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ON THE NEW WEAK HERZ SPACES AND THE BOUNDEDNESS OF SOME SUBLINEAR OPERATOR

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First, we state the definitions of the non-homogeneous Herz space $K_{p,r}^{\alpha}(\mathbb{R}^n)$ and the non-homogeneous weak Herz space $WK_p^{\alpha,r}(\mathbb{R}^n)$.

Now, for a measurable set $E\subset\mathbb{R}^n$, we denote the Lebesgue measure of E by $|E|$ and the characteristic function of the set E by χ_E. Also, let for $k\in\mathbb{Z}$, $B_k=\{x\in\mathbb{R}^n:|x|\le 2^k\}$, $C_k=B_k\backslash B_{k-1}$ and $\tilde{\chi}_k=\chi_{C_k}$. And let for $k\in\mathbb{N}$, $P_k=C_k$, $\chi_k=\chi_{P_k}$ and $P_0=B_0$, $\chi_0=\chi_{P_0}$.

Definition 1. *For $\alpha\in\mathbb{R}$, $0<p\le\infty$, $0<r<\infty$,*

$$K_{p,r}^{\alpha}(\mathbb{R}^n)=\left\{f\in L_{loc}^p(\mathbb{R}^n):\|f\|_{K_{p,r}^{\alpha}}=\left(\sum_{k=0}^{\infty}2^{k\alpha r}\|f\chi_k\|_{L^p}^r\right)^{1/r}<\infty\right\};$$

For $\alpha\in\mathbb{R}$, $0<p\le\infty$,

$$K_{p,\infty}^{\alpha}(\mathbb{R}^n)=\left\{f\in L_{loc}^p(\mathbb{R}^n):\|f\|_{K_{p,\infty}^{\alpha}}=\sup_{k\ge0}2^{k\alpha}\|f\chi_k\|_{L^p}<\infty\right\}.$$

Definition 2. *For $\alpha\in\mathbb{R}$, $0<p\le\infty$, $0<r<\infty$,*

$$WK_{p,r}^{\alpha}(\mathbb{R}^n)=\left\{f\in L_{loc}^p(\mathbb{R}^n):\|f\|_{WK_{p,r}^{\alpha}}<\infty\right\},$$

where

$$\|f\|_{WK_{p,r}^{\alpha}}=\sup_{\lambda>0}\lambda\left(\sum_{k=0}^{\infty}2^{k\alpha r}\left|\{x\in P_k:|f(x)|>\lambda\}\right|^{r/p}\right)^{1/r};$$

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For $\alpha \in \mathbb{R}$, $0 < p \leq \infty$,

$$WK^{\alpha}_{p,\infty}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{WK^{\alpha}_{p,\infty}} < \infty \right\},$$

where

$$\|f\|_{WK^{\alpha}_{p,\infty}} = \sup_{\lambda > 0} \lambda \sup_{k \geq 0} 2^{k\alpha} \left|\{x \in P_k : |f(x)| > \lambda\}\right|^{1/p}.$$

Next, let T be a sublinear operator satisfying that for any integrable function f with a compact support,

$$(*) \qquad |Tf(x)| \leq c \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n} dy, \quad x \notin \operatorname{supp} f,$$

where $c > 0$ is independent of f and x.

Then, for the boundedness of T on the non-homogeneous Herz space $K^{\alpha}_{p,r}(\mathbb{R}^n)$, the following theorems were proved.

Theorem A (X. Li and D. Yang [LY]). *Let* $1 < p < \infty$, $0 < r \leq \infty$ *and* $-n/p < \alpha < n/p'$, *and let* T *be a sublinear operator satisfying* $(*)$. *If* T *is bounded on* $L^p(\mathbb{R}^n)$, *then*

$$T : K^{\alpha}_{p,r}(\mathbb{R}^n) \to K^{\alpha}_{p,r}(\mathbb{R}^n).$$

Theorem B (Y. Komori [K]). *Let* $0 < r \leq \infty$ *and* $-n < \alpha < 0$, *and let* T *be a sublinear operator satisfying* $(*)$. *If* T *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$, *then*

$$T : K^{\alpha}_{1,r}(\mathbb{R}^n) \to WK^{\alpha}_{1,r}(\mathbb{R}^n).$$

Furthermore, we introduce the new definition of the non-homogeneous weak Herz space $\widetilde{WK}^{\alpha}_{p,r}(\mathbb{R}^n)$.

Definition 3. *For* $\alpha \in \mathbb{R}$, $1 \leq p < \infty$ *and* $0 < r < \infty$,

$$\widetilde{WK}^{\alpha}_{p,r}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{\widetilde{WK}^{\alpha}_{p,r}} = \left(\sum_{k=0}^{\infty} 2^{k\alpha r} \|f\chi_k\|^r_{L^{p,\infty}} \right)^{1/r} < \infty \right\};$$

For $\alpha \in \mathbb{R}$, $1 \leq p < \infty$,

$$\widetilde{WK}^{\alpha}_{p,\infty}(\mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{\widetilde{WK}^{\alpha}_{p,\infty}} = \sup_{k \geq 0} 2^{k\alpha} \|f\chi_k\|_{L^{p,\infty}} < \infty \right\}.$$

Then, note that the following result holds.

Proposition 4 (with J. Soria). *Let* $\alpha \in \mathbb{R}$, $1 \leq p < \infty$ *and* $0 < r \leq \infty$. *If* $\alpha \neq -n/p$, *then* $\widetilde{WK}^{\alpha}_{p,r}(\mathbb{R}^n)$ *is proper subset of* $WK^{\alpha}_{p,r}(\mathbb{R}^n)$.

Sketch of proof. Clearly,

$$\widetilde{W}K^{\alpha}_{p,r}(\mathbb{R}^n) \subseteq WK^{\alpha}_{p,r}(\mathbb{R}^n).$$

Now, for $\beta \in \mathbb{R}$, we put

$$f = \sum_{k=0}^{\infty} 2^{\beta k} \chi_k.$$

Then, under the conditions $\alpha + \beta + n/p = 0$ and $\alpha \neq -n/p$,

$$\|f\|_{\widetilde{W}K^{\alpha}_{p,r}} = \infty \qquad \text{and} \qquad \|f\|_{WK^{\alpha}_{p,r}} < \infty.$$

Hence,

$$f \in WK^{\alpha}_{p,r}(\mathbb{R}^n) \qquad \text{and} \qquad f \notin \widetilde{W}K^{\alpha}_{p,r}(\mathbb{R}^n),$$

i.e.

$$WK^{\alpha}_{p,r}(\mathbb{R}^n) \setminus \widetilde{W}K^{\alpha}_{p,r}(\mathbb{R}^n) \neq \phi.$$

□

Then, for the boundedness of T on the new non-homogeneous weak Herz space $K^{\alpha}_{1,r}(\mathbb{R}^n)$, we can show the following weak-type estimate.

Theorem 5 (with J. Soria). *Let $0 < r \leq \infty$ and $-n < \alpha < 0$, and let T be a sublinear operator satisfying $(*)$. If T is bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, then*

$$T : K^{\alpha}_{1,r}(\mathbb{R}^n) \to \widetilde{W}K^{\alpha}_{1,r}(\mathbb{R}^n).$$

Before proving this theorem, we observe the interpolation theorem for a quasi-Banach space (see [P]).

Definition 6. *Let A be any quasi-Banach space. Then, we define for $\alpha \in \mathbb{R}$ and $0 < r < \infty$,*

$$\ell^{\alpha}_r(A) = \left\{ (a_k)^{\infty}_{-\infty} : a_k \in A, \|(a_k)^{\infty}_{-\infty}\|_{\ell^{\alpha}_r(A)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha r} \|a_k\|^r_A \right)^{1/r} < \infty \right\};$$

for $\alpha \in \mathbb{R}$,

$$\ell^{\alpha}_{\infty}(A) = \left\{ (a_k)^{\infty}_{-\infty} : a_k \in A, \|(a_k)^{\infty}_{-\infty}\|_{\ell^{\alpha}_{\infty}(A)} = \sup_{k \in \mathbb{Z}} 2^{k\alpha} \|a_k\|_A < \infty \right\}.$$

Then, the following theorem for the real interpolation method holds.

Theorem C . *Let A be any quasi-Banach space, and let $\alpha \in \mathbb{R}$ and $0 < r_0, r_1 \leq \infty$. Then*

$$\left(\ell^{\alpha}_{r_0}(A), \ell^{\alpha}_{r_1}(A)\right)_{\theta,r} = \ell^{\alpha}_r(A),$$

where $1/r = (1-\theta)/r_0 + \theta/r_1$ $(0 < \theta < 1)$.

Sketch of proof of Theorem 5. First, we prove that when $0 < r \leq 1$,

$$\|Tf\|_{\widetilde{W}K^{\alpha}_{1,r}} \leq C\|f\|_{K^{\alpha}_{1,r}},$$

i.e. T is bounded from $K^{\alpha}_{1,r}(\mathbb{R}^n)$ to $\widetilde{W}K^{\alpha}_{1,r}(\mathbb{R}^n)$.

Next, we prove the case of $r = \infty$, i.e. T is bounded from $K^{\alpha}_{1,\infty}(\mathbb{R}^n)$ to $\widetilde{W}K^{\alpha}_{1,\infty}(\mathbb{R}^n)$. This case is clear by Theorem B, Definitions 2 and 3.

Finally, we prove the case of $1 < r < \infty$. From the cases of $0 < r \leq 1$ and $r = \infty$,

$$T : K^{\alpha}_{1,1}(\mathbb{R}^n) \to \widetilde{W}K^{\alpha}_{1,1}(\mathbb{R}^n)$$

and

$$T : K^{\alpha}_{1,\infty}(\mathbb{R}^n) \to \widetilde{W}K^{\alpha}_{1,\infty}(\mathbb{R}^n),$$

respectively. Furthermore, by applying Theorem C,

$$\left(K^{\alpha}_{1,1}(\mathbb{R}^n), K^{\alpha}_{1,\infty}(\mathbb{R}^n)\right)_{\theta,r} = \ell^{\alpha}_{r}(L^1(\mathbb{R}^n)) = K^{\alpha}_{1,r}(\mathbb{R}^n)$$

and

$$\left(\widetilde{W}K^{\alpha}_{1,1}(\mathbb{R}^n), \widetilde{W}K^{\alpha}_{1,\infty}(\mathbb{R}^n)\right)_{\theta,r} = \ell^{\alpha}_{r}(L^{1,\infty}(\mathbb{R}^n)) = \widetilde{W}K^{\alpha}_{1,r}(\mathbb{R}^n),$$

where

$$\frac{1}{r} = 1 - \theta, \quad \text{i.e.} \quad r = \frac{1}{1-\theta} \quad (0 < \theta < 1).$$

Thus, when $1 < r < \infty$,

$$T : K^{\alpha}_{1,r}(\mathbb{R}^n) \to \widetilde{W}K^{\alpha}_{1,r}(\mathbb{R}^n),$$

i.e. T is bounded from $K^{\alpha}_{1,r}(\mathbb{R}^n)$ to $\widetilde{W}K^{\alpha}_{1,r}(\mathbb{R}^n)$. □

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