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Vector valued Siegel modular forms of degree 2 with small levels

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1 Introduction

On the structure theorem of Siegel modular forms of degree 2, Igusa [Ig1, Ig2] determined the structure of Siegel modular forms with respect to the full modular group $\mathrm{Sp}(2,\mathbb{Z})$. There are five generators of weight $4, 6, 10, 12$ and 35. First four generators are algebraically independent and the square of the last generator is in the subring generated by first four.

Recently, Aoki and Ibukiyama [AI] indicated that the ring of Siegel modular forms with small level has similar structure. That is, on the ring of Siegel modular forms of degree 2 with respect to the congruent subgroup of level $N = 1, 2, 3, 4$ (for $N = 3, 4$, taking Neven-type case with character), there are five generators, among which four generators are algebraically independent and the square of the last generator is in the subring generated by first four.

On the structure of vector valued Siegel modular forms of degree 2 with respect to the symmetric tensor of degree 2, Satoh [Sa] and Ibukiyama [Ib3] determined the structure with respect to the full modular group. There are ten generators with some relations.

The original proofs of above structure theorems are various. However, now we can prove all of them by using the elementary estimation of the dimension of the space of Siegel modular forms. In this exposition, we study this method.

By this method, we also determined the structure of vector valued Siegel modular forms with small level. This structure is similar to the structure with respect to the full modular group.

2 Main theorem

2.1 Complex scalar valued case

We denote the Siegel upper half plane of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) \;\middle|\; \operatorname{Im} Z > 0 \right\}.$$

The symplectic group

$$\mathrm{Sp}(2,\mathbb{R}) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}_4(\mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O_2 & -E_2 \\ E_2 & O_2 \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ transitively by

$$\mathbb{H}_2 \ni Z \mapsto M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} \in \mathbb{H}_2.$$

For $M \in \mathrm{Sp}(2,\mathbb{R})$, $k \in \mathbb{Z}$ and a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$, we write

$$(F|_kM)(Z) := \det(CZ+D)^{-k}F(M\langle Z\rangle).$$

Let

$$\mathrm{Sp}(2,\mathbb{Z}) := \mathrm{Sp}(2,\mathbb{R}) \cap \mathrm{M}_4(\mathbb{Z})$$

and $\Gamma \subset \mathrm{Sp}(2,\mathbb{R})$ be a commensurable subgroup with $\mathrm{Sp}(2,\mathbb{Z})$, namely, $\Gamma \cap \mathrm{Sp}(2,\mathbb{Z})$ is a finite index subgroup of Γ and also a finite index subgroup of $\mathrm{Sp}(2,\mathbb{Z})$.

Definition 1. *For a holomorphic function $F : \mathbb{H}_2 \to \mathbb{C}$ and $k \in \mathbb{Z}$, we say F is a Siegel modular forms of weight k with respect to Γ if F satisfies the condition $F(Z) = (F|_kM)(Z)$ for any $M \in \Gamma$.*

We remark that this F is bounded at each cusps by Köcher principle. We denote by $A_k(\Gamma)$ the space of all Siegel modular forms of weight k with respect to Γ. The space $A_*(\Gamma) := \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma)$ is a graded ring.

Put

$$\Gamma_0(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2,\mathbb{Z}) \;\middle|\; C \equiv O_2 \pmod{N} \right\}$$

for any natural number $N \in \mathbb{N} := \{1,2,3,\ldots\}$.

In this exposition, our interest is the case $N = 1,2,3,4$. When $N = 3,4$, we take a character because the structure theorem become simple. That is, for $N = 1,2$, we assume $\Gamma := \Gamma_0(N)$ and for $N = 3,4$, we assume

$$\Gamma := \Gamma_{0,\psi_N}(N) := \left\{ M \in \Gamma_0(N) \;\middle|\; \psi_N(M) = 1 \right\},$$

where we denote by ψ_3 the character defined by $\psi_3(M) = \left(\frac{-3}{\det(D)}\right)$ and by ψ_4 the character defined by $\psi_4(M) = \left(\frac{-1}{\det(D)}\right)$.

In these cases, the structure of $A_*(\Gamma)$ is already known.

Theorem 1. *For each $\Gamma = \mathrm{Sp}(2,\mathbb{Z}), \Gamma_0(2), \Gamma_{0,\psi_3}(3)$ or $\Gamma_{0,\psi_4}(4)$, the graded ring $A_*(\Gamma)$ is generated by five modular forms. First four generators are algebraically independent and the square of the last generator is in the subring generated by first four.*

Γ	The weights of first four generators	The weights of the last generators	References
$\mathrm{Sp}(2,\mathbb{Z})$	$4, 6, 10, 12$	35	Igusa [Ig1, Ig2]
$\Gamma_0(2)$	$2, 4, 4, 6$	19	Ibukiyama [Ib1]
$\Gamma_{0,\psi_3}(3)$	$1, 3, 3, 4$	14	Ibukiyama [Ib1] Aoki-Ibukiyama [AI]
$\Gamma_{0,\psi_4}(4)$	$1, 2, 2, 3$	11	Hayashida-Ibukiyama [HI]

We remark that, in all cases, the last generators are obtained from the first four using by Rankin-Cohen-Ibukiyama differential operators in [AI].

2.2 Vector valued case

Let s be a non-negative integer, V be a $(s+1)$-dimensional $\mathbb{C}$-vector space and $\rho : \mathrm{GL}(2,\mathbb{C}) \longrightarrow \mathrm{GL}(V)$ be a rational representation. It is well-known that ρ is a rational irreducible representation if and only if $\rho = \rho_{k,s} := \mathrm{Sym}^s \otimes \det^k$. For the sake of simplicity, in this exposition, we fix a coordinate of $\mathrm{Sym}^s \otimes \det^k$. Namely, put $V := \mathbb{C}^{s+1}$ and $\rho_{k,s}(A) := (\det A)^k \rho_{0,s}(A)$, where $\rho_{0,s}(A)$ is defined by

$$(u^s, u^{s-1}v, \ldots, v^s) = (x^s, x^{s-1}y, \ldots, y^s)\rho_{0,s}(A) \quad ((u,v) = (x,y)A).$$

For $M \in \mathrm{Sp}(2,\mathbb{R})$ and a holomorphic function $F : \mathbb{H}_2 \longrightarrow \mathbb{C}^{s+1}$, we write

$$(F|_\rho M)(Z) := \rho(CZ+D)^{-1}F(M\langle Z\rangle).$$

Definition 2. *We say F is a Siegel modular forms of weight ρ with respect to Γ if F satisfies the condition $F(Z) = (F|_\rho M)(Z)$ for any $M \in \Gamma$.*

We remark that this F is bounded at each cusps by Köcher principle. We denote by $A_{k,s}(\Gamma)$ the space of all Siegel modular forms of weight $\rho_{k,s}$ with respect to Γ. We remark $A_{k,0}(\Gamma) = A_k(\Gamma)$. It is easy to show that if s is odd and if $-E_4 \in \Gamma$, then $A_{k,s}(\Gamma) = \{0\}$. Put $A_{*,s}(\Gamma) := \bigoplus_{k\in\mathbb{Z}} A_{k,s}(\Gamma)$. The space $A_{*,s}(\Gamma)$ is a graded module of $A_*(\Gamma)$ or R, where R is a subring of $A_*(\Gamma)$ generated by the first four generators in **Theorem 1**.

The aim of this exposition is to determine the structure of $A_{*,2}(\Gamma)$ as a graded module of R. The structure of $A_{*,2}(\mathrm{Sp}(2,\mathbb{Z}))$ was already determined by Satoh [Sa] and Ibukiyama [Ib3]. There are ten generators, whose weights are

$$\begin{array}{lllr} 10 = 4+6, & 16 = 6+10, & & 21 = 4+6+10+1, \\ 14 = 4+10, & 18 = 6+12, & & 23 = 4+6+12+1, \\ 16 = 4+12, & 22 = 10+12, & & 27 = 4+10+12+1 \\ & & \text{and} & 29 = 6+10+12+1. \end{array}$$

To show this, they used the dimension formula of modular forms. In this exposition we will give this result by another way. By our way, we can determine the module structure of $A_{*,2}(\Gamma)$ for $\Gamma = \Gamma_0(2), \Gamma_{0,\psi_3}(3)$ or $\Gamma_{0,\psi_4}(4)$.

Theorem 2. *For each $\Gamma = \mathrm{Sp}(2,\mathbb{Z}), \Gamma_0(2), \Gamma_{0,\psi_3}(3)$ or $\Gamma_{0,\psi_4}(4)$, the graded module $A_{*,2}(\Gamma)$ is generated by ten modular forms.*

Γ	The weights of generators (Type 1)	The weights of generators (Type 2)	References
$\mathrm{Sp}(2,\mathbb{Z})$	$10, 14, 16, 16, 18, 22$	$21, 23, 27, 29$	Satoh [Sa] Ibukiyama [Ib3]
$\Gamma_0(2)$	$6, 6, 8, 8, 10, 10$	$11, 13, 13, 15$	Aoki [Ao]
$\Gamma_{0,\psi_3}(3)$	$4, 4, 5, 6, 7, 7$	$8, 9, 9, 11$	
$\Gamma_{0,\psi_4}(4)$	$3, 3, 4, 4, 5, 5$	$6, 7, 7, 8$	

We remark two points. The first point is, in all cases, these generators are obtained from the generators of R using by differential operators. Indeed, the first six generators are obtained from two generators of R using by Rankin-Cohen type differential operators in [Sa]. And the last four generators are obtained from two generators of R using by Rankin-Cohen-Ibukiyama type differential operators in [Ib3]. The second point is, in all cases, these modules are not free. There are relations called Jacobi Identities.

3 Proof

For the sake of simplicity, in this exposition, we give a proof only on the simplest case: scalar valued full modular case. Hence, from now on, we assume $\Gamma := \mathrm{Sp}(2, \mathbb{Z})$ and $s = 0$. But we insist that our proof is available for all cases in **Theorem 1** and **Theorem 2.**

Anyway, to prove the theorem, we prepare some notations. Let $\widetilde{\Gamma} := \mathrm{SL}(2, \mathbb{Z})$, $q := \mathbf{e}(\tau) := \exp(2\pi\sqrt{-1}\tau)$, $\zeta := \mathbf{e}(z)$ and $p := \mathbf{e}(\omega)$.

3.1 Elliptic modular forms

We denote the complex upper half plane by

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}.$$

For a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$, we say f is an elliptic modular form of weight k with respect to $\widetilde{\Gamma}$ if f satisfies the following two conditions:

(1) For any $M \in \widetilde{\Gamma}$, $f|_k M = f$.

(2) f is bounded at all the cusps.

Let $a(n)$ be the Fourier coefficients of f defined by

$$f(\tau) = \sum_{n=0}^{\infty} a(n) q^n.$$

We denote by $\mathrm{M}_k(\widetilde{\Gamma})$ the space of all elliptic modular forms of weight k with respect to $\widetilde{\Gamma}$. Put $\mathrm{M}_*(\widetilde{\Gamma}) := \bigoplus_{k\in\mathbb{Z}} \mathrm{M}_k(\widetilde{\Gamma})$. The space $\mathrm{M}_*(\widetilde{\Gamma})$ is a graded ring. For $r \in \mathbb{N} \cup \{0\}$, define subspaces of $\mathrm{M}_k(\widetilde{\Gamma})$ by

$$\mathrm{M}_k(\widetilde{\Gamma}; r) := \left\{ f \in \mathrm{M}_k(\widetilde{\Gamma}) \;\middle|\; a(n) = 0 \text{ if } n < r \right\}.$$

the structure of $\mathrm{M}_*(\widetilde{\Gamma})$ is already known. Namely, the graded ring $\mathrm{M}_*(\widetilde{\Gamma})$ is generated by algebraically independent two modular forms of weight 4 and 6. Its Poincaré series is given by

$$P_r(x) := \sum_{k\in\mathbb{N}\cup\{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}_k(\widetilde{\Gamma}; r)\right) x^k := \frac{x^{12r}}{(1-x^4)(1-x^6)}.$$

3.2 Witt modular forms

For a holomorphic function $f : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$ and $k, l \in \mathbb{Z}$, we say f is a Witt modular form of weight (k, l) with respect to $\widetilde{\Gamma}$ if f satisfies the following two

conditions:
(1) For any fixed $\omega_0 \in \mathbb{H}$, the function $f(\tau, \omega_0)$ on $\tau \in \mathbb{H}$ belongs to $\mathrm{M}_k(\widetilde{\Gamma})$.
(2) For any fixed $\tau_0 \in \mathbb{H}$, the function $f(\tau_0, \omega)$ on $\omega \in \mathbb{H}$ belongs to $\mathrm{M}_l(\widetilde{\Gamma})$.
We denote by $\mathrm{M}_{k,l}(\widetilde{\Gamma})$ the space of all Witt modular forms of weight (k, l) with respect to $\widetilde{\Gamma}$. For $r, s \in \mathbb{N} \cup \{0\}$, define subspaces of $\mathrm{M}_{k,l}(\widetilde{\Gamma})$ by

$$\mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) := \left\{ f \in \mathrm{M}_{k,l}(\widetilde{\Gamma}) \;\middle|\; \begin{array}{ll} f(\tau, \omega_0) \in \mathrm{M}_k(\widetilde{\Gamma}; r) & \text{for any } \omega_0 \in \mathbb{H} \\ f(\tau_0, \omega) \in \mathrm{M}_l(\widetilde{\Gamma}; s) & \text{for any } \tau_0 \in \mathbb{H} \end{array} \right\}.$$

By Witt [Wi, Satz A], we have

$$\mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) = \mathrm{M}_k(\widetilde{\Gamma}; r) \otimes_{\mathbb{C}} \mathrm{M}_l(\widetilde{\Gamma}; s).$$

Hence its Poincaré series is given by

$$\begin{aligned} P_{(\widetilde{\Gamma}; r, s)}(x, y) :=& \sum_{k,l \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}_{k,l}(\widetilde{\Gamma}; r, s) \right) x^k y^l \\ =& P_{(\widetilde{\Gamma}; r)}(x) P_{(\widetilde{\Gamma}; s)}(y) \\ =& \frac{x^{12r} y^{12s}}{(1 - x^4)(1 - x^6)(1 - y^4)(1 - y^6)}. \end{aligned}$$

Put $\mathrm{M}_{k,l}(\widetilde{\Gamma}; r) := \mathrm{M}_{k,l}(\widetilde{\Gamma}; r, r)$. We say $f \in \mathrm{M}_{k,k}(\widetilde{\Gamma}; r)$ is symmetric or skew-symmetric if $f(\tau, \omega) = f(\omega, \tau)$ or $f(\tau, \omega) = -f(\omega, \tau)$ and denote by $f \in \mathrm{M}^{\mathrm{sym}}_{k,k}(\widetilde{\Gamma}; r)$ or $f \in \mathrm{M}^{\mathrm{skew}}_{k,k}(\widetilde{\Gamma}; r)$, respectively. The structure of these spaces are easily determined. Their Poincaré series are given by

$$\begin{aligned} P^{\mathrm{sym}}_{(\widetilde{\Gamma}; r)}(x) :=& \sum_{k \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k,k}(\widetilde{\Gamma}; r) \right) x^k \\ =& \frac{x^{12r}}{(1 - x^4)(1 - x^6)(1 - x^{12})}, \\ P^{\mathrm{skew}}_{(\widetilde{\Gamma}; r)}(x) :=& \sum_{k \in \mathbb{N} \cup \{0\}} \left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{skew}}_{k,k}(\widetilde{\Gamma}; r) \right) x^k \\ =& \frac{x^{12(r+1)}}{(1 - x^4)(1 - x^6)(1 - x^{12})}. \end{aligned}$$

3.3 Differential operator

For a complex domain X, we denote by $\mathrm{Hol}(X, \mathbb{C})$ the set of all holomorphic functions from X to $\mathbb{C}$. For $r \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, define a differential operator $D_r : \mathrm{Hol}(\mathbb{H}_2, \mathbb{C}) \to \mathrm{Hol}(\mathbb{H}^2, \mathbb{C})$ by

$$(D_r(F))(\tau, \omega) := \left(\frac{\partial^r F}{\partial z^r}\right)\begin{pmatrix} \tau & 0 \\ 0 & \omega \end{pmatrix}.$$

and put

$$A_k(\Gamma; r) := \{\, F \in A_k(\Gamma) \mid D_t(F) = 0 \text{ for any } t < r \}.$$

We remark that there is a descent sequence of vector spaces

$$A_k(\Gamma) = A_k(\Gamma; 0) \supset A_k(\Gamma; 1) \supset A_k(\Gamma; 2) \supset A_k(\Gamma; 3) \supset \cdots$$

and

$$\bigcap_{r \in \mathbb{N}_0} A_k(\Gamma; r) = \{0\}.$$

Lemma 3. *There exists an exact sequence*

$$0 \longrightarrow A_k(\Gamma; r+1) \longrightarrow A_k(\Gamma; r) \xrightarrow{D_r} \mathrm{Hol}(\mathbb{H}^2, \mathbb{C}).$$

This lemma insists that, if we can know the dimension of $D_r(A_k(\Gamma; r))$ possibly, we have the dimension of $A_k(\Gamma)$ by

$$\dim_{\mathbb{C}} A_k(\Gamma) = \sum_{r=0}^{\infty} \dim_{\mathbb{C}} D_r(A_k(\Gamma; r)).$$

Indeed, from the next section, we will calculate the upper bound of the dimension of $D_r(A_k(\Gamma; r))$. Hence we will have the upper bound of the dimension of $A_k(\Gamma)$. Therefore, by constructing sufficiently many modular forms, we can show this upper bound is the true dimension of $A_k(\Gamma)$.

3.4 Estimation

The following lemma is easy to show from the transformation formula of modular forms.

Lemma 4. *The image by D_r has the following properties.*
(1) If k is even and if r is even, $D_r(A_k(\Gamma; r)) \subset \mathrm{M}^{\mathrm{sym}}_{k+r}(\widetilde{\Gamma})$.
(2) If k is even and if r is odd, $D_r(A_k(\Gamma; r)) = \{0\}$.
(3) If k is odd and if r is even, $D_r(A_k(\Gamma; r)) = \{0\}$.
(4) If k is odd and if r is odd, $D_r(A_k(\Gamma; r)) \subset \mathrm{M}^{\mathrm{skew}}_{k+r}(\widetilde{\Gamma})$.

Hence we can improve the exact sequence given by Lemma 3.

Corollary 5. *There exist two exact sequences.*
(1) If k is even, $A_k(\Gamma) = A_k(\Gamma; 0)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+2) \longrightarrow A_k(\Gamma; 2r) \xrightarrow{D_{2r}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma}).$$

(2) If k is odd, $A_k(\Gamma) = A_k(\Gamma; 1)$ and

$$0 \longrightarrow A_k(\Gamma; 2r+3) \longrightarrow A_k(\Gamma; 2r+1) \xrightarrow{D_{2r+1}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma}).$$

To study the image $D_r(A_k(\Gamma; r))$ more precisely, we will investigate the Fourier coefficients of modular forms. Let $F \in A_k(\Gamma)$. Put the Fourier coefficients of F by

$$F(Z) = \sum_{n,l,m\in\mathbb{Z}} a(n,l,m) q^n \zeta^l p^m.$$

Because

$$(D_r(F))(\tau,\omega) := \sum_{n,m\in\mathbb{Z}} \left(\sum_{l\in\mathbb{Z}} \left(2\pi\sqrt{-1}l\right)^r a(n,l,m) \right) q^n p^m,$$

if $F \in A_k(\Gamma; r)$, for any $n \in \mathbb{Z}, m \in \mathbb{Z}$ and $t < r$,

$$\sum_{l\in\mathbb{Z}} l^t a(n,l,m) = 0.$$

Lemma 6. *The Fourier coefficients of F satisfy the following equations:*
(1) $a(n,-l,m) = (-1)^k a(n,l,m)$.
(2) $a(m,l,n) = (-1)^k a(n,l,m)$.
(3) $a(n+xl+x^2m, l+2xm, m) = a(n,l,m)$ for any $x \in \mathbb{Z}$.
(4) $a(n, l+2xn, m+xl+x^2n) = a(n,l,m)$ for any $x \in \mathbb{Z}$.
(5) If k is odd, then $a(n,0,m) = 0$ and $a(n,l,n) = 0$.
(6) If $4nm - l^2 < 0$, $n < 0$ or $m < 0$, then $a(n,l,m) = 0$.

Proof. The equations (1)-(5) are easy to show from the transformation formula of modular forms. The equation (6) is well-known as Köcher principle. □

Next lemma is easy, but this is a key of our story.

Lemma 7. *If $|l| > \min\{n,m\}$ and $a(n,l,m) \neq 0$, there exist n', l', m' such that $\min\{n',m'\} < \min\{n,m\}$ and $a(n',l',m') \neq 0$.*

Proof. It is obvious from Lemma 6 (3)(4). □

Lemma 8. *The Fourier coefficients of F has the following properties:*
(1) If k is even, $F \in A_k(\Gamma; 2r)$ and $\min\{n,m\} < r$, then $a(n,l,m) = 0$.
(2) If k is odd, $F \in A_k(\Gamma; 2r+1)$ and $\min\{n,m\} < r+2$, then $a(n,l,m) = 0$.

Proof. First, we show (1). Assume k is even and $F \in A_k(\Gamma; 2r)$. Put

$$b(n,l,m) := \begin{cases} 2a(n,l,m) & (\text{if } l \neq 0) \\ a(n,0,m) & (\text{if } l = 0) \end{cases}$$

Then for any $n, m \in \mathbb{Z}$ and $t \in \{0, 1, \dots, r-1\}$, we have

$$\sum_{l=0}^{2\sqrt{nm}} l^{2t} b(n,l,m) = 0$$

It is sufficient to show $b(n,l,m) = 0$ if $\min\{n,m\} < r$. We will show this by induction on $\min\{n,m\}$. If $\min\{n,m\} = 0$, this lemma is trivial. Now we assume that $b(n,l,m) = 0$ if $\min\{n,m\} \leq u < r-1$ and consider the case $\min\{n,m\} = u+1$. From Lemma 7, $b(n,l,m) = 0$ if $l > u+1$. Then we have

$$\sum_{l=0}^{u+1} l^{2t} b(n,l,m) = 0$$

for any $t \in \{0, 1, \dots, r-1\}$. Hence, by the Vandermonde formula, we have $b(n,l,m) = 0$.

Second, we show (2). Assume k is odd and $F \in A_k(\Gamma; 2r+1)$. Put $b(n,l,m) := 2a(n,l,m)$. We remark that $a(m,l,m) = 0$, $a(n,n,m) = 0$ and $a(n,m,m) = 0$. Then for any $n, m \in \mathbb{Z}$ and $t \in \{0, 1, \dots, r-1\}$, we have

$$\sum_{l=1}^{2\sqrt{nm}} l^{2t+1} b(n,l,m) = 0$$

It is sufficient to show $b(n,l,m) = 0$ if $\min\{n,m\} < r$. We will show this by induction on $\min\{n,m\}$. If $\min\{n,m\} = 0$, this lemma is trivial. Now we assume that $b(n,l,m) = 0$ if $\min\{n,m\} \leq u < r+1$ and consider the case $\min\{n,m\} = u+1$. From Lemma 6, $b(n,l,m) = 0$ if $l > u+1$. Then we have

$$\sum_{l=1}^{u+1} l^{2t+1} b(n,l,m) = 0$$

for any $t \in \{0, 1, \dots, r-1\}$. Hence, by the Vandermonde formula, we have $b(n,l,m) = 0$. $\square$

Corollary 9. *The image by D_r has the following properties.*
(1) If k is even, $D_{2r}(A_k(\Gamma;2r)) \subset \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma};r)$.
(2) If k is odd, $D_{2r+1}(A_k(\Gamma;2r+1)) \subset \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma};r+2)$.

Corollary 10. *There exist two exact sequences.*
(1) If k is even, $A_k(\Gamma) = A_k(\Gamma;0)$ and

$$0 \longrightarrow A_k(\Gamma;2r+2) \longrightarrow A_k(\Gamma;2r) \xrightarrow{D_{2r}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma};r).$$

(2) If k is odd, $A_k(\Gamma) = A_k(\Gamma;1)$ and

$$0 \longrightarrow A_k(\Gamma;2r+3) \longrightarrow A_k(\Gamma;2r+1) \xrightarrow{D_{2r+1}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma};r+2).$$

Corollary 11. *We have the upper bounds for the dimensions of $A_k(\Gamma)$.*
(1) If k is even, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma};r)$.
(2) If k is odd, $\dim_{\mathbb{C}} A_k(\Gamma) \leq \sum_{r=0}^{\infty} \dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r+1}(\widetilde{\Gamma};r+2)$.

Now we calculate the Poincaré series of this upper bound. If k is even, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}\left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{sym}}_{k+2r}(\widetilde{\Gamma};r)\right)x^k &= \sum_{r=0}^{\infty}\frac{x^{12r-2r}}{(1-x^4)(1-x^6)(1-x^{12})}\\ &= \frac{1}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

If k is odd, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{r=0}^{\infty}\left(\dim_{\mathbb{C}} \mathrm{M}^{\mathrm{skew}}_{k+2r+1}(\widetilde{\Gamma};r+2)\right)x^k &= \sum_{r=0}^{\infty}\frac{x^{12(r+2+1)-(2r+1)}}{(1-x^4)(1-x^6)(1-x^{12})}\\ &= \frac{x^{35}}{(1-x^4)(1-x^6)(1-x^{10})(1-x^{12})}.\end{aligned}$$

Hence, if we construct algebraically independent modular forms of weight $4, 6, 10, 12$, and if we construct a modular forms of weight 35, we finish the proof of Theorem 1 for $N = 1$. Indeed, Igusa [Ig1, Ig2] constructed these modular forms from the theta functions.

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