<table>
<thead>
<tr>
<th>Title</th>
<th>PERIODS OF AUTOMORPHIC FORMS AND $L$-VALUES (Automorphic Representations, Automorphic Forms, $L$-functions, and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>IKEDA, TAMOTSU</td>
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Kyoto University
PERIODS OF AUTOMORPHIC FORMS AND 
L-VALUES

TAMOTSU IKEDA

1. Introduction

This article is a proceeding of an expository talk, in which I discussed a possibility to relate a period integral to some L-values.

Let $G$ be a connected reductive algebraic group defined over an algebraic number field $k$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let $H \subset G$ be a connected algebraic subgroup. Let $\theta : H(\mathbb{A}) \to \mathbb{C}^\times$ a character which is trivial on $H(k)$.

Definition 1.1. An integral of the form

$$\mathcal{P}_{H,\theta}(\varphi) = \int_{H(k)\backslash H(\mathbb{A})} \varphi(h) \overline{\theta(h)} \, dh$$

is called an $(H, \theta)$-period.

Remark 1.2. Some people say that the terminology “period” is inadequate in this context.

The automorphic representation $\pi$ is said to be $(H, \theta)$-distinguished if $\mathcal{P}_{H,\theta}(\phi) \neq 0$ for some $\varphi \in \pi$. If there is no fear of confusion, we simply say that $\pi$ is distinguished.

If $\dim_{\mathbb{C}} Hom_{H_v} (\pi_v, \theta_v) < \infty$ for all $v$, then it is believed that the period integral $\mathcal{P}_{H,\theta}(\varphi)$ is related to some L-values. More precisely, we are looking for a formula, which is of the form

$$\frac{|\mathcal{P}_{H,\theta}(\varphi)|^2}{\langle \varphi, \varphi \rangle} = \frac{1}{#S_\pi} \cdot C_H \frac{\Delta_G}{\Delta_H} \frac{L(1/2, \pi, \rho)}{L(1, \pi, \text{Ad})} \prod_v l_v(\varphi_{1,v}, \overline{\varphi}_{1,v}).$$

Here, $S_\pi$ is a certain finite group depending only on the $L$-packet of $\pi$. The constant $\Delta_H$ (resp. $\Delta_G$) is a product of certain L-value determined by the motive (see Gross [8]) of reductive part of $H$ (resp. $G$). The constant $C_H$ is a constant depending only on the choice of the local and global Haar measure on $H(\mathbb{A})$. The representation $\rho$ is a finite dimensional symplectic representation of $^L G$. The local homomorphism $l_v \in Hom_{H_v \times H_v} (\pi_v \times \tilde{\pi}_v, \theta \times \overline{\theta})$ should depends only on local data. We call this kind of equation a “period formula” in this manuscript. A
typical (conjectural) example of a period formula is the Gross-Prasad type conjecture for orthogonal groups (joint work with Ichino [15]), which we recall in the next section.

2. Gross-Prasad type conjectures

Let $k$ be a global field with $\text{char}(k) \neq 2$. Let $(V_1, Q_1)$ and $(V_0, Q_0)$ be quadratic forms over $k$ with rank $n + 1$ and $n$, respectively. We assume $n \geq 2$. When $n = 2$, we also assume $(V_0, Q_0)$ is not isomorphic to the hyperbolic plane over $k$. We denote the special orthogonal group of $(V_i, Q_i)$ by $G_i$ ($i = 0, 1$). In this section, the subscript $i$ will indicate either 0 or 1, except for some obvious situation. We assume there is an embedding $\iota : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal group $\iota : G_0 \hookrightarrow G_1$. We regard $G_0$ as a subgroup of $G_1$ by this embedding. The group $G_i(k_v)$ of $k_v$-valued points of $G_i$ is denoted by $G_{i,v}$.

For even-dimensional quadratic form $(V, Q)$, the discriminant field $K_Q$ is defined by $K_Q = k(\sqrt{(-1)^{\dim V/2} \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if $\dim V_0$ is even (resp. if $\dim V_1$ is even). We call $K$ the discriminant field for the pair $(V_1, V_0)$. Let $\chi = \chi_{K/k}$ be the Hecke character associated to $K/k$ by the class field theory.

Put

$$\Delta_{G_i,v} = \begin{cases} \zeta_v(2) \zeta_v(4) \cdots \zeta_v(2l) & \text{if } \dim V_i = 2l + 1, \\ \zeta_v(2) \zeta_v(4) \cdots \zeta_v(2l - 2) \cdot L_v(l, \chi) & \text{if } \dim V_i = 2l, \end{cases}$$

$$\Delta_{G_i} = \begin{cases} \zeta(2) \zeta(4) \cdots \zeta(2l) & \text{if } \dim V_i = 2l + 1, \\ \zeta(2) \zeta(4) \cdots \zeta(2l - 2) \cdot L(l, \chi) & \text{if } \dim V_i = 2l. \end{cases}$$

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(A)$. There is a canonical inner product $\langle *, * \rangle$ on forms on $G_i(k) \backslash G_i(A)$ defined by

$$\langle \varphi_i, \varphi'_i \rangle = \int_{G_i(k) \backslash G_i(A)} \varphi_i(g_i) \overline{\varphi'_i(g_i)} \, dg_i,$$

where $dg_i$ is the Tamagawa measure on $G_i(A)$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each $v$. There exist a positive numbers $C_i$ such that $dg_i = C_i \prod_v dg_{i,v}$, when the right hand side is well-defined. Since $\pi_{i,v}$ is an unitary representation, there is an inner product $\langle *, * \rangle_v$ on $\pi_{i,v}$ for any place $v$ of $k$. We put $\| \varphi_{i,v} \| = \langle \varphi_{i,v}, \varphi_{i,v} \rangle_v^{1/2}$, as usual. There exists a positive constant $C_{\pi_i}$ such that $\langle \varphi_i, \varphi'_i \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi'_{i,v} \rangle_v$ for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi'_i = \otimes_v \varphi'_{i,v} \in \otimes_v \pi_{i,v}$.
We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a $\mathcal{K}_i$-finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. In this section, we consider the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of $\varphi_1$ to $G_0(\mathbb{A})$.

Let $S$ be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

(U1) $G_i$ is unramified over $k_v$.
(U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
(U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
(U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
(U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
(U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When $G_i$ is unramified over $k_v$, we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The $L$-group $^LG_i$ of $G_i$ is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, $W_k$ is the Weil group of $k$ and

$$\hat{G}_i = \begin{cases} \text{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l + 1, \\ \text{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by st the standard representation of $^LG_i$. The completed standard $L$-function for $\pi_i$ is denoted by $L(s, \pi_i, \text{st})$ for an irreducible automorphic representation $\pi_i$ of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, \text{st})$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - \text{st}(A_{\pi_i,v}) \cdot q_v^{-s})^{-1}$, where, $A_{\pi_i,v}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product $L$-function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - \text{st}(A_{\pi_1,v}) \otimes \text{st}(A_{\pi_0,v}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\text{Ad} : ^LG_i \to \text{GL}(%(\text{Lie}(\hat{G}_i)))$. The associated $L$-function $L(s, \pi_i, \text{Ad})$ is called the adjoint $L$-function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \text{Ad})$ can be analytically continued to the whole $s$-plane.

We put

$$P_{\pi_1,\pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \text{Ad})L(s + (1/2), \pi_0, \text{Ad})}.$$
Let $\pi_{i,v}$ be an irreducible admissible representation of $G_{i,v}$. We denote the complex conjugate of $\pi_{i,v}$ by $\bar{\pi}_{i,v}$. It is believed that

\[(MF) \quad \dim_{C} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \leq 1\]

for non-archimedean place $v$ of $k$. Recently, Aizenbud, Gourevitch, Rallis, and Schiffmann wrote a preprint, in which they obtained closely related results. For archimedean place, (MF) is verified in many cases, but not in general.

We consider the matrix coefficient

$$\Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_{i}) = \langle \pi_{i,v}(g_{i})\varphi_{i,v}, \varphi'_{i,v}\rangle_{v}, \quad g_{i} \in G_{i,v}$$

for a $K_{1,v}$-finite vector $\varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v}$ and a $K_{0,v}$-finite vector $\varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v}$. Put

$$I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \int_{G_{0,v}} \Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_{i})\overline{\Phi_{\varphi_{0,v},\varphi'_{0,v}}(g_{0,v})} \, dg_{0,v},$$

$$\alpha_{v}(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \Delta_{G_{1,v}}^{-1} P_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}).$$

When $\varphi_{1,v} = \varphi'_{1,v}$ and $\varphi_{0,v} = \varphi'_{0,v}$, we simply denote these objects by $I(\varphi_{1,v}, \varphi_{0,v})$ and $\alpha_{v}(\varphi_{1,v}, \varphi_{0,v})$, respectively. If both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered, then the integral $I(\varphi_{1,v}, \varphi_{0,v})$ is absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $K_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$. Moreover, if $v$ is a non-archimedean place, and the conditions (U1), (U2), (U3), (U4), (U5), and (U6) hold, then we can show that $\alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) = 1$.

**Conjecture 2.1.** Assume that both $\pi_{1,v}$ and $\pi_{0,v}$ are tempered. Then

$$\dim_{C} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$$

if and only if $\alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $K_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.

Now let $\pi_{i} \simeq \otimes_{v} \pi_{i,v}$ be irreducible cuspidal automorphic representation of $G_{i}(\mathbb{A})$. We shall say that $\pi_{i}$ is almost locally generic if $\pi_{i}$ satisfies the following condition (ALG).

**\text{(ALG)}** For almost all $v$, the constituent $\pi_{i,v}$ is generic.

It is believed that $\pi_{i}$ is almost locally generic if and only if $\pi_{i}$ is tempered (generalized Ramanujan conjecture).

**Conjecture 2.2.** Let $\pi_{i} \simeq \otimes_{v} \pi_{i,v}$ be an irreducible cuspidal automorphic representation of $G_{i}(\mathbb{A})$. We assume both $\pi_{1}$ and $\pi_{0}$ are almost locally generic. Then

1. The integral $I(\varphi_{1,v}, \varphi_{0,v})$ should be absolutely convergent and $I(\varphi_{1,v}, \varphi_{0,v}) \geq 0$ for any $K_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$.
2. $\dim_{C} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \bar{\pi}_{0,v}, \mathbb{C}) \neq \{0\}$ if and only if $\alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) > 0$ for some $K_{i,v}$-finite vector $\varphi_{i,v} \in \pi_{i,v}$. 


Now we state our global conjecture.

**Conjecture 2.3.** Let $\pi_1 \simeq \otimes_v \pi_{1,v}$ and $\pi_0 \simeq \otimes_v \pi_{0,v}$ are irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume $\pi_1$ and $\pi_0$ are almost locally generic. Then there should be an integer $\beta$ such that

$$\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1} \rho_{\pi_1, \pi_0} (1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

It seems that the integer $\beta$ is related to the order of the groups, which appear in the theory of endoscopy.

It is possible to formulate a similar conjecture for non-tempered automorphic representations (cf. [15]).

### 3. The relative trace formula

For low rank groups, some periods formula are proved by using theta correspondence and Rankin-Selberg formulas (see, e.g., [3], [12], [13], [14], [19], [22]). For higher rank groups, it seems some sophisticated tool such as relative trace formula is necessary. In this section, we will discuss how a relative trace formula can be applied to period formulas.

Let $G$ be a connected reductive algebraic group defined over $k$. We assume, for simplicity, $G(k) \backslash G(\mathbb{A})$ is compact.

We recall the Selberg trace formula. Let $f \in C_0^\infty(G(\mathbb{A}))$ be a test function. The kernel function $K_f(g_1, g_2)$ is defined by

$$K_f(g_1, g_2) = \sum_{\gamma \in G(k)} f(g_1^{-1}\gamma g_2).$$

For an automorphic form $\varphi$ on $G(\mathbb{A})$,

$$\rho(f) \varphi(g_2) = (\varphi \ast f)(g_2) = \int_{G(\mathbb{A})} \varphi(g_1) f(g_1^{-1}g_2) \, dg_1$$

$$= \int_{G(k) \backslash G(\mathbb{A})} \varphi(g_1) \sum_{\gamma \in G(k)} f(g_1^{-1}\gamma g_2) \, dg_1$$

$$= \int_{G(k) \backslash G(\mathbb{A})} \varphi(g_1) K_f(g_1, g_2) \, dg_1.$$
It follows that
\[
\text{tr}\rho(f) = \int_{G(k)\backslash G(A)} K_f(g_1, g_2) \, dg
\]
\[
= \int_{G(k)\backslash G(A)} \sum_{\gamma \in G(k)} f(g^{-1}\gamma g) \, dg
\]
\[
= \sum_{\{\gamma\}} \int_{G(k)\backslash G(A)} \sum_{\gamma' \in G_{\gamma}(k)\backslash G(k)} f(g^{-1}\gamma'^{-1}\gamma\gamma'g) \, dg
\]
\[
= \sum_{\{\gamma\}} \text{Vol}(G_{\gamma}(k)\backslash G_{\gamma}(A)) \int_{G_{\gamma}(A)\backslash G(A)} f(g^{-1}\gamma g) \, dg.
\]

Here, \(\{\gamma\}\) is a conjugacy class of \(\gamma \in G(k)\) and \(G_{\gamma}\) is the centralizer of \(\gamma\). Set \(a(\gamma) = \text{Vol}(G_{\gamma}(k)\backslash G_{\gamma}(A))\).

Note that the orbital integral \(O(\gamma, f) = \int_{G_{\gamma}(A)\backslash G(A)} f(g^{-1}\gamma g) \, dg\) is decomposed as a local product
\[
\int_{G_{\gamma}(A)\backslash G(A)} f(g^{-1}\gamma g) \, dg = \prod_{v} \int_{G_{\gamma}(k_v)\backslash G(k_v)} f(g_v^{-1}\gamma g_v) \, dg_v.
\]

The right regular representation \(\rho\) is a sum of automorphic representations \(\rho = \oplus_{\pi} m_{\rho}(\pi) \cdot \pi\). Here, \(m_{\rho}(\pi)\) is the multiplicity of \(\pi\). The distribution character \(\chi_{\pi}(f)\) is defined by \(\chi_{\pi}(f) = \text{tr}\pi(f)\) for a test function \(f \in C^\infty_0(G(A))\). Then we have
\[
\text{tr}\rho(f) = \sum_{\pi} m_{\rho}(\pi) \chi_{\pi}(f).
\]

Thus we have the Selberg trace formula
\[
\sum_{\{\gamma\}} a(\gamma) O(\gamma, f) = \sum_{\pi} m_{\rho}(\pi) \chi_{\pi}(f).
\]

Note that in the right hand side, \(\pi\) extends over the isomorphism classes of irreducible automorphic representations.

Now, we consider the relative trace formula. Let \(H_1, H_2 \subset G\) be connected algebraic subgroups of \(G\). Let \(\theta_i : H_i(A) \to \mathbb{C}^\times\) be a character which is trivial on \(H_i(k)\) for \(i = 1, 2\). As before, the kernel function \(K_f(g_1, g_2)\) is defined by
\[
K_f(g_1, g_2) = \sum_{\gamma \in G(k)} f(g_1^{-1}\gamma g_2)
\]
for a test function \(f \in C^\infty_0(G(A))\).
Consider the integral
\[
\int_{H_1(k)\backslash H_1(A)} \int_{H_2(k)\backslash H_2(A)} K_f(h_1, h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \sum_{\gamma \in H_1(k)\backslash G(k) / H_2(k)} \int_{H_1(A)} \int_{H_2,\gamma(k)\backslash H_2(A)} f(h_1^{-1} \gamma h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2.
\]

Here, $H_{2,\gamma} = \gamma^{-1} H_1 \gamma \cap H_2$. In this sum, $\gamma$ contributes only when $\theta_1(\gamma h_2 \gamma^{-1}) = \theta_2(h_2)$ for any $h_2 \in H_{2,\gamma}(A)$, in which case $\gamma$ is said to be $(\theta_1, \theta_2)$-relevant (or simply "relevant"). Set
\[
a(\gamma) = \text{Vol}(H_{2,\gamma}(k)\backslash H_{2,\gamma}(A)),
\]
\[
I_\gamma(\theta_1, \theta_2; f) = \int_{H_1(A)} \int_{H_2,\gamma(A)\backslash H_2(A)} f(h_1^{-1} \gamma h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2.
\]

Then we have
\[
\int_{H_1(k)\backslash H_1(A)} \int_{H_2(k)\backslash H_2(A)} K_f(h_1, h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \sum_{\gamma \in H_1(k)\backslash G(k) / H_2} a(\gamma) I_\gamma(\theta_1, \theta_2; f).
\]

On the other hand, note that
\[
\rho(f) \varphi_1(g_2) = \int_{G(k)\backslash G(A)} K_f(g_1, g_2) \varphi_1(g_1) \, dg_1
\]
\[
= \sum_{\pi} \sum_{\varphi_2 \in \pi} \int_{G(k)\backslash G(A)} K_f(g_1, g_2) \varphi_1(g_1) \varphi_2(g'_2) \, dg_1 \, dg'_2 \cdot \overline{\varphi_2(g_2)}
\]
\[
= \sum_{\pi} \sum_{\varphi_2 \in \pi} \langle K_f, \varphi_1 \times \varphi_2 \rangle \cdot \overline{\varphi_2(g_2)}.
\]

Here, $\varphi_2$ extends over a complete orthonormal system (CONS) for $\pi$. It follows that
\[
K_f(g_1, g_2) = \sum_{\pi} \sum_{\varphi_1, \varphi_2 \in \pi} \langle K_f, \varphi_1 \times \varphi_2 \rangle \cdot \overline{\varphi_1(g_1) \varphi_2(g_2)}
\]
\[
= \sum_{\pi} \sum_{\varphi_1 \in \pi} \overline{\varphi(g_1)} \cdot \rho(f) \varphi(g_2).
\]
Therefore we have
\[
\int_{H_{1}(k)\backslash H_{1}(A)}\int_{H_{2}(k)\backslash H_{2}(A)} K_f(h_1, h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \int_{H_{1}(k)\backslash H_{1}(A)}\int_{H_{2}(k)\backslash H_{2}(A)} \left[ \sum_{\pi} \sum_{\varphi \in \pi} \overline{\varphi(g_1)} \cdot \rho(f) \varphi(g_2) \right] \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \sum_{\pi} \sum_{\varphi \in \pi} \mathcal{P}_{H_{1}, \theta_1}(\varphi) \mathcal{P}_{H_{2}, \theta_2}(\rho(f) \varphi).
\]

Set
\[
I_\pi(\theta_1, \theta_2; f) = \sum_{\varphi \in \pi} \mathcal{P}_{H_{1}, \theta_1}(\varphi) \mathcal{P}_{H_{2}, \theta_2}(\rho(f) \varphi).
\]

The automorphic representation \( \pi \) is said to be \((\theta_1, \theta_2)\)-distinguished (or simply "distinguished") if it is \((H_{1}, \theta_1)\)-distinguished and \((H_{2}, \theta_2)\)-distinguished. Then we have the relative trace formula
\[
\sum_{\gamma \in H_{1}\backslash G/H_{2} \text{relevant}} a(\gamma) I_\gamma(\theta_1, \theta_2; f) = \sum_{\pi \text{distinguished}} I_\pi(\theta_1, \theta_2; f).
\]

Note that in the right hand side, \( \pi \) extends over some orthogonal decomposition \( \rho = \sum_\pi \pi \). (Therefor different \( \pi \)'s can be isomorphic.)

**Remark 3.1.** Assume that \( G \) is the product \( G = G' \times G' \). Let \( H_1 \) be the diagonal subgroup \( H_1 = \Delta(G') = \{(g', g') | g' \in G'\} \) and \( H_2 \) be the second factor \( H_2 = \{(1, g') | g' \in G'\} \). Set \( \theta_1 = \theta_2 = 1 \). Then the double coset \( H_1 \backslash G/H_2 \) can be identified with the conjugacy classes of \( G' \). If \( \gamma \in H_1 \backslash G/H_2 \) correspond to the conjugacy class \( \gamma' \) of \( G' \), then we have
\[
I_\gamma(\theta_1, \theta_2; f) = O(\gamma', f'),
\]
where
\[
f'(g') = \int_{G'(A)} f(g_1', g_1'g') \, dg_1.
\]

Moreover, an irreducible automorphic representation \( \pi = \pi_1 \boxtimes \pi_2 \) is \((\theta_1, \theta_2)\)-distinguished if and only if \( \pi_1 \simeq \tilde{\pi}_2 \). In this case, we have \( I_\pi(\theta_1, \theta_2; f) = \text{tr} \pi'_2(f') \). Thus the Selberg trace formula can be considered as a special case of the relative trace formula.

Let \( G', H_1', \theta_1', H_2', \) and \( \theta_2' \) be another set of data. We assume there exists a bijection
\[
\{ \gamma \in H_1 \backslash G/H_2 \mid \gamma : \text{relevant} \} \simeq \{ \gamma' \in H_1' \backslash G'/H_2' \mid \gamma' : \text{relevant} \}.
\]
with the following properties:

(1) (matching) For each test function \( f \in C_0(G(\mathbb{A})) \), there exists a test function \( f' \in C_0(G'(\mathbb{A})) \) such that \( I_\gamma(\theta_1, \theta_2; f) = I_{\gamma'}(\theta_1, \theta_2; f') \).

(2) (fundamental lemma) For almost all unramified \( v \), there exists a Hecke algebra homomorphism

\[
\mathcal{H}(K_{G,v} \backslash G_{v}/K_{G,v}) \rightarrow \mathcal{H}(K_{G',v} \backslash G'_{v}/K_{G',v})
\]

which is compatible with the matching.

Then it is expected that there exists a correspondence for the \( L \)-packets of \( G(\mathbb{A}) \) and \( G'(\mathbb{A}) \) such that

\[
I_{\Pi}^\kappa(\theta_1, \theta_2; f) = I_{\Pi'}^{\kappa'}(\theta'_1, \theta'_2; f').
\]

Here, \( \Pi \) is an \( L \)-packet for \( G(\mathbb{A}) \), and \( \kappa \) is certain function on the \( L \)-packet and

\[
I_{\Pi}^\kappa(\theta_1, \theta_2; f) = \sum_{\pi \in \Pi} \kappa(\pi) I_\pi(\theta_1, \theta_2; f).
\]

In the right hand side, \( \Pi' \) is the \( L \)-packet of \( G'(\mathbb{A}) \) corresponding to \( \Pi \), and \( I_{\Pi'}^{\kappa'}(\theta'_1, \theta'_2; f') \) is defined in a similar way.

This equation would imply that there exists a certain relation between period integrals for \( G(\mathbb{A}) \) and \( G'(\mathbb{A}) \). In this way, it would be possible to reduce a period formula for \( G(\mathbb{A}) \) to an analogous formulas for \( G'(\mathbb{A}) \).

Recently, H. Jacquet [16] proposed a program to attack an analogue of the Gross-Prasad type conjecture for the unirary groups.

REFERENCES

TAMOTSU IKEDA


GRADUATE SCHOOL OF MATHEMATICS, KYOTO UNIVERSITY, KITASHIRAKAWA, KYOTO, 606-8502, JAPAN

E-mail address: ikeda@math.kyoto-u.ac.jp