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Renormalization Group Method and its Application to Coupled Oscillators

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Abstract: The renormalization group (RG) method for differential equations is one of the perturbation methods which provides not only approximate solutions but also approximate vector fields. Some topological properties of an original equation, such as the existence of a normally hyperbolic invariant manifold and its stability are shown to be inherited from those of the RG equation. This fact is applied to the Kuramoto model and the stability of the invariant torus will be determined.

1 Introduction

The renormalization group (RG) method for differential equations is one of the perturbation methods for obtaining solutions which approximate exact solutions for a long time interval. In their papers [1,2], Chen, Goldenfeld, Oono have established the RG method for ordinary differential equations of the form

$$\dot{x} = \frac{dx}{dt} = f(t,x) + \varepsilon g(t,x), \quad x \in \mathbf{R}^n, \tag{1}$$

where $\varepsilon > 0$ is a small parameter. For this equation, the method for deriving approximate solutions of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots \tag{2}$$

is called the *naive expansion* or the *regular perturbation method*, where $x_i(t)$'s are governed by inhomogeneous linear ODEs obtained by putting Eq.(2) into Eq.(1) and equating the coefficients of ε^i of the both sides of Eq.(1). It is well known that approximate solutions constructed by the naive expansion are valid only in a time interval of $O(1)$ in general, since secular terms diverge as $t \to \infty$. Many techniques for obtaining approximate solutions which are valid in a long time interval have been developed until now, which are collectively called singular perturbation methods.

The RG method proposed by Chen *et al.* is one of the singular perturbation methods looking like the variation-of-constant method, in which the secular terms included in $x_1(t), x_2(t), \cdots$ of Eq.(2) are renormalized into the integral constant of $x_0(t)$. The ODE to be satisfied by the renormalized integral constant is called the *RG equation*.

In their papers [1,2], it is not clear why the RG method works well. Kunihiro [9,10] revealed the reason by characterizing the RG equation as an equation for obtaining an envelope of a family of curves

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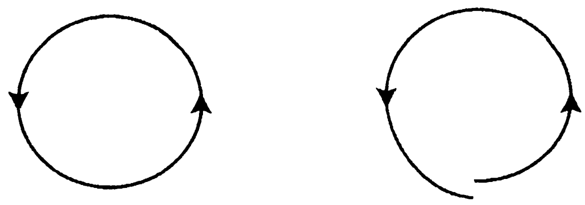
constructed by the naive expansion. His idea gave an intuitive concept of the RG method, however, the problems below remained to be solved.

(i) An explicit definition of the RG equation was not given.
(ii) It was not clear whether we can detect the existence of an invariant manifold and its stability.

The problem (i) was solved by Ziane [11], DeVille *et al.* [7], and Chiba [3]. Ziane and DeVille *et al.* gave the definition of the first order RG equation by using the averaging operator. Further, Chiba gave the formula for the m-th order RG equation by calculating the envelope of naive expansion solutions up to arbitrary order of ε. With these formulas, one can obtain a higher order RG equation by using a computer software like Mathematica.

While the problem (i) is a computational issue, the problem (ii) is more essential. It was shown by Ziane [11], DeVille *et al.* [7], and Chiba [3] that an approximate solution constructed by the RG method is close to an exact solution in a time interval $-T/\varepsilon < t < T/\varepsilon$, where T is some positive constant (see Thm.10 for the exact statement). Now consider the following situation : Suppose that an approximate solution obtained by the RG method is a periodic solution. However, the original equation may not have a periodic orbit because of the error of the approximate solution (see the figure below). In general, since approximate solutions have small errors, it is not obvious that the RG method can show the existence of an invariant manifold. Further, we can not understand an asymptotic behavior of an exact solution because the time interval $-T/\varepsilon < t < T/\varepsilon$ is finite in general. In particular, we may not understand the stability of an invariant manifold.

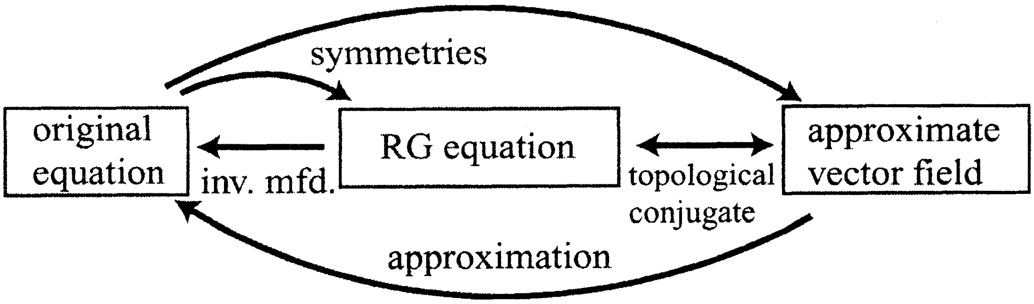


approximate solution　　　　exact solution

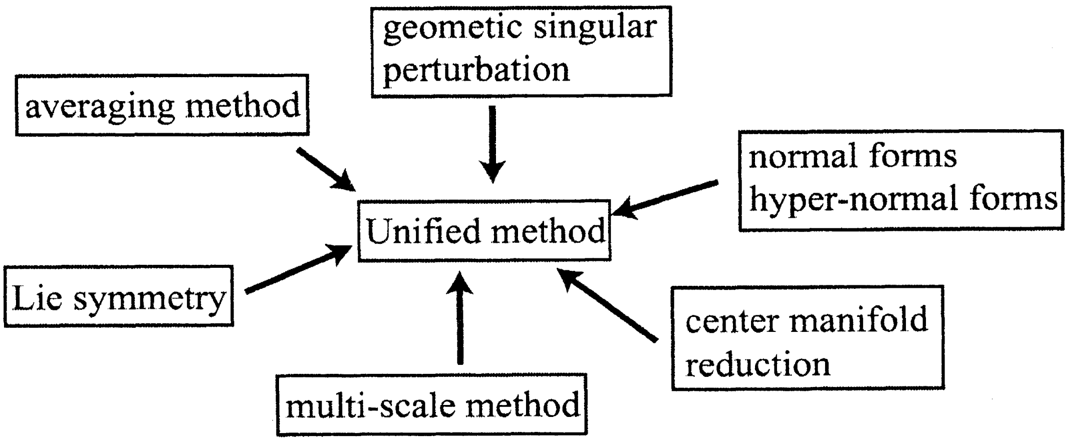
This problem was solved by Chiba [3]. He considered constructing an approximate vector field instead of an approximate solution. It was considered that whether there exists an approximate differential equation satisfied by a family of approximate solutions. It is not obvious because two approximate solutions may intersect with each other and the uniqueness of solutions is violated. However, under appropriate assumptions, Chiba [3] proved that a family of approximate solutions defines a vector field which is sufficiently close to the original vector field associated with the original equation (Thm.9).

Once we obtain the approximate vector field, we can use powerful tools of dynamical systems theory. In particular, by using the invariant manifold theory, we can prove that if the RG equation has a normally hyperbolic invariant manifold N, then the original equation also has an invariant manifold N_ε, which is diffeomorphic to N. Further the stability of N_ε coincides with that of N (Thm.11).

For the reason above, we hope that the RG equation is easier to solve than the original equation. In fact, it is shown that symmetries of the original equation (group invariance) are inherited to those of the RG equation. Furthermore, the RG equation is invariant under the action of the 1-parameter group defined by the flow of the unperturbed vector field of the original equation (Thm.12). It means that the RG equation has more symmetries than the original equation has, and it is easier to solve than the original equation.



The fact that the RG method unifies the traditional singular perturbation methods was already suggested by Chen, Goldenfeld, Oono [1,2] (without mathematical proofs). Ei, Fujii, Kunihiro [8] suggested that the RG method provides an approximate center manifold, and this fact was proved by Chiba [5] by using invariant manifold theory. The equivalence of the normal forms of vector fields and the RG equations was proved by DeVille *et al.* [7] for the case of the first order RG equation. This result was extended to the higher order RG equation by Chiba [4]. Further, by focusing the nonuniqueness of the higher order RG equation, Chiba [4] showed that we can construct the higher order RG equation so that it is equivalent to the hyper-normal form. In addition, it is known that the RG method unifies the averaging method, the multi-scale method, the geometric singular perturbation, and the Lie symmetry method [1,2,6].



2 Definitions

Let f be a time independent C^r vector field on a C^r manifold M and $\varphi : \mathbf{R} \times M \to M$ its flow. We denote by $\varphi_t(x_0) \equiv x(t)$, $t \in \mathbf{R}$, a solution to the ODE $\dot{x} = f(x)$ through $x_0 \in M$, which satisfies $\varphi_t \circ \varphi_s = \varphi_{t+s}$, $\varphi_0 = id_M$, where id_M denotes the identity map of M. For a fixed $t \in \mathbf{R}$, $\varphi_t : M \to M$ defines a diffeomorphism of M. We assume φ_t is defined for all $t \in \mathbf{R}$.

For a time-dependent vector field, let $x(t, \tau, \xi)$ denote a solution to an ODE $\dot{x}(t) = f(t, x)$ through ξ at $t = \tau$, which defines a flow $\varphi : \mathbf{R} \times \mathbf{R} \times M \to M$ by $\varphi_{t,\tau}(\xi) = x(t, \tau, \xi)$. For fixed $t, \tau \in \mathbf{R}$, $\varphi_{t,\tau} : M \to M$ is a diffeomorphism of M satisfying

$$\varphi_{t,t'} \circ \varphi_{t',\tau} = \varphi_{t,\tau}, \quad \varphi_{t,t} = id_M. \tag{3}$$

Conversely, a family of diffeomorphisms $\varphi_{t,\tau}$ of M, which are C^1 with respect to t and τ, satisfying the above equality for any $t, \tau \in \mathbf{R}$ defines a time-dependent vector field on M through

$$f(t, x) = \frac{d}{d\tau}\Big|_{\tau=t} \varphi_{\tau,t}(x). \tag{4}$$

Definition 1. Let f be a vector field on M, and φ_t its flow. A submanifold N of M is called *f-invariant* if $\varphi_t(N) = N$ for $\forall t \in \mathbf{R}$. An f-invariant manifold N is called *hyperbolic*, if there are vector bundles $\boldsymbol{E}^s, \boldsymbol{E}^u$ over N s.t.

(i) $TM|_N = \boldsymbol{E}^s \oplus \boldsymbol{E}^u \oplus TN$,

(ii) both $\boldsymbol{E}^s \oplus TN$ and $\boldsymbol{E}^u \oplus TN$ are $D\varphi_t$-invariant,

(iii) there exist constants $C \geq 1$, $\alpha, \beta > 0$ s.t. for $\forall p \in N$,

$$v \in \boldsymbol{E}^s_p \quad \Rightarrow \quad \|\pi^s \circ (D\varphi_t)_p v\| \leq Ce^{-\alpha t}, \quad t \geq 0, \tag{5}$$

$$v \in \boldsymbol{E}^u_p \quad \Rightarrow \quad \|\pi^u \circ (D\varphi_{-t})_p v\| \leq Ce^{-\beta t}, \quad t \geq 0, \tag{6}$$

where π^s, π^u are projections from $TM|_N$ to $\boldsymbol{E}^s, \boldsymbol{E}^u$, respectively.

Definition 2. A hyperbolic invariant manifold N is called *r-normally hyperbolic*, if there exist an integer $r \geq 1$ and constants $C \geq 1, \gamma > 0$, such that for $\forall p \in N, v \in \boldsymbol{E}^s_p$, $w \in \boldsymbol{E}^u_p$, $u \in T_pN$, the following inequalities hold.

$$\|(D\varphi_t)_p u\|^k \|\pi^s \circ (D\varphi_t)_p v\| \leq Ce^{-\gamma t}\|u\|^k\|v\|, \quad k = 0, 1, \cdots, r, \quad t \geq 0, \tag{7}$$

$$\|(D\varphi_{-t})_p u\|^k \|\pi^u \circ (D\varphi_{-t})_p w\| \leq Ce^{-\gamma t}\|u\|^k\|w\|, \quad k = 0, 1, \cdots, r, \quad t \geq 0. \tag{8}$$

Next theorem is one of the fundamental theorem of the invariant manifold theory.

Theorem 3. (Fenichel, 1971)

Let M be a C^r manifold ($r \geq 1$), and $\mathcal{X}^r(M)$ the set of C^r vector fields on M with C^1 topology. Let f be a C^r vector field on M and suppose that $N \subset M$ is a compact connected r-normally hyperbolic

f-invariant manifold. Then, the following holds:

(i) There is a neighborhood $\mathcal{U} \subset \mathcal{X}^r(M)$ of f s.t. there exists an r-normally hyperbolic g-invariant C^r manifold $N_g \subset M$ for $\forall g \in \mathcal{U}$.
(ii) N_g is diffeomorphic to N and the diffeomorphism $h : N_g \to N$ is close to the identity $id : N \to N$ in the C^1 topology. In particular, N_g lies within an $O(\varepsilon)$ neighborhood of N if $||f - g|| \sim O(\varepsilon)$.

3 RG method

In this section, we give the definition of the RG equation and fundamental theorems of the RG method. All proofs are given in Chiba [3].

Consider an ODE on $\mathbf{R}^n$ of the form

$$\begin{aligned} \dot{x} &= Fx + \varepsilon g(t, x, \varepsilon) \\ &= Fx + \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots, \quad x \in \mathbf{R}^n, \end{aligned} \tag{9}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter. For this system, we suppose that

(A1) the matrix F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues lie on the imaginary axis.
(A2) the function $g(t, x, \varepsilon)$ is of C^∞ class with respect to t, x and ε. The formal power series expansion of $g(t, x, \varepsilon)$ in ε is given as above.
(A3) each $g_i(t, x)$ is periodic in $t \in \mathbf{R}$ and polynomial in x.

Remark 4. These assumptions can be weakened in various ways. For example, if F has eigenvalues on the left half plane, our method provides the center manifold reduction (see Sec.5.3). The case that the unperturbed term is nonlinear is treated in Sec.5.4. The assumption (A3) can be replaced as :
(A3') each $g_i(t, x)$ is almost periodic functions such that the set of whose Fourier exponents does not have accumulation points.
If the set of Fourier exponents of $g_i(t, x)$ has accumulation points, the RG transformation defined below may diverge as $t \to \infty$ (see Chiba [3]).

At first, let us attempt the naive expansion. Replacing x in (9) by $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$, we rewrite (9) as

$$\dot{x}_0 + \varepsilon \dot{x}_1 + \varepsilon^2 \dot{x}_2 + \cdots = F(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \sum_{i=1}^{\infty} \varepsilon^i g_i(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots). \tag{10}$$

Expanding the right hand side of the above equation with respect to ε and equating the coefficients of each ε^i of the both sides, we obtain ODEs of $x_0, x_1, x_2, \cdots$ as

$$\dot{x}_0 = Fx_0, \tag{11}$$
$$\dot{x}_1 = Fx_1 + G_1(t, x_0), \tag{12}$$
$$\vdots$$
$$\dot{x}_i = Fx_i + G_i(t, x_0, x_1, \cdots, x_{i-1}), \tag{13}$$
$$\vdots$$

where the inhomogeneous term G_i is a smooth function of $t, x_0, x_1, \cdots, x_{i-1}$. For instance, G_1, G_2, G_3 and G_4 are given by

$$G_1(t, x_0) = g_1(t, x_0), \tag{14}$$
$$G_2(t, x_0, x_1) = \frac{\partial g_1}{\partial x}(t, x_0)x_1 + g_2(t, x_0), \tag{15}$$
$$G_3(t, x_0, x_1, x_2) = \frac{1}{2}\frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_1}{\partial x}(t, x_0)x_2 + \frac{\partial g_2}{\partial x}(t, x_0)x_1 + g_3(t, x_0), \tag{16}$$
$$\begin{aligned} G_4(t, x_0, x_1, x_2, x_3) = {} & \frac{1}{6}\frac{\partial^3 g_1}{\partial x^3}(t, x_0)x_1^3 + \frac{\partial^2 g_1}{\partial x^2}(t, x_0)x_1x_2 + \frac{\partial g_1}{\partial x}(t, x_0)x_3 \\ & + \frac{1}{2}\frac{\partial^2 g_2}{\partial x^2}(t, x_0)x_1^2 + \frac{\partial g_2}{\partial x}(t, x_0)x_2 + \frac{\partial g_3}{\partial x}(t, x_0)x_1 + g_4(t, x_0), \end{aligned} \tag{17}$$

respectively. We can verify the equality (see Lemma A.2 of Chiba [3] for the proof)

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial G_{i-1}}{\partial x_{j-1}} = \cdots = \frac{\partial G_{i-j}}{\partial x_0}, \quad i > j \geq 0, \tag{18}$$

and it may help in deriving G_i and proving Prop.5 below.

Solving the system above and constructing the curve $x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$ is called the naive expansion as is mentioned in Sec.1. According to the Kunihiro's idea, the RG equation is given as an equation for obtaining an envelope of a family of the naive expansion solutions. Now let us derive the naive expansion solutions.

In what follows, we denote the fundamental matrix e^{Ft} as $X(t)$. Define the functions $R_i, h_t^{(i)}$, $i = 1, 2, \cdots$ on $\mathbf{R}^n$ by

$$R_1(y) := \lim_{t\to\infty} \frac{1}{t}\int^t X(s)^{-1}G_1(s, X(s)y)ds, \tag{19}$$
$$h_t^{(1)}(y) := X(t)\int^t \Big(X(s)^{-1}G_1(s, X(s)y) - R_1(y)\Big)ds, \tag{20}$$
$$\begin{aligned} R_i(y) := \lim_{t\to\infty} \frac{1}{t}\int^t \Big(& X(s)^{-1}G_i(s, X(s)y, h_s^{(1)}(y), \cdots, h_s^{(i-1)}(y)) \\ & - X(s)^{-1}\sum_{k=1}^{i-1}(Dh_s^{(k)})_y R_{i-k}(y)\Big)ds, \quad i = 2, 3, \cdots, \end{aligned} \tag{21}$$
$$\begin{aligned} h_t^{(i)}(y) := X(t)\int^t \Big(& X(s)^{-1}G_i(s, X(s)y, h_s^{(1)}(y), \cdots, h_s^{(i-1)}(y)) \\ & - X(s)^{-1}\sum_{k=1}^{i-1}(Dh_s^{(k)})_y R_{i-k}(y) - R_i(y)\Big)ds, \quad i = 2, 3, \cdots, \end{aligned} \tag{22}$$

respectively. Then, the following statement holds.

Proposition 5. (Chiba [3]) Let $x_0(t) = X(t)y$ be the solution to Eq.(11) whose initial value is $y \in \mathbf{R}^n$. Then, for arbitrary $\tau \in \mathbf{R}$ and $i = 1, 2, \cdots$, the curve

$$x_i := x_i(t, \tau; y) = h_t^{(i)}(y) + p_1^{(i)}(t, y)(t-\tau) + p_2^{(i)}(t, y)(t-\tau)^2 + \cdots + p_i^{(i)}(t, y)(t-\tau)^i \tag{23}$$

gives a solution to Eq.(13), where the functions $p_1^{(i)}, \cdots, p_i^{(i)}$ are given by

$$p_1^{(i)}(t, y) = X(t)R_i(y) + \sum_{k=1}^{i-1} (Dh_t^{(k)})_y R_{i-k}(y), \tag{24}$$

$$p_j^{(i)}(t, y) = \frac{1}{j} \sum_{k=1}^{i-1} \frac{\partial p_{j-1}^{(k)}}{\partial y}(t, y) R_{i-k}(y), \quad (j = 2, 3, \cdots, i-1), \tag{25}$$

$$p_i^{(i)}(t, y) = \frac{1}{i} \sum_{k=1}^{i-1} \frac{\partial p_{i-1}^{(k)}}{\partial y}(t, y) R_{i-k}(y) = \frac{1}{i} \frac{\partial p_{i-1}^{(i-1)}}{\partial y}(t, y) R_1(y), \tag{26}$$

$$p_j^{(i)}(t, y) = 0, \quad (j > i). \tag{27}$$

Further, the functions $h_t^{(i)}(y)$ are bounded uniformly in t.

Remark 6. Note that we gave the solution to Eq.(13) so that it is split into the bounded term $h_t^{(i)}$ and the divergence terms. In particular, the linearly increasing term $p_1^{(i)}(t, y)(t-\tau)$ is called the *secular term*.

To see what we did, let us derive the functions R_1 and $h_t^{(1)}$. With the 0-th order solution $x_0(t) = X(t)y$, the first order equation (12) is written as

$$\dot{x}_1 = F x_1 + G_1(t, X(t)y). \tag{28}$$

The solution to this equation is given by

$$x_1 = X(t)X(\tau)^{-1} h + X(t) \int_\tau^t X(s)^{-1} G_1(s, X(t)y) ds, \tag{29}$$

where h is an initial value and τ is an initial time. The integrand in the right hand side is a almost periodic function because of the assumptions (A1) to (A3). In particular, it is written as "constant term" + "almost periodic term" by virtue of the Fourier expansion. The linearly increasing term, namely secular term, arises from the integral of "the constant". On the other hand, the integral of the "almost periodic term" is almost periodic yet. Our purpose is to rewrite Eq.(29) so that the right hand side is split explicitly into the secular term and the bounded term. Since the integral in the right hand side is "secular term" + "almost periodic term", if we divide the integral by t and take the limit $t \to \infty$, the almost periodic part vanishes and the coefficient of the secular term remains. This coefficient of the secular term is just the function R_1 defined by Eq.(19). To obtain the bounded term, we subtract the secular part R_1 from the integrand of Eq.(29) as

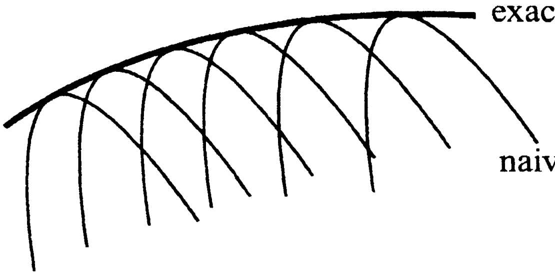
$$x_1 = X(t)X(\tau)^{-1} h + X(t) \int_\tau^t (X(s)^{-1} G_1(s, X(t)y) - R_1(y)) ds + X(t)R_1(y)(t-\tau). \tag{30}$$

Now we define $h_t^{(1)}$ by Eq.(20) and put $h = h_\tau^{(1)}(y)$ in the above. Then we obtain

$$x_1 = h_t^{(1)}(y) + X(t)R_1(y)(t-\tau) \tag{31}$$

and this proves Eq.(23) for $i = 1$.

Now that we know the naive expansion solutions up to the first order $x(t,\tau,y) = X(t)y + \varepsilon(h_t^{(1)}(y) + X(t)R_1(y)(t-\tau))$, let us calculate the envelope of the family of these curves parameterized by τ. We vary the parameter y, which is an initial value of the 0-th order equation, along an exact solution when we vary the initial time τ (see the figure below).



Then, it seems that the envelope gives a good approximate solution. To do this, put $y = y(\tau)$, and the envelope is given as follows: At first, we differentiate the family by τ at t and determine $y(\tau)$ so that the derivative is equal to zero.

$$\frac{d}{d\tau}\bigg|_{\tau=t} x(t,\tau,y(\tau)) = X(t)\frac{dy}{dt}(t) + \varepsilon\left(\frac{\partial h_t^{(1)}}{\partial y}\frac{dy}{dt}(t) - X(t)R_1(y)\right) = 0. \tag{32}$$

It is easy to verify that if we put

$$\frac{dy}{dt} = \varepsilon R_1(y) + O(\varepsilon^2), \tag{33}$$

then the equality (32) is satisfied. Let $y(t)$ be an solution to this equation. Then, the envelope for the family of the naive expansion solutions is given by

$$x(t,t,y(t)) = X(t)y(t) + \varepsilon h_t^{(1)}(y(t)). \tag{34}$$

Calculating the higher order case in a similar manner, we find the definition of the RG equation.

Definition 7. Along with $R_1(y), \cdots, R_m(y)$ defined in Eqs.(19), (21), we define the *m-th order RG equation* for Eq.(9) to be

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots + \varepsilon^m R_m(y), \quad y \in \mathbf{R}^n. \tag{35}$$

Using $h_t^{(1)}(y), \cdots, h_t^{(m)}(y)$ defined in Eqs.(20), (22), we define the *m-th order RG transformation* α_t : $\mathbf{R}^n \to \mathbf{R}^n$ to be

$$\alpha_t(y) = X(t)y + \varepsilon h_t^{(1)}(y) + \cdots + \varepsilon^m h_t^{(m)}(y). \tag{36}$$

Remark 8. Since $X(t)$ is nonsingular and $h_t^{(1)}(y), \cdots, h_t^{(m)}(y)$ are bounded uniformly in $t \in \mathbf{R}$, for sufficiently small $|\varepsilon|$, there exists an open set $U = U(\varepsilon)$ such that $\overline{U}$ is compact and the restriction of α_t to U is diffeomorphism from U into $\mathbf{R}^n$.

An approximate solution is constructed as above, however, we want to construct an approximate vector field to investigate topological properties of the original equation as discussed in Sec.1. Fundamental theorems of the RG method are listed below.

Theorem 9. (Approximation of Vector Fields)

Let φ_t^{RG} be the flow of the m-th order RG equation for Eq.(9) and α_t the m-th order RG transformation. Then, there exists a positive constant ε_0 such that the following holds for $\forall|\varepsilon| < \varepsilon_0$:

(i) The map

$$\Phi_{t,t_0} := \alpha_t \circ \varphi_{t-t_0}^{RG} \circ \alpha_{t_0}^{-1} : \alpha_{t_0}(U) \to \mathbf{R}^n \tag{37}$$

defines a local flow on $\alpha_{t_0}(U)$ for each $t_0 \in \mathbf{R}$, where $U = U(\varepsilon)$ is an open set on which α_{t_0} is a diffeomorphism (see Rem.8). This Φ_{t,t_0} induces a time-dependent vector field F_ε through

$$F_\varepsilon(t,x) := \frac{d}{da}\Big|_{a=t} \Phi_{a,t}(x), \quad x \in \alpha_t(U). \tag{38}$$

(ii) There exists a time-dependent vector field $\widetilde{F}_\varepsilon(t,x)$ such that

$$F_\varepsilon(t,x) = Fx + \varepsilon g_1(t,x) + \cdots + \varepsilon^m g_m(t,x) + \varepsilon^{m+1}\widetilde{F}_\varepsilon(t,x), \tag{39}$$

where $\widetilde{F}_\varepsilon(t,x)$ is a C^∞ function with respect to ε, x, t and bounded uniformly in $t \in \mathbf{R}$ with its derivatives. In particular, the vector field $F_\varepsilon(t,x)$ is close to the original vector field $Fx+\varepsilon g_1(t,x)+\cdots$ within of $O(\varepsilon^{m+1})$.

Theorem 10. (Error Estimate)

(i) Let $y(t)$ be a solution to the m-th order RG equation for Eq.(9) and α_t the m-th order RG transformation. Then, integral curves of the approximate vector field $F_\varepsilon(t,x)$ are given by

$$\widetilde{x}(t) = \alpha_t(y(t)) = X(t)y(t) + \varepsilon h_t^{(1)}(y(t)) + \cdots + \varepsilon^m h_t^{(m)}(y(t)). \tag{40}$$

(ii) There exist positive constants ε_0, C, T, and a compact subset $V = V(\varepsilon) \subset \mathbf{R}^n$ including the origin such that for $\forall|\varepsilon| < \varepsilon_0$, every solution $x(t)$ of Eq.(9) and $\widetilde{x}(t)$ defined by Eq.(40) with $x(0) = \widetilde{x}(0) \in V$ satisfy the inequality

$$||x(t) - \widetilde{x}(t)|| < C\varepsilon^m, \quad \text{for } 0 \le t \le T/\varepsilon. \tag{41}$$

The following two theorems are concerned with an autonomous equation

$$\dot{x} = Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \tag{42}$$

where $\varepsilon \in \mathbf{R}$ is a small parameter, F is a diagonalizable $n \times n$ matrix all of whose eigenvalues lie on the imaginary axis, and $g_i(x)$ are C^∞ vector fields on $\mathbf{R}^n$.

Theorem 11. (Existence of Invariant Manifolds)

Let $\varepsilon^k R_k(y)$ be a first non zero term in the RG equation (35). If the vector field $\varepsilon^k R_k(y)$ has a normally hyperbolic invariant manifold N, then the original equation (9) also has a normally hyperbolic invariant manifold N_ε, which is diffeomorphic to N, for sufficiently small $|\varepsilon|$. In particular, the stability of N_ε coincides with that of N.

Theorem 12. (Inheritance of the Symmetries)
(i) If vector fields Fx and $g_1(x), g_2(x), \cdots$ are invariant under the action of a Lie group G, then the m-th order RG equation is also invariant under the action of G.
(ii) The m-th order RG equation commutes with the linear vector field Fx with respect to Lie bracket product. Equivalently, each $R_i(y)$, $i = 1, 2, \cdots$, satisfies

$$X(t)R_i(y) = R_i(X(t)y), \quad y \in \mathbf{R}^n. \tag{43}$$

Theorem 9 means that the family of approximate solutions (40) defines the vector field $F_\varepsilon(t, x)$ and it approximates to the original vector field well. In other words, the curves (40) are solutions to the "approximate differential equation"

$$\frac{d\widetilde{x}}{dt} = F_\varepsilon(t, \widetilde{x}). \tag{44}$$

Once we obtain the approximate differential equation, subtracting the above equation from the original equation (9) and using the Gronwall's inequality, we can prove the Theorem 10. Since the approximate vector field $F_\varepsilon(t, x)$ is close to the original vector field, by virtue of the Fenichel's theorem (Thm.3), it is expected that an invariant manifold of the original vector field is inherited from that of the approximate vector field $F_\varepsilon(t, x)$. In fact, we can show that it is inherited from an invariant manifold of the RG equation (Thm.11). It is because Eqs.(37,38) show that the flow of the RG equation is topological conjugate to that of the approximate vector field. Thus, we hope that the RG equation is easier to analyze than the original equation. Theorem 12 assures it. Actually, it means that if the original equation is autonomous and invariant under the action of a k dimensional Lie group, then its RG equation is invariant under the action of a $k + 1$ dimensional Lie group. It is worth pointing out that Thm.12 (i) holds even if for non-autonomous equations, while Thm.12 (ii) holds only for autonomous equations. However, since the RG equation is an autonomous equation even if the original equation is non-autonomous, the RG equation has simpler structure than the original equation yet.

Remark 13. The infinite order RG equation and the infinite order RG transformation do not converge as power series of ε in general. However, the necessary and sufficient condition for the convergence is obtained by Chiba [6]. Roughly speaking, the infinite order RG equation converges if and only if the original equation is invariant under the action of some Lie group which is diffeomorphic to S^1. This fact is understood as follows : By Thm.12 (ii), the RG equation is invariant under the action of the fundamental matrix e^{Ft}, which is diffeomorphic to S^1. Since the approximate vector field $F_\varepsilon(t, x)$ is given as transforming the RG equation by the RG transformation (see Eqs.(37,38)), it is invariant under the S^1 action, which is obtained by transforming the e^{Ft} by the RG transformation. If the infinite order RG equation and the infinite order RG transformation converge, then the left hand side of Eq.(39) is well-defined as $m \to \infty$ and it is invariant under the action of S^1. Therefore the right hand side, which corresponds to the original equation as $m \to \infty$, is also invariant under the action of S^1.

4 Examples

In this section, we give two simple examples.

Example 14. consider the system on $\mathbf{R}^2$

$$\begin{cases} \dot{x} = y - x^3 + \varepsilon x, \\ \dot{y} = -x. \end{cases} \tag{45}$$

To bring the nonlinear term x^3 into the first order term with respect to ε, put $(x, y) = (\varepsilon^{1/2}X, \varepsilon^{1/2}Y)$. Then we obtain

$$\begin{cases} \dot{X} = Y + \varepsilon(X - X^3), \\ \dot{Y} = -X. \end{cases} \tag{46}$$

Introduce the complex variable z by $X = z + \bar{z}$, $Y = i(z - \bar{z})$ to diagonalize the unperturbed term as

$$\begin{cases} \dot{z} = iz + \dfrac{\varepsilon}{2}(z + \bar{z}) - \dfrac{\varepsilon}{2}(z + \bar{z})^3, \\ \dot{\bar{z}} = -i\bar{z} + \dfrac{\varepsilon}{2}(z + \bar{z}) - \dfrac{\varepsilon}{2}(z + \bar{z})^3. \end{cases} \tag{47}$$

For this system, the first order RG equation is given by

$$\dot{z} = \frac{\varepsilon}{2}(z - 3|z|^2 z). \tag{48}$$

Putting $z = re^{i\theta}$ yields

$$\begin{cases} \dot{r} = \dfrac{\varepsilon r}{2}(1 - 3r^2), \\ \dot{\theta} = 0. \end{cases} \tag{49}$$

Now Thm.12 (ii) means that the RG equation is invariant under the flow e^{Ft}, $F = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ defined by the harmonic oscillator (rotation invariance). Thus, written in the polar coordinate, the RG equation is split into the equation of r direction and the equation of θ direction. Generally, the RG equation for a perturbed harmonic oscillator is easily solved in the polar coordinate.

It is easy to show that this RG equation has a stable periodic orbit $r = \sqrt{1/3}$ if $\varepsilon > 0$. Now Thm.11 proves that the system (46) also has a stable periodic orbit, whose radius is of $O(1)$. By transforming it into the original (x, y) coordinate, it is shown that the system (45) has a stable periodic orbit whose radius is of $O(\varepsilon^{1/2})$. This result coincides with the classical Hopf theorem.

Example 15. Consider the system on $\mathbf{R}^2$

$$\begin{cases} \dot{x} = y + y^2, \\ \dot{y} = -x + \varepsilon^2 y - xy + y^2. \end{cases} \tag{50}$$

Changing the coordinates by $(x, y) = (\varepsilon X, \varepsilon Y)$ yields

$$\begin{cases} \dot{X} = Y + \varepsilon Y^2, \\ \dot{Y} = -X + \varepsilon(Y^2 - XY) + \varepsilon^2 Y. \end{cases} \tag{51}$$

We introduce a complex variable z by $X = z + \bar{z}$, $Y = i(z - \bar{z})$. Then, the above system is rewritten as

$$\begin{cases} \dot{z} = iz + \dfrac{\varepsilon}{2}\left(i(z-\bar{z})^2 - 2z^2 + 2z\bar{z}\right) + \dfrac{\varepsilon^2}{2}(z - \bar{z}), \\ \dot{\bar{z}} = -i\bar{z} + \dfrac{\varepsilon}{2}\left(-i(z-\bar{z})^2 - 2\bar{z}^2 + 2z\bar{z}\right) - \dfrac{\varepsilon^2}{2}(z - \bar{z}). \end{cases} \tag{52}$$

For this system, the *second order* RG equation is given by

$$\dot{z} = \frac{1}{2}\varepsilon^2(z - 3|z|^2 z - \frac{16i}{3}|z|^2 z). \tag{53}$$

Note that the first order term $R_1(y)$ vanishes. Putting $z = re^{i\theta}$ results in

$$\begin{cases} \dot{r} = \dfrac{1}{2}\varepsilon^2 r(1 - 3r^2), \\ \dot{\theta} = -\dfrac{8}{3}\varepsilon^2 r^2. \end{cases} \tag{54}$$

It is easy to verify that this RG equation has a stable periodic orbit $r = \sqrt{1/3}$. Since $R_1(y) = 0$, Thm.11 for $k = 2$ implies that the original system (50) also has a stable periodic orbit, whose radius is of $O(\varepsilon)$, for small ε. Such a case is known as the degenerate Hopf bifurcation.

5 Relation to the traditional singular perturbation methods

It is shown in [1,2,6] that the RG method unifies the traditional singular perturbation methods. In this section, we give a brief review of this fact.

5.1 Multi-scale method [18]

The multi-scale method is one of the most famous perturbation methods which is widely used. This method is based on ideas of introducing various time scales and removing secular terms. Consider the system on $\mathbf{R}^n$

$$\frac{dx}{dt} = \dot{x} = Fx + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots, \tag{55}$$

where the matrix F is a diagonalizable $n \times n$ constant matrix all of whose eigenvalues lie on the imaginary axis and where the functions $g_i(x)$'s are polynomial in x. Assume that there exist many time scales $t_0, t_1, \cdots, t_m$ satisfying

$$t_0 = t,\ t_1 = \varepsilon t, \cdots, t_m = \varepsilon^m t. \tag{56}$$

Then, d/dt is rewritten as

$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \cdots + \varepsilon^m \frac{\partial}{\partial t_m}. \tag{57}$$

Further we suppose that the dependent variable x is expanded in ε as

$$x(t) = x_0(t_0, t_1, \cdots, t_m) + \varepsilon x_1(t_0, t_1, \cdots, t_m) + \varepsilon^2 x_2(t_0, t_1, \cdots, t_m) + \cdots . \tag{58}$$

Substituting Eqs.(57, 58) into Eq.(55) yields

$$\frac{\partial x_0}{\partial t_0} = F x_0, \tag{59}$$

$$\frac{\partial x_1}{\partial t_0} + \frac{\partial x_0}{\partial t_1} = F x_1 + g_1(x_0). \tag{60}$$

The solution to the former is given by $x_0 = e^{Ft_0} y$, where the initial value $y = y(t_1, \cdots, t_m)$ depends on $t_1, \cdots, t_m$. Substituting the x_0 into Eq.(60), we obtain the general solution of x_1 as

$$x_1(t_0, \cdots, t_m) = e^{F(t_0 - \tau)} h + e^{Ft_0} \int_\tau^{t_0} e^{-Fs} \left(g_1(e^{Fs} y) - e^{Fs} \frac{\partial y}{\partial t_1} \right) ds.$$

Since the secular term arising from the integral in the right hand side is given by

$$\lim_{t_0 \to \infty} \frac{1}{t_0} \int_\tau^{t_0} e^{-Fs} \left(g_1(e^{Fs} y) - e^{Fs} \frac{\partial y}{\partial t_1} \right) ds = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_\tau^{t_0} e^{-Fs} g_1(e^{Fs} y) ds - \frac{\partial y}{\partial t_1}, \tag{61}$$

if we determine y so that the secular term vanishes, we obtain the first order RG equation $\partial y / \partial t_1 = R_1(y) \Rightarrow \partial y / \partial t = \varepsilon R_1(y)$. We can obtain the higher order RG equation in a similar manner.

5.2 Normal forms [15],[16]

For Eq.(55), if there exists a time independent (local) coordinate transformation $x \mapsto z$ such that Eq.(55) is brought into

$$\begin{cases} \dot{z} = Fz + \varepsilon \tilde{g}_1(z) + \varepsilon^2 \tilde{g}_2(z) + \cdots \\ \text{s.t. } \tilde{g}_i(e^{Ft} z) = e^{Ft} \tilde{g}_i(z), \text{ for } i = 1, 2, \cdots, \end{cases} \tag{62}$$

then Eq.(62) is called the *normal form* of Eq.(55). Since a purpose of normal forms is not obtaining approximate solutions but transforming vector fields by coordinate transformations, it answers our purpose of investigating topological properties of a given vector field.

By Thm.9, if we apply the RG transformation $x = \alpha_t(y)$ to Eq.(55), we obtain the RG equation (35). Further, because of Thm.12 (ii), changing the coordinates by $y = e^{-Ft} z$ yields the system

$$\dot{z} = Fz + \varepsilon R_1(z) + \varepsilon^2 R_2(z) + \cdots . \tag{63}$$

Since R_i commutes with e^{Ft}, this system seems to be the normal form of Eq.(55). However, we have to show that the coordinate transformation $x = \alpha_t(e^{-Ft} z)$ from Eq.(55) to Eq.(63) is independent of t. In fact, it immediately follows from the next lemma.

Lemma 16. The RG transformation α_t satisfies the equality $\alpha_t(e^{Ft'} y) = \alpha_{t+t'}(y)$.

Since this lemma shows that $\alpha_t(e^{-Ft} z) = \alpha_0(z)$ is independent of t, Eq.(63) is proved to be the normal form of Eq.(55) and the RG equation is equivalent to the normal form. Note that the RG method is applicable to non-autonomous equations while the normal forms are defined only for autonomous equations.

Remark 17. It is known that for a given equation (55), its normal forms are not unique in general. Thus, the RG equations are also not unique. The non-uniqueness results from undetermined integral constants in Eqs.(20, 22) (note that integral constants in Eqs.(19, 21) vanish as the limit $t \to \infty$). Since the integrating variable is s, we can choose arbitrary functions of y as integral constants in Eqs.(20, 22). An integral constant in the definition of $h_t^{(i)}$ affects the definitions of $R_{i+1}, R_{i+2}, \cdots$. If we choose integral constants appropriately so that $R_2, R_3, \cdots$ take the simplest forms in some sense, the resultant RG equation is proved to be equivalent to the hyper-normal forms [4].

5.3 Center manifold reduction [14]

For Eq.(55), we suppose that

(C1) all eigenvalues of the matrix F are on the imaginary axis or the left half plane. The Jordan block corresponding to eigenvalues on the imaginary axis is diagonalizable.
(C2) each $g_i(x)$ is polynomial in x.

If all eigenvalues of F are on the left half plane, the origin is stable and the flow near the origin is trivial. In what follows, we suppose that at least one eigenvalue is on the imaginary axis. In this case, Eq.(55) has a center manifold which is tangent to the center subspace at the origin. Since nontrivial phenomena such as bifurcations occur on a center manifold and orbits out of the center manifold approach to the center manifold as $t \to \infty$, it is important to investigate the flow on the center manifold. The purpose of this section is to construct a center manifold and a flow on it by the RG method.

Let N_0 be the center subspace of F, which is spanned by the eigenvectors associated with the eigenvalues on the imaginary axis. For Eq.(55), we define the functions $R_i : N_0 \to \mathbf{R}^n$ and $h_t^{(i)} : N_0 \to \mathbf{R}^n$, $i = 1, 2, \cdots$ to be

$$R_1(y) := \lim_{t \to -\infty} \frac{1}{t} \int^t X(s)^{-1} g_1(s, X(s)y) ds, \tag{64}$$

$$h_t^{(1)}(y) := X(t) \int^t \Big(X(s)^{-1} g_1(s, X(s)y) - R_1(y) \Big) ds, \tag{65}$$

and

$$\begin{aligned} R_i(y) := \lim_{t \to -\infty} \frac{1}{t} \int^t & \Big(X(s)^{-1} G_i(s, X(s)y, h_s^{(1)}(y), \cdots, h_s^{(i-1)}(y)) \\ & - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) \Big) ds, \end{aligned} \tag{66}$$

$$\begin{aligned} h_t^{(i)}(y) := X(t) \int^t & \Big(X(s)^{-1} G_i(s, X(s)y, h_s^{(1)}(y), \cdots, h_s^{(i-1)}(y)) \\ & - X(s)^{-1} \sum_{k=1}^{i-1} (Dh_s^{(k)})_y R_{i-k}(y) - R_i(y) \Big) ds, \end{aligned} \tag{67}$$

for $i = 2, 3, \cdots$, respectively. They are the same as Eqs.(19,20,21,22), except that the domain is restricted to the center subspace N_0. In the above definitions, the integral constants are assumed to be zero. We can prove the next lemma.

Lemma 18. The functions $R_1(y), R_2(y), \cdots$ are well-defined (namely, the limits converge) and the following holds.
(i) $R_i(y) \in N_0$ for all $y \in N_0$, $i = 1, 2, \cdots$.
(ii) $h_t^{(i)}(y)$ is bounded uniformly in $t \in \mathbf{R}$ for all $y \in N_0$, $i = 1, 2, \cdots$.

With these $R_i, h_t^{(i)}$, we define the *m-th order RG equation on* N_0 to be

$$\dot{y} = \varepsilon R_1(y) + \varepsilon^2 R_2(y) + \cdots + \varepsilon^m R_m(y), \quad y \in N_0, \tag{68}$$

and define the *m-th order RG transformation on* N_0 $\alpha_t : N_0 \to \mathbf{R}^n$ by

$$\alpha_t(y) = X(t)y + \varepsilon h_t^{(1)}(y) + \cdots + \varepsilon^m h_t^{(m)}(y). \tag{69}$$

They are the same as Eqs.(35,36), except that the domain is restricted to the center subspace N_0 as before. Since Lem.18 shows that the domain and the range of $R_i(y)$ are N_0, Eq.(68) defines the differential equations on N_0. It means that the system (68) has $\dim N_0$ linearly independent equations.

In this situation, Thm.9 to Thm.12 hold on N_0. Further, we can prove the next theorem.

Theorem 19. (Approximation of Center Manifolds, [5])
Let α_t be the m-th order RG transformation on N_0 and W a compact neighborhood of the origin such that α_t is diffeomorphism on $W \cap N_0$ (see Rem.8). Then, the set $\alpha_t(W \cap N_0)$ lies within an $O(\varepsilon^{m+1})$ neighborhood of the center manifold of Eq.(55).

5.4 Averaging method [17]

Consider the system on a manifold M

$$\dot{x} = \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots, \tag{70}$$

where each g_i is a time-dependent smooth vector field on M, which is almost periodic in t, the set of whose Fourier exponents has no accumulation points on $\mathbf{R}$. For this system, we define the maps $R_i, u_t^{(i)} : M \to M$ to be

$$R_1(y) = \lim_{t\to\infty} \frac{1}{t} \int^t g_1(s, y) ds, \tag{71}$$

$$u_t^{(1)}(y) = \int^t (g_1(s, y) - R_1(y))\, ds, \tag{72}$$

and

$$R_i(y) = \lim_{t\to\infty} \frac{1}{t} \int^t \Big(G_i(s, y, u_s^{(1)}(y), \cdots, u_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} (Du_s^{(k)})_y R_{i-k}(y)\Big) ds, \tag{73}$$

$$u_t^{(i)}(y) = \int^t \Big(G_i(s, y, u_s^{(1)}(y), \cdots, u_s^{(i-1)}(y)) - \sum_{k=1}^{i-1} (Du_s^{(k)})_y R_{i-k}(y) - R_i(y)\Big) ds, \tag{74}$$

for $i = 2, 3, \cdots$, respectively. Define the m-th order RG equation by Eq.(35) and the m-th order RG transformation by

$$\alpha_t(y) = y + \varepsilon u_t^{(1)}(y) + \cdots + \varepsilon^m u_t^{(m)}(y). \tag{75}$$

In this situation, we can prove Thm.9 to Thm.12(i).

Remark 20. Consider the system of the form

$$\dot{x} = f(x) + \varepsilon g_1(t, x) + \varepsilon^2 g_2(t, x) + \cdots . \tag{76}$$

Let φ_t be the flow of the vector field f and suppose that it is almost periodic in t. Then, if we change the coordinates as $x = \varphi_t(X)$, the above system is brought into the system

$$\dot{X} = \varepsilon(D\varphi_t)_X^{-1} g_1(t, \varphi_t(X)) + \varepsilon^2 (D\varphi_t)_X^{-1} g_2(t, \varphi_t(X)) + \cdots , \tag{77}$$

which is of the form of Eq.(70). Thus, the RG method in the present section gives extension of those in the cases of the previous sections. The RG equation in this section is proved to be equivalent to the averaging equation in the averaging method.

6 Analysis of the Kuramoto model

The Kuramoto model of coupled phase oscillators

$$\dot{\theta}_i = \omega_i + \frac{\varepsilon}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \; i = 1, \cdots , N \tag{78}$$

is one of the most studied models of nonlinear phenomena of globally coupled limit-cycle oscillators [19], where $\theta_i \in S^1$ is a particle on a circle, ω_i is a constant called the *natural frequency*, N is a number of particles, and ε is the coupling strength. It is known numerically that if ε is larger than the threshold ε_0, then $< \dot{\theta}_i > - < \dot{\theta}_j >$ tends to zero as $t \to \infty$ for all i, j, where $< >$ denotes averaging over time. Such a phenomenon is called the *synchronization*. However, mechanism of the transition from the coupled harmonic oscillators, namely $\varepsilon = 0$ (uncoupled system), to the synchronization state is not well understood [20]. Most recently, bifurcation diagrams of the Kuramoto model for small N and small ε were investigated by Maistrenko *et al.* [21,22] and Popovych *et al.* [23], in which natural frequencies are assumed to be distributed symmetrically around a mean frequency Ω :

$$\omega_i - \Omega = -(\omega_{N-i+1} - \Omega), \; i = 1, 2, \cdots , N. \tag{79}$$

Note that we can put $\Omega = 0$ without loss of generality because the Kuramoto model is invariant under the rotation $\theta_i \mapsto \theta_i + \Omega$. In this case, it is easy to show that the Kuramoto model has the invariant torus M defined by

$$M = \{\theta_i = -\theta_{N-i+1}, \; i = 1, \cdots , N\}. \tag{80}$$

It is important to determine the stability of M because the synchronization solutions are always restricted to the torus [22]. Since M is neutrally stable when $\varepsilon = 0$, the (in)stability of M is quite weak when ε is small and in this case it is difficult to determine the stability by numerical simulation. In this article, we apply the RG method to the Kuramoto model to determine the stability of M for small ε. We can prove the following results.

Theorem 21. Suppose that $N = 2M - 1$ is an odd number. Suppose the natural frequencies satisfy the symmetric condition (79) (we put $\Omega = 0$) and the following *nonresonance condition* :

$$
\begin{aligned}
&\omega_i \neq \omega_j \text{ for all } i, j,\\
&\omega_k + \omega_j = 2\omega_i \text{ if and only if } i = k = j \text{ or } j = 2M - k, i = M,\\
&\omega_i + \omega_j = \omega_k + \omega_l \text{ if and only if } i = j = k = l \text{ or } j = 2M - i, l = 2M - k,\\
&3\omega_i = \omega_j + \omega_k + \omega_l \text{ if and only if } i = j = k = l,\\
&\omega_i + 2\omega_k = \omega_j + 2\omega_l \text{ if and only if } i = j, k = l \text{ or } j = 2M - i, k = M, l = i.
\end{aligned}
$$

Then there exists positive constant ε_0, which depends on the natural frequencies, such that if $0 < \varepsilon < \varepsilon_0$, the invariant torus M is stable and the transverse Lyapunov exponents of M is of $O(\varepsilon^3)$.

When $N = 3$, the nonresonance condition is violated if and only if $\omega_1 = \omega_2 = \omega_3 = 0$. However, in this case, the phase portrait of the Kuramoto model is independent of ε because we can divide the right hand side of Eq.(78) by ε by changing the time scale so that the system is independent of ε. Similarly, if $\omega_i = 0$ for all i, the phase portrait of the Kuramoto model is independent of ε. Otherwise, for $N = 5$, the nonresonance condition is violated if and only if

$$\omega_1 = 0,\ \omega_2,\ 3\omega_2/2,\ 2\omega_2,\ 3\omega_2,\ 4\omega_2,\ 5\omega_2. \tag{81}$$

In these cases, the RG equations take different forms from Eq.(83), which is the RG equation for the nonresonance case. To determine the stability of M for the above cases, deriving the RG equations for individual resonance cases and investigating them, we can prove the next theorem.

Theorem 22. Suppose that $N = 5$ and the natural frequencies satisfy the symmetric condition (79). Then the invariant torus M is stable for sufficiently small ε. In particular, if all of the natural frequencies are not identical, the transverse Lyapunov exponents of M is of $O(\varepsilon^3)$.

In this article, we give the proof of Thm.21. The proof of Thm.22 needs more hard analysis and it is omitted here.

Proof of Thm.21 To write down the system (78) in the Cartesian coordinate, put $x_i = \cos\theta_i$, $y_i = \sin\theta_i$. Further putting $x_i = z_i + \bar{z}_i$, $y_i = i(z_i - \bar{z}_i)$, we obtain the system of the form

$$
\left\{
\begin{aligned}
\dot{z}_i &= -i\omega_i z_i + \frac{2\varepsilon}{N}\sum_{j=1}^{N} z_i(\bar{z}_i z_j - z_i \bar{z}_j),\\
\dot{\bar{z}}_i &= i\omega_i \bar{z}_i + \frac{2\varepsilon}{N}\sum_{j=1}^{N} \bar{z}_i(z_i \bar{z}_j - \bar{z}_i z_j).
\end{aligned}
\right. \tag{82}
$$

Since this system is the perturbed harmonic oscillators whose unperturbed term has eigenvalues $\pm i\omega_1, \cdots, \pm i\omega_N$, we can apply the RG method to this system. After deriving the 3-rd order RG equation for Eq.(82), we put $z_i = e^{i\theta_i}$ to change to the polar coordinate (For the RG equation, we use the same notations z_i, θ_i with those of the original system). Then we obtain the RG equation of the form

$$\left\{\begin{array}{l} \dot{\theta}_M = -\dfrac{16\varepsilon^3}{N^3}\displaystyle\sum_{k\neq M}\dfrac{1}{\omega_k^2}\sin(2\theta_M - \theta_k - \theta_{2M-k}), \\ \dot{\theta}_i = \dfrac{8\varepsilon^2}{N^2}\left(2\displaystyle\sum_{k\neq i}\dfrac{1}{\omega_i - \omega_k} - \dfrac{1}{\omega_i}\cos(\theta_i - 2\theta_M + \theta_{2M-i})\right) \\ \quad +\dfrac{16\varepsilon^3}{N^3}\left(\displaystyle\sum_{k\neq i,2M-i}\dfrac{1}{\omega_i^2-\omega_k^2}\sin(\theta_i - \theta_k - \theta_{2M-k} + \theta_{2M-i}) - 2\displaystyle\sum_{k\neq i,2M-i}\dfrac{1}{\omega_i(\omega_i-\omega_k)}\sin(\theta_i - \theta_k - \theta_{2M-k} + \theta_{2M-i})\right. \\ \quad +2\displaystyle\sum_{k\neq i,M}\dfrac{1}{\omega_k(\omega_i - \omega_k)}\sin(2\theta_M - \theta_k - \theta_{2M-k}) - \displaystyle\sum_{k\neq M,i,2M-i}\dfrac{1}{\omega_k(\omega_i + \omega_k)}\sin(\theta_i - \theta_k - \theta_{2M-k} + \theta_{2M-i}) \\ \quad \left. -2\displaystyle\sum_{k\neq M,2M-i}\dfrac{1}{\omega_i(\omega_i + \omega_k)}\sin(\theta_i - 2\theta_M + \theta_{2M-i})\right), \quad (i \neq M). \end{array}\right. \tag{83}$$

Note that the first order term vanishes. Since the invariant torus M corresponds to the solution $\theta_i + \theta_{2M-i} = 0$, we put $\phi_i = \theta_i + \theta_{2M-i}$ and $\phi_M = 2\theta_M$. Then we obtain the system of ϕ_i

$$\left\{\begin{array}{l} \dot{\phi}_M = -\dfrac{64\varepsilon^3}{N^3}\displaystyle\sum_{k=1}^{M-1}\dfrac{1}{\omega_k^2}\sin(\phi_M - \phi_k), \\ \dot{\phi}_i = \dfrac{32\varepsilon^3}{N^3}\left(-\dfrac{1}{\omega_i^2}\sin(\phi_i - \phi_M) - 4\displaystyle\sum_{k\neq i}^{M-1}\dfrac{1}{\omega_i^2 - \omega_k^2}\sin(\phi_i - \phi_M)\right. \\ \qquad \left. +4\displaystyle\sum_{k\neq i}^{M-1}\dfrac{1}{\omega_i^2 - \omega_k^2}\sin(\phi_M - \phi_k)\right), \quad (i = 1, \cdots, M-1). \end{array}\right. \tag{84}$$

Now that the second order term vanishes, Thm.11 for $k = 3$ is applicable to this system. We can prove that the eigenvalues of the Jacobian matrix at the fixed point $\phi_i = 0$ $(i = 1, \cdots, M)$ of the right hand side of Eq.(84) have negative real parts except to a zero eigenvalue, which results from the rotation invariance of Eq.(78). Thus, the solution $\phi_i = \theta_i + \theta_{2M-i} = 0$ $(i = 1, \cdots, M)$ of the RG equation is stable and this proves that the invariant torus M is stable for small $\varepsilon > 0$.

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