$ by $\text{Lip}_{\alpha(\cdot)}(X)$. Then

$$\|f\|_{\text{Lip}_{\alpha(\cdot)}} = \sup_{x, y \in X, x \neq y} \frac{2|f(x) - f(y)|}{d(x, y)^{\alpha(x)} + d(y, x)^{\alpha(y)}}.$$  

If $\alpha(\cdot)$ satisfies a certain condition, then $\text{Lip}_{\alpha(\cdot)}(X) = L_{p, \phi}(X)$ for all $p \in [1, \infty)$. Using atoms, Coifman and Weiss [2] defined the Hardy space $H^{p}(X)$ as a subspace of the dual of $\text{Lip}_{\alpha}(X)$ and they proved that $\text{Lip}_{\alpha}(X)$ is the dual of $H^{p}(X)$. Their results are generalization of the case $X = \mathbb{R}^{n}$. In [2] $\text{Lip}_{\alpha}(X)$ was regarded as the space of functions modulo constants. Therefore, we denote by $(H^{p}(X))^{*} = \text{Lip}_{\alpha}(X)/C$ the fact above.

In this report, using $[\phi, q]$-atoms, we define a generalized Hardy space $H_{U}^{\phi,q}(X)$ as a subspace of the dual of $L_{q', \phi}(X)/C$ and we prove that $L_{q', \phi}(X)/C$ is the dual of $H_{U}^{\phi,q}(X)$, i.e. $H_{U}^{\phi,q}(X)^{*} = L_{q', \phi}(X)/C$, where $1 < q \leq \infty$, $1/q + 1/q' = 1$, $U$ is a concave strictly increasing function from $[0, \infty)$ to itself and $U(0) = 0$ (see the third section for the precise definition of $H_{U}^{\phi,q}(X)$). The definition of $H^{p}(X)$ in [2] is a special case of ours, since $\text{Lip}_{\alpha}(X)$ is a special case of $L_{q', \phi}(X)$.

Coifman and Weiss [2] first defined $H^{p,q}(X)$, and then proved $H^{p,q}(X) = H^{p,\infty}(X)$, which was denoted by $H^{p}(X)$. We will prove that $H_{U}^{\phi,q}(X) = H_{U}^{\phi,\infty}(X)$ under a certain condition. In particular, for Hardy spaces with variable exponent $p(x)$, we use the condition that $p(x)$ is log-H"older continuous (see Corollary 4.2).

The log-H"older continuity was used to prove boundedness of the Hardy-Littlewood maximal operator on $L^{p(x)}$, Lebesgue spaces with variable exponent, as follows.

Let $G \subset \mathbb{R}^{n}$ be bounded. For a function $p : G \rightarrow [1, \infty)$, let

$$L^{p(x)}(G) = \left\{ f \in L^{1}(G) : \int_{G} (c |f(x)|)^{p(x)} dx < \infty \text{ for some } c > 0 \right\}.$$  

For $f \in L^{p(x)}(G)$, let

$$\|f\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{G} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$  

Then $\| \cdot \|_{p(x)}$ is a norm and thereby $L^{p(x)}(G)$ is a Banach space. For a function $f$ on $G$, the Hardy-Littlewood maximal function of $f$ is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B \cap G} |f(y)| dy,$$

where the supremum is taken over all balls $B$ containing $x$. By the definition we have

$$\|Mf\|_{\infty} \leq \|f\|_{\infty}.$$
We say that \( p(x) \) is log-Hölder continuous if
\[
|p(x) - p(y)| \leq \frac{c}{|\log |x - y||} \quad \text{for } |x - y| \leq \frac{1}{2}.
\]

**Theorem 1.1** (Diening [3]). If \( p(x) \) is log-Hölder continuous, then the operator \( M \) is bounded on \( L^{p(x)}(G) \).

**Remark 1.1.** Let
\[
p(x) = \begin{cases} 
4 & (-1 < x \leq 0) \\
2 & (0 < x < 1).
\end{cases}
\]
If \( f(x) = \begin{cases} 
0 & (-1 < x \leq 0) \\
x^{-1/3} & (0 < x < 1),
\end{cases} \) then \( Mf(x) \geq c|x|^{-1/3} \). In this case \( f \in U^{(x)}(-1,1)\times^{-1/3} (0 < x < 1) \), and \( Mf \notin U^{(x)}(-1,1) \).

## 2. Space of homogeneous type

Let \( X = (X, d, \mu) \) be a space of homogeneous type, i.e. \( X \) is a topological space endowed with a quasi-distance \( d \) and a nonnegative measure \( \mu \) such that
\[
d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,
\]
\[
d(x, y) = d(y, x),
\]
(2.1)
\[
d(x, y) \leq K_1 (d(x, z) + d(z, y)),
\]
the balls (\( d \)-balls) \( B(x, r) = B^d(x, r) = \{ y \in X : d(x, y) < r \} \), \( r > 0 \), form a basis of neighborhoods of the point \( x \), \( \mu \) is defined on a \( \sigma \)-algebra of subsets of \( X \) which contains the balls, and
(2.2)
\[
0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,
\]
If there are constants \( \theta \) (\( 0 < \theta \leq 1 \)) and \( K_3 \geq 1 \) such that
(2.3)
\[
|d(x, z) - d(y, z)| \leq K_3 (d(x, z) + d(y, z))^{1-\theta}d(x, y)^{\theta}, \quad x, y, z \in X,
\]
then the balls are open sets. Note that (2.1) for some \( K_1 \geq 1 \) follows from (2.3) (Lemarié [4]). Conversely, from (2.1) it follows that there exist \( \theta > 0 \), \( K_3 \geq 1 \) and a quasi-distance which is equivalent to the original \( d \) such that (2.3) holds (Macías and Segovia [5]). Therefore We always assume (2.3) in this report.

It is known that, if \( \mu(X) < +\infty \), then there is a constant \( R_0 > 0 \) such that
(2.4)
\[
X = B(x, R_0) \quad \text{for all } x \in X
\]
(see [12, Lemma 5.1]).
3. Definitions

**Definition 3.1** ([φ, q]-atom (resp. (p(·), q)-atom)). Let $\phi : X \times (0, \infty) \to (0, \infty)$ and $1 < q \leq \infty$. A function $a$ on $X$ is called a [φ, q]-atom (resp. (p(·), q)-atom) if there exists a ball $B$ such that

1. $\text{supp} \, a \subset B$,
2. $\|a\|_q \leq \frac{1}{\mu(B)^{1/q'} \phi(B)}$

(resp. $\|a\|_q \leq \mu(B)^{1/q - 1/p(x)}$, where $x$ is the center of $B$),
3. $\int_X a(x) \, d\mu(x) = 0$,

where $\|a\|_q$ is the $L^q$ norm of $a$ and $1/q + 1/q' = 1$. We denote by $A[\phi, q]$ the set of all [φ, q]-atoms. (We denote by $A(p(\cdot), q)$ the set of all $(p(\cdot), q)$-atoms.)

We note that $(p(\cdot), q)$-atoms are special cases of [φ, q]-atoms. If $p(x) \equiv p$, then the $(p(\cdot), q)$-atom is the usual $(p, q)$-atom. Let $p_-= \inf p(x)$ and $p_+ = \sup p(x)$.

**Remark 3.1.** Assume that $\mu(B(x, r)) \sim r^Q$ ($Q > 0$) for $x \in X$ and $0 < r < \infty$ ($0 < r < R_0$ if $\mu(X) < \infty$). Let $\alpha(x) = Q(1/p(x) - 1)$. If $Q/(\theta + Q) \leq p_- \leq p_+ < 1$, then $0 < \alpha_- \leq \alpha_+ \leq \theta$ and $\text{Lip}_{\alpha(\cdot)}(X) = \mathcal{L}_{q', \phi}(X)$ for all $q' \in [1, \infty)$.

If $a$ is a [φ, q]-atom and a ball $B$ satisfies (i)-(iii), then

$$| \int_X a(x)g(x) \, d\mu(x) | = \left| \int_B a(x)(g(x) - g_B) \, d\mu(x) \right|$$

$$\leq \|a\|_q \left( \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'}$$

$$\leq \frac{1}{\phi(B)} \left( \frac{1}{\mu(B)} \int_B |g(x) - g_B|^{q'} \, d\mu(x) \right)^{1/q'}$$

$$\leq \|g\|_{\mathcal{L}_{q', \phi}}.$$

That is, the mapping $g \mapsto \int_X a g \, d\mu$ is a bounded linear functional on $\mathcal{L}_{q', \phi}(X)/C$ with norm not exceeding 1.

**Definition 3.2** ($H_U^{[\phi,q]}(X)$). Let $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$, $1 < q \leq \infty$ and $1/q + 1/q' = 1$. Let $U$ be a continuous, concave, increasing and bijective function from $[0, +\infty)$ to itself. Assume that $\mathcal{L}_{q', \phi}(X)/C \neq \{0\}$. We define the space $H_U^{[\phi,q]}(X) \subset (\mathcal{L}_{q', \phi}(X)/C)^*$ as follows:
$f \in H_{U}^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_{j}\} \subset A[\phi,q]$ and positive numbers $\{\lambda_{j}\}$ such that

$$f = \sum_{j} \lambda_{j}a_{j} \text{ in } (\mathcal{L}_{q',\phi}(X)/C)^{*} \quad \text{and} \quad \sum_{j} U(\lambda_{j}) < \infty.$$  

(3.2)

From $U(0) = 0$ and the concavity of $U$ it follows that

$$U(Cr) \leq CU(r), \quad 1 \leq C < \infty, \quad 0 \leq r < \infty,$$

(3.3)

$$U(r + s) \leq U(r) + U(s), \quad 0 \leq r, s < \infty.$$  

(3.4)

Then $H_{U}^{[\phi,q]}(X)$ is a linear space. (3.4) implies

$$\sum_{j} \lambda_{j} \leq U^{-1}\left(\sum_{j} U(\lambda_{j})\right).$$

(3.5)

Therefore, if $\sum_{j} U(\lambda_{j}) < \infty$, then $\sum_{j} \lambda_{j} < \infty$ and $\sum_{j} \lambda_{j}a_{j}$ converges in $(\mathcal{L}_{q',\phi}(X)/C)^{*}$. In general, the expression (3.2) is not unique. We define

$$\|f\|_{H_{U}^{[\phi,q]}} = \inf \left\{ U^{-1}\left(\sum_{j} U(\lambda_{j})\right) \right\},$$

where the infimum is taken over all expressions as in (3.2). We note that $\|f\|_{H_{U}^{[\phi,q]}}$ is not a norm in general. Let $d(f, g) = U(\|f - g\|_{H_{U}^{[\phi,q]}})$ for $f, g \in H_{U}^{[\phi,q]}(X)$. Then $d(f, g)$ is a metric and $H_{U}^{[\phi,q]}(X)$ is complete with respect to this metric. If $I(r) = r$, then $\|f\|_{H_{I}^{[\phi,q]}}$ is a norm and $H_{I}^{[\phi,q]}(X)$ is a Banach space.

In the case of $(p(\cdot), q)$-atoms instead of $[\phi,q]$-atoms, we denote $H_{U}^{[\phi,q]}(X)$ by $H_{U}^{p(\cdot),q}(X)$.

4. RESULTS

**Theorem 4.1.** If there exists a constant $C_{*} > 0$ such that

$$U(rs) \leq C_{*}U(r)U(s) \quad \text{for} \quad 0 < r, s \leq 1,$$

(4.1)

$$U\left(\frac{\mu(B_{1})\phi(B_{1})}{\mu(B_{2})\phi(B_{2})}\right) \leq C_{*}\frac{\mu(B_{1})}{\mu(B_{2})} \quad \text{for all balls } B_{1} \text{ and } B_{2} \text{ with } B_{1} \subset B_{2},$$

(4.2)

then

$$H_{U}^{[\phi,q]}(X) = H_{U}^{[\phi,\infty]}(X),$$

with equivalent topologies.
**Corollary 4.2.** Let $Q > 0$. Assume that $\mu(X) < \infty$ and that $\mu(B(x,r)) \sim r^Q$ for all $x \in X$ and $0 < r < R_0$, where $R_0$ is the constant in (2.4). Let $U(r) = r^{p+}$ with $0 < p_- \leq p_+ \leq 1$, where $p_- = \inf p(x)$ and $p_+ = \sup p(x)$. If there exists a constant $C_0 > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_0}{\log(1/d(x,y))} \quad \text{for} \quad d(x,y) < 1/2,$$

then

$$H^{p(\cdot),q}(X) = H^{p(\cdot),\infty}(X),$$

with equivalent topologies.

In this case we denote $H^{p(\cdot),q}(X)$ by $H^{p(\cdot)}(X)$ simply, which is a kind of Hardy spaces with variable exponent.

**Proof of Corollary 4.2.** The inequality (4.1) holds clearly. We show (4.2).

For $B(x,r) \subset B(y,s)$,

$$\frac{U(s)}{U(r)} \sim \left(\frac{r}{s}\right)^{Qp_+(1/p(x)-1/p+)} s^{Qp_+(1/p(x)-1/p(y))} \leq s^{Qp_+(1/p(x)-1/p(y))},$$

since $r/s \leq 1$. If $1/2 < s < R_0$, then

$$s^{Qp_+(1/p(x)-1/p(y))} \leq R_0^{Qp_+/p_-}.$$

If $s \leq 1/2$, then $d(x,y) < s$ and

$$\log s^{Qp_+(1/p(x)-1/p(y))} \leq Qp_+ \left| \frac{1}{p(y)} - \frac{1}{p(x)} \right| \log(1/s)$$

$$\leq Qp_+ \left| \frac{p(x) - p(y)}{p(x)p(y)} \right| \log(1/d(x,y)) \leq \frac{C_0Qp_+}{p_-^2}. \quad \square$$

**Lemma 4.3.** Let $E = H^{[\phi,q]}(X)$. If

$$\sup_{0<s \leq 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0),$$

then

$$\|\ell\|_{E^*} = \sup \{ |\ell(f)| : \|f\|_E \leq 1 \}$$

is finite for all $\ell \in E^*$, and $\|\ell\|_{E^*}$ is a norm.

**Remark 4.1.** If (4.1) holds, then (4.4) holds. If (4.4) holds, then there exist constants $C > 0$ and $p > 0$ such that $U(r) \leq Cr^p$ for $r \in (0,1]$. If $\alpha > 0$ and $U(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$, then $U$ does not satisfy (4.4).
Let $L_{c}^{q}(X)$ be the set of all $L^{q}$-functions with bounded support, and let

$$L_{c}^{q,0}(X) = \left\{ f \in L_{c}^{q}(X) : \int_{X} f \, d\mu = 0 \right\}.$$ 

Then, for $1 < q \leq \infty$, $L_{c}^{q,0}(X)$ is dense in $H_{U}^{[\phi,q]}(X)$.

**Theorem 4.4.** If $U$ satisfies (4.4), then

$$\left( H_{U}^{[\phi,q]}(X) \right)^{*} = \mathcal{L}_{q',\phi}(X)/C.$$ 

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)/C$, then the mapping $\ell : f \mapsto \int_{X} f(g+c) \, d\mu$, for $f \in L_{c}^{q,0}(X)$, can be extended to a continuous linear functional on $H_{U}^{[\phi,q]}(X)$. Conversely, if $\ell$ is a continuous linear functional on $H_{U}^{[\phi,q]}(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)/C$ such that $\ell(f) = \int_{X} f(g+c) \, d\mu$ for $f \in L_{c}^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

**Corollary 4.5.** Assume the conditions in Remark 3.1 and Corollary 4.2. Then

$$\left( H^{p(\cdot)}(X) \right)^{*} = \text{Lip}_{\alpha(\cdot)}(X)/C.$$ 

**References**


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