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Some vector-valued theta series on $U(2, 2)$ and $Sp(1, 1)$

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This is an overview of the paper [5], which is a joint work with Hiro-aki NARITA.
In this talk, we will first construct vector-valued (holomorphic) singular modular forms
on $U(2, 2)$, which can be regarded as generalizations of theta series. The restrictions of these
singular forms to $Sp(1, 1)$ are modular forms which generate quaternionic discrete series.

1 The construction of vector-valued singular forms on $U(2, 2)$

First of all, let us construct vector-valued singular modular forms on $U(2, 2)$. Put

$$U(2, 2) = \left\{ g \in GL(4, \mathbb{C}) \mid g \begin{pmatrix} 0 & -12 \\ 12 & 0 \end{pmatrix} \overline{g} = \begin{pmatrix} 0 & -12 \\ 12 & 0 \end{pmatrix} \right\},$$

$$\mathcal{H}_{U(2,2)} = \{ z \in M_{2}(\mathbb{C}) \mid \sqrt{-1} (\overline{tz} - z) > 0 \}.$$ 

As is well-known, the group $U(2, 2)$ acts on $\mathcal{H}_{U(2,2)}$ as $\begin{pmatrix} \alpha_{1} \alpha_{2} \\ \alpha_{3} \alpha_{4} \end{pmatrix} (z) = (\alpha_{1}z + \alpha_{2})(\alpha_{3}z + \alpha_{4})^{-1}$, and we put $\mu_{1} \begin{pmatrix} \alpha_{1} \alpha_{2} \\ \alpha_{3} \alpha_{4} \end{pmatrix}, z) = \alpha_{3}z + \alpha_{4}$.

For any non-negative integer $\kappa$, we denote by $V_{\kappa}$ the space of homogeneous polynomials
of degree $\kappa$ with two variables. (Note that $\dim_{\mathbb{C}} V_{\kappa} = \kappa + 1$.) We define a map $\sigma_{\kappa} : M_{2}(\mathbb{C}) \rightarrow \text{End}(V_{\kappa})$ by

$$(\sigma_{\kappa}(h) v) (X, Y) = v(aX + cY, bX + dY) \quad \left( h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2}(\mathbb{C}) \right).$$

This induces an irreducible rational representation $(\sigma_{\kappa}, V_{\kappa})$ of $GL_{2}(\mathbb{C})$. The space $V_{\kappa}$ has a
basis $\{ v_{i} = v_{i}(X, Y) := X^{i}Y^{\kappa-i} \mid 0 \leq i \leq \kappa \}$. The vector $v_{\kappa}$ is a highest weight vector.

Take an imaginary quadratic field $K$ and fix it. For any $r, s \in K^{2}$ and any $Z$-lattice $L$
in $K^{2}$, we define a $V_{\kappa}$-valued holomorphic function $\theta^{(\kappa)}(z; L, r, s)$ on $z \in \mathcal{H}_{U(2,2)}$ as

$$\theta^{(\kappa)}(z; L, r, s) = \sum_{X \in \mathbb{Z} L + r} \exp \left( \pi \sqrt{-1} \text{Tr}_{K/Q}(t \overline{x}s) \right) \exp \left( \pi \sqrt{-1} \overline{x}zx \right) \sigma_{\kappa}( (x, *) ) \cdot v_{\kappa}.$$
Here \((x, *)\) denotes an arbitrary \(2 \times 2\)-matrix whose first column is \(x\), since \(\sigma_\kappa((x, *)) \cdot v_\kappa\) does not depend on \(*\).

**Theorem 1.1.** The function \(\theta^{(\kappa)}(z; L, r, s)\) is a modular form of weight \(\sigma_\kappa \otimes \det\). It means, there exists a congruence subgroup \(\Gamma\) of \(U(2, 2 : K)(:= U(2, 2) \cap GL_4(K))\) so that

\[
\theta^{(\kappa)}(\gamma(z); L, r, s) = \det(\mu_1(\gamma, z))\sigma_\kappa(\mu_1(\gamma, z))\theta^{(\kappa)}(z; L, r, s)
\]

for any \(\gamma \in \Gamma\).

**Remark 1.** Note that \(\theta^{(\kappa)}(z; L, r, s)\) is so-called a singular form since all of its Fourier coefficients at non-degenerate indices are 0.

**Remark 2.** In case \(\kappa = 0\), the scalar valued function \(\theta^{(0)}(z; L, r, s)\) is a pull-back of a theta series on \(Sp(4, \mathbb{Q})\), and a modular form of weight 1.

This theorem can be proved in the same way as the proof of automorphy of scalar-valued theta series.

## 2 Restriction to \(Sp(1, 1)\)

Let \(B\) be a definite quaternion algebra over \(\mathbb{Q}\), and \(\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}ij\) be the Hamilton quaternion algebra. Since \(B \otimes_\mathbb{Q} \mathbb{R} = \mathbb{H}\), we regard \(B\) as a dense subset of \(\mathbb{H}\). The main involution of \(\mathbb{H}\) is written as \(\tau \rightarrow \overline{\tau}\) for \(\tau \in \mathbb{H}\), and the reduced norm and the reduced trace are defined as \(N(\tau) = \tau \overline{\tau}\) and \(tr(\tau) = \tau + \overline{\tau}\). We write \(\mathbb{H}^- = \{\tau \in \mathbb{H} | tr(\tau) = 0\}\) and \(B^- = B \cap \mathbb{H}^-\).

Put

\[
Sp(1, 1)_\mathbb{R} := \left\{ g \in M_2(\mathbb{H}) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t \overline{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},
\]

\[
Sp(1, 1)_\mathbb{Q} = Sp(1, 1)_\mathbb{R} \cap M_2(B).
\]

The canonical maximal compact subgroup of \(Sp(1, 1)_\mathbb{R}\) is given as

\[
K_{Sp(1, 1)} = \left\{ g \in Sp(1, 1)_\mathbb{R} \mid g = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ with } a, b \in \mathbb{H} \right\}.
\]

Hence the symmetric space of \(Sp(1, 1)_\mathbb{R}\) is

\[
\mathfrak{H}_{Sp(1, 1)} = \{\tau \in \mathbb{H} | tr(\tau) > 0\} \cong Sp(1, 1)_\mathbb{R} / K_{Sp(1, 1)},
\]

and the factor of automorphy \(\mu_2(g, \tau)\) is given as

\[
\mu_2 \left( \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \tau \right) = \alpha_3 \tau + \alpha_4 \quad \text{for} \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \in Sp(1, 1)_\mathbb{R} \text{ and } \tau \in \mathfrak{H}_{Sp(1, 1)}.
\]
For given integer $\kappa \geq 2$, $\xi \in B^-$, $p, q \in B$ and $\mathbb{Z}$-lattice $\Lambda \subset B$, we define a $V_\kappa$-valued function $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ on $g \in Sp(1,1)_R$ as
\[
\theta_{\xi}^{(\kappa)} \left( \begin{bmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{bmatrix} k_\infty; \Lambda, p, q \right) 
:= \sum_{\lambda \in \Lambda + p} \exp(\pi \sqrt{-1 \text{tr}(\lambda q)}) y^{\frac{\kappa}{2} + \frac{1}{2}} \exp(-4\pi \sqrt{N(\lambda \xi \overline{\lambda})} y) \exp(2\pi \sqrt{-1 \text{tr}(\lambda \xi \overline{\lambda} x))}\sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1} \sigma_\kappa(\iota(\lambda u_\xi)) u_\kappa,
\]
where $x \in \mathbb{H}^-$, $y \in \mathbb{R}^+$ and $k_\infty \in K_{Sp(1,1)}$. The symbol $\iota$ denotes the canonical embedding of $\mathbb{H}$ into $M_2(\mathbb{C})$, and $u_\xi \in \mathbb{H}$ is as $u_\xi i\overline{u_\xi} = \frac{\xi}{\sqrt{N(\xi)}}$.

**Theorem 2.1.** The function $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ is a modular form of weight $\sigma_\kappa$ with respect to some congruence subgroup of $Sp(1,1)_\mathbb{Q}$, generating a quaternionic discrete series.

**Sketch of proof.** The automorphy of $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ comes from the fact that it is a pull-back of $\theta^{(\kappa)}(z; L, r, s)$ on $U(2,2 : K)$ with $K := \mathbb{Q}(\sqrt{-N(\xi)}) \cong \mathbb{Q}(\xi)$ and suitable $L, r, s$.

**Remark 1.** Note that the Fourier coefficients of $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ are 0 outside the single-orbit of indices $\{\lambda \xi \overline{\lambda} | \lambda \in B^x\}$, (and so are those of $\theta_{\xi}^{(\kappa)}(\alpha g; \Lambda, p, q)$ for any $\alpha \in Sp(1,1)_\mathbb{Q}$.) We should add that the subalgebra $\mathbb{Q}(\lambda \xi \overline{\lambda})$ is isomorphic to $\mathbb{Q}(\xi) \cong K$ for any $\lambda \in B^x$.

**Remark 2.** We can take $p \in B$ so that $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q) \neq 0$ for given $\xi, \Lambda$ and $q$. In fact, there is $p \in B \setminus \{0\}$ which satisfies $N(p) < \frac{1}{4} N(l)$ for any $l \in \Lambda \setminus \{0\}$. Then the Fourier coefficient indexed by $p\xi \overline{p}$ is $\exp(\pi \sqrt{-1 \text{tr}(pq)})\sigma_\kappa(\iota(p u_\xi)) u_\kappa \neq 0$ since $N(p) < N(p + l)$ holds for any $l \in \Lambda \setminus \{0\}$.

In the end, we should note that the Fourier coefficients of $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ are essentially algebraic. Take any $0 \neq \eta \in B^-$ such that $\xi \eta \in B^-$. Then the set $\{1, \xi, \eta, \xi \eta\}$ is a basis of $B$ over $\mathbb{Q}$. We take $u_{\xi, \eta} \in \mathbb{H}$ which satisfies $u_{\xi, \eta} i \overline{u_{\xi, \eta}} = \frac{\xi}{\sqrt{N(\xi)}}$ and $u_{\xi, \eta} j \overline{u_{\xi, \eta}} = \frac{\eta}{\sqrt{N(\eta)}}$. Note that such $u_{\xi, \eta}$ exists and is uniquely determined up to the multiple of $\{\pm 1\}$. From now on, we replace $u_\xi$ by $u_{\xi, \eta}$ in the definition of $\theta_{\xi}^{(\kappa)}$. (By this replacement, the function $\theta_{\xi}^{(\kappa)}$ changes just a constant multiple.) We put $V_\kappa(\mathbb{Q}) = \sum_{i=0}^\kappa \mathbb{Q} v_i$ and $V_{\kappa, B}(\mathbb{Q}) = \sigma_\kappa(\iota(u_{\xi, \eta})) V_\kappa(\mathbb{Q})$. It can be verified that $V_{\kappa, B}(\mathbb{Q})$ does not depend on the choice of $(\xi, \eta)$. Then we have the following.

**Corollary 2.2.** If we write the Fourier expansion of $\theta_{\xi}^{(\kappa)}(g; \Lambda, p, q)$ as
\[
\theta_{\xi}^{(\kappa)} \left( \begin{bmatrix} \sqrt{y} & x(\sqrt{y})^{-1} \\ 0 & (\sqrt{y})^{-1} \end{bmatrix} k_\infty; \Lambda, p, q \right) 
= \sum_{\delta \in B^-} \sigma_\kappa(\iota(\mu_2(k_\infty, 1)))^{-1} C_{\delta y} y^{\frac{\kappa}{2} + 1} \exp(-4\pi \sqrt{N(\delta)} y) \exp(2\pi \sqrt{-1 \text{tr}(\delta x))},
\]
with $C_{\delta} \in V_{\kappa}$, then $C_{\delta} \in V_{\kappa,B}(\overline{\mathbb{Q}})$ for any $\delta \in B^{-}$.

References


[4] H.Narita, Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$, to appear in Math. Z.


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