QUICK REVIEW ON PROPERTY (X)

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ABSTRACT. We will review some materials that are useful to prove the uniqueness of preduals. Those were used crucially in our recent work on the uniqueness of predual of any ‘finite’ non-commutative $H^\infty$.

1. INTRODUCTION

In [12] we established, among other things, the uniqueness of predual of any ‘finite’ non-commutative $H^\infty$-algebra $H^\infty(M, \tau)$, which was introduced by Bill Arveson modeled after the usual pair $H^\infty(D) \hookrightarrow L^\infty(T)$ with the aid of operator algebra theory. The class of finite non-commutative $H^\infty$-algebras contains $H^\infty(D)$ as well as its abstract generalizations. Thus [12, Theorem 2] covers any existing generalization of the famous result due to Tsuyoshi Ando [3].

The most key ingredient of our proof of the uniqueness of predual of $H^\infty(M, \tau)$ is to provide a non-commutative analog of Amar–Lederer’s peak set result [2] (also see [4]), which we fully explained in [12]. However, our proof of the uniqueness of predual also uses two purely Banach space theoretic techniques – Property (X) due to Godefroy and Talagrand and a very clever trick, both of which we just borrowed from some references without any detailed explanation. Here we will give detailed accounts (for non-experts like us) on those techniques as supplements to [12, Theorem 2].

In closing, we should mention our sincere thanks to Professor Kichi-Suke Saito for giving this opportunity.

2. GODEFROY–TALAGRAND’S Property (X)

This section mainly follows Godefroy and Talagrand’s elegant work [6]. The key ingredient behind Godefroy–Talagrand’s property (X) is the next proposition.

**Proposition 2.1.** Let $E$ and $G$ be Banach spaces with $E^* = G^*$. If a sequence $\{x_n\} \subset E^*$ satisfies

(i) $x_n \to 0$ in $\sigma(E^*, E)$; and
(ii) $\sum_{n=1}^\infty |\psi(x_{n+1} - x_n)| < +\infty$ for all $\psi \in E^{**},$

then $x_n \to 0$ in $\sigma(E^*, G)$.

**Proof.** Set $u_0 := x_1$, $u_1 := x_2 - x_1$, and $u_n := x_{n+1} - x_n$, and then by (i)

$$\sum_{k=0}^n u_k = x_{n+1} \to 0 \text{ in } \sigma(E^*, E).$$

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For each $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots \}$ we consider the map $T_n : \alpha = (\alpha_k) \in \ell^\infty(\mathbb{N}_0) \mapsto \sum_{k=0}^{n} \alpha_k u_k \in E^*$ ($\hookrightarrow E'''$ via the canonical embedding). Then one has, by (ii),

$$\sup\{(T_n \alpha)(\phi) : \|\alpha\|_\infty \leq 1, n \in \mathbb{N}_0\} \leq \sum_{k=0}^{\infty} |\phi(u_k)| < +\infty$$

for all $\phi \in E''$, and hence the uniform boundedness principle shows that there is $K > 0$ such that

$$\left\| \sum_{k=0}^{n} \alpha_k u_k \right\|_{E^*} \leq K$$

for all $n \in \mathbb{N}_0$ and for all $\alpha_k \in \mathbb{C}$ with $|\alpha_k| \leq 1$.

Choose an arbitrary free ultrafilter $\omega \in \beta(\mathbb{N}_0) \setminus \mathbb{N}_0$ and put $\xi_\omega := \lim_{n \to \omega} \sum_{k=0}^{n} u_k$ in $\sigma(E^*, G)$. Let us choose arbitrary $n_1 < n_2 < \cdots < n_{2l-1} < n_{2l}$. Then, using (2) with

$$\alpha_k = \begin{cases} 1 & n_{2j-1} \leq k \leq n_{2j}, \ j = 1, \ldots, l, \\ 0 & \text{otherwise} \end{cases}$$

we get

$$\left\| \sum_{k=n_1}^{n_2} u_k + \sum_{k=n_3}^{n_4} u_k + \cdots + \sum_{k=n_{2l-1}}^{n_{2l}} u_k \right\| \leq K.$$
Definition 2.1. A Banach space $E$ has property (X) if for any $\psi \in E^{**}$ the following conditions are equivalent:

(a) $\psi \in E$ with the canonical embedding $E \hookrightarrow E^{**}$.

(b) For any sequence $\{x_n\} \subset E^*$ with the properties

\[- x_n \longrightarrow 0 \text{ in } \sigma(E^*, E),\]
\[- \sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty \text{ for all } \phi \in E^{**},\]

one has $\psi(x_n) \longrightarrow 0$.

This definition gives, in some sense, a criterion of $w^*$-continuity for bounded linear functionals on the dual $E^*$ of a Banach space $E$ with property (X).

Definition 2.2. A Banach space $E$ is said to be the unique predual of its dual $E^*$ if another Banach space $G$ with $G^* = E^*$ must coincide with $E$ inside the dual $E^{**}$ of $E^* (= G^*)$ via the canonical embedding.

Corollary 2.2. If a Banach space $E$ has property (X), then $E$ must be the unique predual of its dual $E^*$.

Proof. Assume another Banach space $G$ satisfies $G^* = E^*$. Embed $G \hookrightarrow (E^*)^* = E^{**}$ by $g(x) := x(g)$ for $x \in E^* = G^*$ and $g \in G$. Let $\{x_n\} \subset E^*$ be chosen in such a way that $x_n \longrightarrow 0$ in $\sigma(E^*, E)$ and $\sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty$ for all $\phi \in E^{**}$. By Proposition 2.1 we get $x_n \longrightarrow 0$ in $\sigma(E^*, G)$, which shows that $g(x_n) = x_n(g) \longrightarrow 0$ for all $g \in G$. Thus, Property (X) ensures that any $g$ must fall in $E \hookrightarrow E^{**}$, that is, $G \subseteq E$ inside $E^{**}$. If $G \nsubseteq E$ inside $E^{**}$, then by the Hahn–Banach extension theorem there is $x \in E^*$ such that $x \neq 0$ but $x|_G = 0$. (Indeed, there is $e \in E \setminus G$ by the assumption, and thus $[e] \in E/G$ with $[e] \neq 0$. Then by the Hahn–Banach extension theorem there is $\varphi \in (E/G)^*$ sending $[e]$ to $||[e]|| = \inf\{||e - g|| : g \in G\} \neq 0$. Hence the $x := \varphi \circ Q \in E^*$ with the quotient map $Q : E \rightarrow E/G$ becomes a desired element.) This $x$ is a non-zero element in $G^* = E^*$ but it is identically zero on $G$, a contradiction. Hence $G = E$ inside $E^{**}$. □

The next proposition has been known, but we do give one proof, which is a prototype of our proof of the uniqueness of predual of $H^\infty(M, \tau)$.

Proposition 2.3. Let $M$ be a $\sigma$-finite von Neumann algebra and $M_*$ be its predual. Then, $M_*$ has property (X).

Proof. It suffices to show that, if $\varphi \in M^*$ satisfies $\varphi(x_n) \longrightarrow 0$ for any $\{x_n\} \subset M$ with the properties

\[
\begin{align*}
\bullet & \ x_n \longrightarrow 0 \text{ in } \sigma(M, M_*) \text{ and} \\
\bullet & \ \sum_{n=1}^{\infty} |\phi(x_{n+1} - x_n)| < +\infty \text{ for all } \phi \in M^*,
\end{align*}
\]

then $\varphi$ must fall in $M_* \hookrightarrow M^*$. Here we need the following standard facts on von Neumann algebras (see e.g. [9] and [11] for their proofs):

(1) Any $\psi \in M^*$ can be decomposed into $\psi = \psi_{\text{nor}} + \psi_{\text{sing}}$ with $\psi_{\text{nor}} \in M_*$ and $\psi_{\text{sing}} \in M^* \ominus M_*$, and $\|\psi\| = \|\psi_{\text{nor}}\| + \|\psi_{\text{sing}}\|$ holds. (This is the so-called non-commutative Lebesgue decomposition due to Takesaki.) We call $M_*$ the normal part and $M^* \setminus M_*$ the singular part. Remark that the notation here is a little bit different from that in [12].

(2) For any $\psi \in M^*$ (or $\psi \in M_*$) there are a unique positive linear functional $|\psi| \in M_*$ (resp. $|\psi| \in M_*$) and a unique partial isometry $v \in M^{**}$ (resp. $v \in M_*$) such that $\langle \psi, x \rangle = \langle |\psi|, xv \rangle$ as well as $\langle |\psi|, x \rangle = \langle \psi, xv \rangle$ for $x \in M^{**}$, where $\langle \cdot, \cdot \rangle : M^* \times M^{**} \rightarrow \mathbb{C}$ stands for the canonical pairing. (This is the so-called polar decomposition.
of linear functionals due to Sakai and also Tomita.) Remark here that the second dual $M^{**}$ becomes a von Neumann algebra, which naturally contains the original $M$ as a subalgebra via the canonical embedding $M \hookrightarrow M^{**}$.

(3) Both the closed subspaces $M_*$ and $M^* \ominus M_*$ of $M^*$ are closed under the operation $\psi \in M^* \mapsto |\psi| \in M^*$. (This follows from the construction of the decomposition in (1) together with (2).)

(4) For a positive linear functional $\psi \in M^*$ the following are equivalent:

- $\psi \in M^* \ominus M_*$.
- For every nonzero projection $e \in M$ there is a non-zero projection $e_0 \in M$ such that $e_0 \leq e$ and $\psi(e_0) = 0$.

(This is Takesaki’s criterion for ‘singularity’ of linear functionals.)

(5) Any $\psi \in M^*$ (or $M_*$) can be written as a linear combination of four positive linear functionals in $M^*$ (resp. $M_*$).

Let us decompose the given $\varphi$ into $\varphi = \varphi_{nor} + \varphi_{sing}$ as in (1), and what we have to show is $\varphi_{sing} = 0$, i.e., $\varphi = \varphi_{nor} \in M_*$. For contrary we suppose $\varphi_{sing} \neq 0$. Then, by (2) and (3), $|\varphi_{sing}| \neq 0$ and $|\varphi_{sing}| \in M^* \ominus M_*$ still holds. Clearly, the orthogonal families of non-zero projections in $\text{Ker} |\varphi_{sing}|$ forms an inductive set by inclusion, and Zorn’s lemma ensures the existence of a maximal family $\{q_k\}$, which is at most countable since $M$ is $\sigma$-finite. Put $q_0 := \sum_k q_k$ in $M$, and then $q_0 = 1$ since $q_0 \neq 1$ clearly contradicts to the above (4). Also, if $\{q_k\}$ is a finite family, then $|\varphi_{sing}|(1) = \sum_k |\varphi_{sing}|(q_k) = 0$, a contradiction. Therefore, $\{q_k\}$ must be a countably infinite family with $\sum_k q_k = 1$ in $M$. Letting $p_n := 1 - \sum_{k \leq n} q_k$ we have $p_n \Downarrow 0$ in $\sigma(M, M_*)$ but $|\varphi_{sing}|(p_n) = |\varphi_{nor}(1)|$ for all $n$. The latter says that $p_n$ converges a non-zero projection $p \in M^{**}$ in $\sigma(M^{**}, M^*)$ with $|\varphi_{sing}|(p) = |\varphi_{sing}|(1) = |\varphi_{nor}(1)|$ since $p_n$ is a decreasing sequence. Let $u \in M$ and $v \in M^{**}$ be the partial isometries for the polar decompositions of $\varphi_{nor}$ and $\varphi_{sing}$, respectively. Then, for $x \in M^{**}$ one has $|\varphi_{sing}, (1-p)x| = |\varphi_{sing}|((1-p)xv) \leq |\varphi_{sing}|(1-p)^{1/2}|\varphi_{sing}|(u^*x^*xv)^{1/2} = 0$ so that $\varphi_{sing}(x) = \varphi_{sing}(px)$ since $|\psi_{sing}|(p) = |\psi_{sing}|(1)$. Similarly, for $x \in M^{**}$ one has $|\varphi_{nor}, px| = |\varphi_{nor}, pxu\rangle = \varphi_{nor}(px),$ so that $\varphi_{nor}(px) = 0$. Consequently, we get $\varphi(px) = \varphi_{nor} + \varphi_{sing}, px = \varphi_{sing}(x)$ for $x \in M$.

Let $x \in M$ be arbitrary. Clearly, $p_n x \longrightarrow 0$ in $\sigma(M, M_*)$. Let $\varphi \in M^*$ be arbitrary, and decompose $y \in M \mapsto \varphi(yx)$ into a linear combination of four positive linear functionals $\phi_i \in M^*$, $i = 1, 2, 3, 4$, thanks to the above (5). Since $\sum_{n=1}^N |\phi_i(p_n+1-n)| = \sum_{n=1}^N \phi_i(q_{n+1}) = \phi_i(\sum_{n=1}^{N+1} q_n) \leq \phi_i(1) < +\infty$ for all $N \in \mathbb{N}$, it follows that $\sum_{n=1}^N |\phi(p_n+1-n)x| < +\infty$. Therefore, by the assumption here one has $\varphi(p_n x) \longrightarrow 0$. On the other hand, $\varphi(p_n x) = \langle \varphi, p_n x \rangle \longrightarrow \langle \varphi, px \rangle = \varphi_{sing}(x)$ so that $\varphi_{sing} = 0$, a contradiction.

The heart of the above proof is as follows. Although $\varphi_{nor}$ and $\varphi_{sing}$ are ‘orthogonal’, we cannot find a projection in $M$ that distinguishes those. (Of course, we can find such a projection in $M^{**}$ since both functionals can be regarded as ‘normal’ ones on $M^{**}$.) Thus we first construct a projection $p \in M^{**}$ in such a way that it can be ‘nicely’ approximated by projections in $M$ and $p$ is greater than ‘the support of $\varphi_{sing}$’ but ‘disjoint’ from ‘the support of $\varphi_{nor}$’. This essentially says that $M$ ‘remembers’ the decomposition $M^* = M_* \ominus (M^* \ominus M_*)$ of $M^*$ (the second dual of $M_*$). This suggests us that such a decomposition of the second dual should be related to property (X) of a Banach space in question. This was quite recently answered affirmatively by Hermann Pfitzner when a Banach space in question is separable, see [8].

Further accounts on the present topics can be found in [5].
3. ADDENDUM – A CLEVER TRICK DUE TO PELCZYŃSKI

The essential idea of our proof of the uniqueness of predual of $H^\infty(M, \tau)$ is similar to that of Proposition 2.3. However, the luck of self-adjointness of our algebra $H^\infty(M, \tau)$ (thus we cannot use the order structure) makes some trouble, which we overcame with a clever trick borrowed from the proof of [7, Proposition 1.c.3]. (The trick is due to Aleksander Pelczyński, see [10, p.637] for this credit, and it was originally used for proving that if a Banach space has Pelczyński’s property (u) then so does any closed subspace, see [7] or more recent [1].) Here we will explain it. The situation we deal with is as follows. Let $M$ be a von Neumann algebra and $A$ be its $\sigma$-weakly closed (possibly non-self-adjoint) unital subalgebra. Assume that we have two sequences $\{a_n\} \subset A$ and $\{b_n\} \subset M$ such that

(i) both $a_n$ and $b_n$ converge to the same $p \in M^\ast$ in $\sigma(M^\ast, M^\ast)$, and

(ii) $\sum_{n=1}^\infty |\phi(b_{n+1} - b_n)| < +\infty$ for all $\phi \in M^\ast$.

What we want to do is to replace $a_n$ by a new one with keeping (i) and further satisfying (ii). This can be done by utilizing the above-mentioned clever trick in Banach space theory.

**Proposition 3.1.** There is another $\{a'_n\} \subset A$ such that

(i) $a'_n \longrightarrow p$ in $\sigma(M^\ast, M^\ast)$, and

(ii) $\sum_{n=1}^\infty |\phi(a'_{n+1} - a'_n)| < +\infty$ for all $\phi \in M^\ast$.

We need one elementary lemma due to Stanislaw Mazur.

**Lemma 3.2.** Let $E$ be a normed space and $\{x_n\} \subset E$ be such that $x_n \longrightarrow 0$ in $\sigma(E, E^\ast)$. Then, for each $\varepsilon > 0$ and each $m \in \mathbb{N}$ there is a convex combination $y = \sum_{n \geq m} \lambda_n x_n$ with $\|y\| < \varepsilon$.

**Proof.** Let $C_m$ be the closed convex hull of $\{x_n\}_{n \geq m}$ in $E$. It suffice to show $0 \in C_m$. Thus, for contrary, suppose $0 \notin C_m$. Then there is a small open ball $B$ centered at 0 with $C_m \cap B = \emptyset$. The Hahn–Banach separation theorem ensures that there are $\varphi \in E^\ast$ and $t \in \mathbb{R}$ such that $\text{Re}\varphi(b) \leq t \leq \text{Re}\varphi(c)$ for all $b \in B$ and $c \in C_m$. This is impossible since $x_n \longrightarrow 0$ in $\sigma(E, E^\ast)$ (implying $t \leq 0$) and $0 \in B$ (implying $t \geq 0$). Thus $0 \in C_m$, which means the desired assertion.

\[\square\]

**Proof.** (Proposition 3.1) Putting $b_0 := 0$ we have $\sum_{n=1}^\infty |\phi(b_n - b_{n-1})| < +\infty$ for all $\phi \in M^\ast$. Set $u_n := a_n - \sum_{k=1}^n b_k - b_{k-1}$, and then $u_n = a_n - b_n \longrightarrow 0$ in $\sigma(M, M^\ast)$ by (i). By Lemma 3.2 there are convex combinations $u'_j = \sum_{n=p_j-1}^{p_j+1} \lambda_n^{(j)} u_n$ such that $0 = p_0 < p_1 < p_2 < \cdots$ and $\|u'_j\| \leq 2^{-j}$. Then We define $a'_j := \sum_{n=p_j-1}^{p_j+1} \lambda_n^{(j)} a_n \in A$ and put $a'_0 := 0$ for convenience. Let us prove that this $\{a'_j\}$ gives a desired sequence.

Since $a_n \longrightarrow p$ in $\sigma(M^\ast, M^\ast)$, for any $\varepsilon > 0$ and any $\phi \in M^\ast$ there is $n_0 \in \mathbb{N}$ such that $|\langle a_n, \phi \rangle - \langle p, \phi \rangle| < \varepsilon$ for all $n \geq n_0$, where $\langle \cdot, \cdot \rangle : M^\ast \times M^\ast \rightarrow \mathbb{C}$ is the canonical pairing. If $j_0$ is chosen so that $p_{j_0-1} + 1 \geq n_0$, then one has $|\langle a'_j, \phi \rangle - \langle p, \phi \rangle| \leq \sum_{n=p_{j_0-1}+1}^{p_{j_0}} \lambda_n^{(j)} |\langle a_n, \phi \rangle - \langle p, \phi \rangle| < \varepsilon$ for all $j \geq j_0$. Thus $a'_j \longrightarrow p$ in $\sigma(M^\ast, M^\ast)$ as $j \rightarrow \infty$.

One has

\[
\begin{align*}
a'_{j+1} - a'_j &= u'_{j+1} + \sum_{n=p_j+1}^{p_{j+1}} \lambda_n^{(j+1)} (a_n - u_n) - u'_j - \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} (a_n - u_n) \\
&= u'_j + \sum_{n=p_j+1}^{p_{j+1}} \lambda_n^{(j+1)} (b_k - b_{k-1}) - \sum_{n=p_{j-1}+1}^{p_j} \lambda_n^{(j)} (b_k - b_{k-1}) \\
&= u'_j + \mu_n^{(j)} (b_n - b_{n-1})
\end{align*}
\]
with some $0 \leq \mu_{n}^{(j)} \leq 1$. Hence,
\[
\sum_{j=0}^{\infty} |\phi(a_{j+1}' - a_{j}')| 
\leq \sum_{j=0}^{\infty} \|\phi\| \|u_{j+1}'\| + \sum_{j=1}^{\infty} \sum_{n=1}^{p_{j}+1} \mu_{n}^{(j)} |\phi(b_{n} - b_{n-1})| 
\leq 2 \sum_{j=0}^{\infty} \|\phi\| \|u_{j}'\| + \sum_{n=1}^{\infty} |\phi(b_{n} - b_{n-1})| 
\leq 4 \|\phi\| + \sum_{n=1}^{\infty} |\phi(b_{n} - b_{n-1})| < +\infty
\]
by $\|u_{j}'\| \leq 2^{-j}$ and (ii).

Remark here that the argument presented above uses only the linear structure; hence clearly it can be applied to more general situations.

REFERENCES


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