PERIODS OF AUTOMORPHIC FORMS AND $L$-VALUES

(Automorphic Representations, Automorphic Forms, $L$-functions, and Related Topics)

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PERIODS OF AUTOMORPHIC FORMS AND L-VALUES

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1. Introduction

This article is a proceeding of an expository talk, in which I discussed a possibility to relate a period integral to some L-values.

Let $G$ be a connected reductive algebraic group defined over an algebraic number field $k$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let $H \subset G$ be a connected algebraic subgroup. Let $\theta : H(\mathbb{A}) \to \mathbb{C}^\times$ a character which is trivial on $H(k)$.

Definition 1.1. An integral of the form

$$\mathcal{P}_{H, \theta}(\varphi) = \int_{H(k) \backslash H(\mathbb{A})} \varphi(h) \overline{\theta(h)} \, dh$$

is called an $(H, \theta)$-period.

Remark 1.2. Some people say that the terminology “period” is inadequate in this context.

The automorphic representation $\pi$ is said to be $(H, \theta)$-distinguished if $\mathcal{P}_{H, \theta}(\phi) \neq 0$ for some $\varphi \in \pi$. If there is no fear of confusion, we simply say that $\pi$ is distinguished.

If $\dim_{\mathbb{C}} \text{Hom}_{H, v} (\pi_v, \theta_v) < \infty$ for all $v$, then it is believed that the period integral $\mathcal{P}_{H, \theta}(\varphi)$ is related to some L-values. More precisely, we are looking for a formula, which is of the form

$$\frac{|\mathcal{P}_{H, \theta}(\varphi)|^2}{\langle \varphi, \varphi \rangle} = \frac{1}{\# S_{\pi}} \cdot C_H \frac{\Delta_G}{\Delta_H} \frac{L(1/2, \pi, \rho)}{L(1, \pi, \text{Ad})} \prod_v l_v(\varphi_{1,v}, \overline{\varphi}_{1,v}).$$

Here, $S_{\pi}$ is a certain finite group depending only on the $L$-packet of $\pi$. The constant $\Delta_H$ (reps. $\Delta_G$) is a product of certain $L$-value determined by the motive (see Gross [8]) of reductive part of $H$ (resp. $G$). The constant $C_H$ is a constant depending only on the choice of the local and global Haar measure on $H(\mathbb{A})$. The representation $\rho$ is a finite dimensional symplectic representation of $\mathfrak{g}$. The local homomorphism $l_v \in \text{Hom}_{H_v \times H_v}(\pi_v \times \overline{\pi}_v, \theta \times \overline{\theta})$ should depends only on local data. We call this kind of equation a “period formula” in this manuscript.
typical (conjectural) example of a period formula is the Gross-Prasad type conjecture for orthogonal groups (joint work with Ichino [15]), which we recall in the next section.

2. Gross-Prasad type conjectures

Let $k$ be a global field with $\text{char}(k) \neq 2$. Let $(V_1, Q_1)$ and $(V_0, Q_0)$ be quadratic forms over $k$ with rank $n + 1$ and $n$, respectively. We assume $n \geq 2$. When $n = 2$, we also assume $(V_0, Q_0)$ is not isomorphic to the hyperbolic plane over $k$. We denote the special orthogonal group of $(V_i, Q_i)$ by $G_i$ ($i = 0, 1$). In this section, the subscript $i$ will indicate either 0 or 1, except for some obvious situation. We assume there is an embedding $i : V_0 \hookrightarrow V_1$ of quadratic spaces. Then we have an embedding of the corresponding special orthogonal group $i : G_0 \hookrightarrow G_1$. We regard $G_0$ as a subgroup of $G_1$ by this embedding. The group $G_i(k_v)$ of $k_v$-valued points of $G_i$ is denoted by $G_{i,v}$.

For even-dimensional quadratic form $(V, Q)$, the discriminant field $K_Q$ is defined by $K_Q = k(\sqrt{(-1)^{\dim V}/2 \det Q})$. We put $K = K_{Q_0}$ (resp. $K = K_{Q_1}$), if dim $V_0$ is even (resp. if dim $V_1$ is even). We call $K$ the discriminant field for the pair $(V_1, V_0)$. Let $\chi = \chi_{K/k}$ be the Hecke character associated to $K/k$ by the class field theory.

Let $\pi_i \simeq \otimes_v \pi_{i,v}$ be an irreducible square-integrable automorphic representation of $G_i(\mathbb{A})$. There is a canonical inner product $\langle *, * \rangle$ on forms on $G_i(k) \backslash G_i(\mathbb{A})$ defined by

$$\langle \varphi_i, \varphi'_i \rangle = \int_{G_i(k) \backslash G_i(\mathbb{A})} \varphi_i(g_i) \overline{\varphi'_i(g_i)} dg_i,$$

where $dg_i$ is the Tamagawa measure on $G_i(\mathbb{A})$. We choose a Haar measure $dg_{i,v}$ on $G_{i,v}$ for each $v$. There exist a positive numbers $C_i$ such that $dg_i = C_i \prod_v dg_{i,v}$, when the right hand side is well-defined. Since $\pi_{i,v}$ is an unitary representation, there is an inner product $\langle *, * \rangle_v$ on $\pi_{i,v}$ for any place $v$ of $k$. We put $\| \varphi_{i,v} \| = \langle \varphi_{i,v}, \varphi_{i,v} \rangle_v^{1/2}$, as usual. There exists a positive constant $C_{\pi_i}$ such that $\langle \varphi_i, \varphi'_i \rangle = C_{\pi_i} \prod_v \langle \varphi_{i,v}, \varphi'_{i,v} \rangle_v$ for any decomposable vectors $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$ and $\varphi'_i = \otimes_v \varphi'_{i,v} \in \otimes_v \pi_{i,v}$.
We fix maximal compact subgroups $\mathcal{K}_1 = \prod_v \mathcal{K}_{1,v} \subset G_1(\mathbb{A})$ and $\mathcal{K}_0 = \prod_v \mathcal{K}_{0,v} \subset G_0(\mathbb{A})$ such that $[\mathcal{K}_0 : \mathcal{K}_1 \cap \mathcal{K}_0] < \infty$. We choose a $\mathcal{K}_i$-finite decomposable vector $\varphi_i = \otimes_v \varphi_{i,v} \in \otimes_v \pi_{i,v}$. In this section, we consider the period $\langle \varphi_1|_{G_0}, \varphi_0 \rangle$ where $\varphi_1|_{G_0}$ is the restriction of $\varphi_1$ to $G_0(\mathbb{A})$.

Let $S$ be a finite set of bad places containing all archimedean places. We may and do assume the following conditions hold for $v \notin S$:

(U1) $G_i$ is unramified over $k_v$.
(U2) $\mathcal{K}_{i,v}$ is a hyperspecial maximal compact subgroup of $G_{i,v}$.
(U3) $\mathcal{K}_{0,v} \subset \mathcal{K}_{1,v}$.
(U4) $\pi_{i,v}$ is an unramified representation of $G_{i,v}$.
(U5) The vector $\varphi_{i,v}$ is fixed by $\mathcal{K}_{i,v}$ and $\|\varphi_{i,v}\| = 1$.
(U6) $\int_{\mathcal{K}_{i,v}} dg_{i,v} = 1$.

When $G_i$ is unramified over $k_v$, we shall say that a Haar measure on $G_{i,v}$ is the standard Haar measure if the volume of a hyperspecial maximal compact subgroup is 1. Thus the condition (U6) means that the measure $dg_{i,v}$ is the standard Haar measure.

The $L$-group $^L G_i$ of $G_i$ is a semi-direct product $\hat{G}_i \rtimes W_k$. Here, $W_k$ is the Weil group of $k$ and

$$\hat{G}_i = \begin{cases} \text{Sp}_l(\mathbb{C}) & \text{if } \dim V_i = 2l + 1, \\ \text{SO}(2l, \mathbb{C}) & \text{if } \dim V_i = 2l. \end{cases}$$

We denote by $st$ the standard representation of $^L G_i$. The completed standard $L$-function for $\pi_i$ is denoted by $L(s, \pi_i, st)$ for an irreducible automorphic representation $\pi_i$ of $G_i(\mathbb{A})$. For simplicity, we sometimes denote $L(s, \pi_i, st)$ by $L(s, \pi_i)$. For $v \notin S$, the Euler factor for $L(s, \pi_i)$ is given by $\det(1 - st(A_{\pi_i,v}) \cdot q_v^{-s})^{-1}$, where, $A_{\pi_i,v}$ is the Satake parameter of $\pi_{i,v}$. We consider the tensor product $L$-function $L(s, \pi_1 \boxtimes \pi_0)$. The Euler factor of $L(s, \pi_1 \boxtimes \pi_0)$ for $v \notin S$ is given by $\det(1 - st(A_{\pi_1,v}) \otimes \text{st}(A_{\pi_0,v}) \cdot q_v^{-s})^{-1}$.

Consider the adjoint representation $\text{Ad} : ^L G_i \rightarrow \text{GL}(\text{Lie}(\hat{G}_i))$. The associated $L$-function $L(s, \pi_i, \text{Ad})$ is called the adjoint $L$-function. We assume that $L(s, \pi_1 \boxtimes \pi_0)$ and $L(s, \pi_i, \text{Ad})$ can be analytically continued to the whole $s$-plane.

We put

$$\mathcal{P}_{\pi_1, \pi_0}(s) = \frac{L(s, \pi_1 \boxtimes \pi_0)}{L(s + (1/2), \pi_1, \text{Ad})L(s + (1/2), \pi_0, \text{Ad})}.$$
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Let \( \pi_{i,v} \) be an irreducible admissible representation of \( G_{i,v} \). We denote the complex conjugate of \( \pi_{i,v} \) by \( \overline{\pi}_{i,v} \). It is believed that

\[
\text{(MF)} \quad \dim_{\mathbb{C}} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \overline{\pi}_{0,v}, \mathbb{C}) \leq 1
\]

for non-archimedean place \( v \) of \( k \). Recently, Aizenbud, Gurevitch, Rallis, and Schiffmann wrote a preprint, in which they obtained closely related results. For archimedean place, (MF) is verified in many cases, but not in general.

We consider the matrix coefficient

\[
\Phi_{\varphi_{i,v},\varphi'_{i,v}}(g_{i}) = (\pi_{i,v}(g_{i})\varphi_{i,v}, \varphi'_{i,v})_{v}, \quad g_{i} \in G_{i,v}
\]

for a \( \mathcal{K}_{1,v} \)-finite vector \( \varphi_{1,v}, \varphi'_{1,v} \in \pi_{1,v} \) and a \( \mathcal{K}_{0,v} \)-finite vector \( \varphi_{0,v}, \varphi'_{0,v} \in \pi_{0,v} \). Put

\[
I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \int_{G_{0,v}} \Phi_{\varphi_{1,v},\varphi'_{1,v}}(g_{0,v}) \overline{\Phi_{\varphi_{0,v},\varphi'_{0,v}}(g_{0,v})} dg_{0,v},
\]

\[
\alpha_{v}(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}) = \Delta_{G_{1,v}}^{-1} \mathcal{P}_{\pi_{1,v},\pi_{0,v}}(1/2)^{-1} I(\varphi_{1,v}, \varphi'_{1,v}; \varphi_{0,v}, \varphi'_{0,v}).
\]

When \( \varphi_{1,v} = \varphi'_{1,v} \) and \( \varphi_{0,v} = \varphi'_{0,v} \), we simply denote these objects by \( I(\varphi_{1,v}, \varphi_{0,v}) \) and \( \alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) \), respectively. If both \( \pi_{1,v} \) and \( \pi_{0,v} \) are tempered, then the integral \( I(\varphi_{1,v}, \varphi_{0,v}) \) is absolutely convergent and \( I(\varphi_{1,v}, \varphi_{0,v}) \geq 0 \) for any \( \mathcal{K}_{i,v} \)-finite vector \( \varphi_{i,v} \in \pi_{i,v} \). Moreover, if \( v \) is a non-archimedean place, and the conditions (U1), (U2), (U3), (U4), (U5), and (U6) hold, then we can show that \( \alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) = 1 \).

**Conjecture 2.1.** Assume that both \( \pi_{1,v} \) and \( \pi_{0,v} \) are tempered. Then

\[
\dim_{\mathbb{C}} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \overline{\pi}_{0,v}, \mathbb{C}) \neq \{0\} \quad \text{if and only if} \quad \alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) > 0
\]

for some \( \mathcal{K}_{i,v} \)-finite vector \( \varphi_{i,v} \in \pi_{i,v} \).

Now let \( \pi_{i} \simeq \otimes_{v} \pi_{i,v} \) be irreducible cuspidal automorphic representation of \( G_{i}(\mathbb{A}) \). We shall say that \( \pi_{i} \) is almost locally generic if \( \pi_{i} \) satisfies the following condition (ALG).

\[
(\text{ALG}) \quad \text{For almost all } v, \text{ the constituent } \pi_{i,v} \text{ is generic.}
\]

It is believed that \( \pi_{i} \) is almost locally generic if and only if \( \pi_{i} \) is tempered (generalized Ramanujan conjecture).

**Conjecture 2.2.** Let \( \pi_{i} \simeq \otimes_{v} \pi_{i,v} \) be an irreducible cuspidal automorphic representation of \( G_{i}(\mathbb{A}) \). We assume both \( \pi_{1} \) and \( \pi_{0} \) are almost locally generic. Then

1. The integral \( I(\varphi_{1,v}, \varphi_{0,v}) \) should be absolutely convergent and \( I(\varphi_{1,v}, \varphi_{0,v}) \geq 0 \) for any \( \mathcal{K}_{i,v} \)-finite vector \( \varphi_{i,v} \in \pi_{i,v} \).
2. \( \dim_{\mathbb{C}} \text{Hom}_{G_{0,v}}(\pi_{1,v} \otimes \overline{\pi}_{0,v}, \mathbb{C}) \neq \{0\} \quad \text{if and only if} \quad \alpha_{v}(\varphi_{1,v}, \varphi_{0,v}) > 0 \quad \text{for some} \quad \mathcal{K}_{i,v} \text{-finite vector} \quad \varphi_{i,v} \in \pi_{i,v} \).
Now we state our global conjecture.

**Conjecture 2.3.** Let $\pi_1 \cong \otimes_v \pi_{1,v}$ and $\pi_0 \cong \otimes_v \pi_{0,v}$ are irreducible cuspidal automorphic representations of $G_1(\mathbb{A})$ and $G_0(\mathbb{A})$, respectively. We assume $\pi_1$ and $\pi_0$ are almost locally generic. Then there should be an integer $\beta$ such that

$$
\frac{|\langle \varphi_1|_{G_0}, \varphi_0 \rangle|^2}{\langle \varphi_1, \varphi_1 \rangle \langle \varphi_0, \varphi_0 \rangle} = 2^\beta C_0 \Delta_{G_1} P_{\pi_1, \pi_0} (1/2) \prod_{v \in S} \frac{\alpha_v(\varphi_{1,v}, \varphi_{0,v})}{\|\varphi_{1,v}\|^2 \cdot \|\varphi_{0,v}\|^2}
$$

for any non-zero vectors $\varphi_1 = \otimes_v \varphi_{1,v} \in \pi_1$ and $\varphi_0 = \otimes_v \varphi_{0,v} \in \pi_0$.

It seems that the integer $\beta$ is related to the order of the groups, which appear in the theory of endoscopy.

It is possible to formulate a similar conjecture for non-tempered automorphic representations (cf. [15]).

3. The relative trace formula

For low rank groups, some periods formula are proved by using theta correspondence and Rankin-Selberg formulas (see, e.g. [3], [12], [13], [14], [19], [22]). For higher rank groups, it seems some sophisticated tool such as relative trace formula is necessary. In this section, we will discuss how a relative trace formula can be applied to period formulas.

Let $G$ be a connected reductive algebraic group defined over $k$. We assume, for simplicity, $G(k) \backslash G(\mathbb{A})$ is compact.

We recall the Selberg trace formula. Let $f \in C_0^\infty(G(\mathbb{A}))$ be a test function. The kernel function $K_f(g_1, g_2)$ is defined by

$$
K_f(g_1, g_2) = \sum_{\gamma \in G(k)} f(g_1^{-1} \gamma g_2).
$$

For an automorphic form $\varphi$ on $G(\mathbb{A})$,

$$
\rho(f) \varphi(g_2) = (\varphi \ast f)(g_2) = \int_{G(\mathbb{A})} \varphi(g_1) f(g_1^{-1} g_2) \, dg_1
$$

$$
= \int_{G(k) \backslash G(\mathbb{A})} \varphi(g_1) \sum_{\gamma \in G(k)} f(g_1^{-1} \gamma g_2) \, dg_1
$$

$$
= \int_{G(k) \backslash G(\mathbb{A})} \varphi(g_1) K_f(g_1, g_2) \, dg_1.
$$
It follows that

$$\operatorname{tr}\rho(f) = \int_{G(k) \backslash G(\mathbb{A})} K_{f}(g, g) \, dg$$

$$= \int_{G(k) \backslash G(\mathbb{A})} \sum_{\gamma \in G(k)} f(g^{-1}\gamma g) \, dg$$

$$= \sum_{\{\gamma\}} \int_{G(k) \backslash G(\mathbb{A})} \sum_{\gamma' \in G_{\gamma}(k) \backslash G(k)} f(g^{-1}\gamma'^{-1}\gamma'g) \, dg$$

$$= \sum_{\{\gamma\}} \operatorname{Vol}(G_{\gamma}(k) \backslash G_{\gamma}(\mathbb{A})) \int_{G_{\gamma}(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g) \, dg.$$ 

Here, \( \{\gamma\} \) is a conjugacy class of \( \gamma \in G(k) \) and \( G_{\gamma} \) is the centralizer of \( \gamma \). Set \( a(\gamma) = \operatorname{Vol}(G_{\gamma}(k) \backslash G_{\gamma}(\mathbb{A})) \).

Note that the orbital integral \( O(\gamma, f) = \int_{G_{\gamma}(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g) \, dg \) is decomposed as a local product

$$\int_{G_{\gamma}(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g) \, dg = \prod_{v} \int_{G_{\gamma}(k_{v}) \backslash G(k_{v})} f(g_{v}^{-1}\gamma g_{v}) \, dg_{v}.$$ 

The right regular representation \( \rho \) is a sum of automorphic representations \( \rho = \oplus_{\pi} m_{\rho}(\pi) \cdot \pi \). Here, \( m_{\rho}(\pi) \) is the multiplicity of \( \pi \). The distribution character \( \chi_{\pi}(f) \) is defined by \( \chi_{\pi}(f) = \operatorname{tr}\pi(f) \) for a test function \( f \in C_{0}^{\infty}(G(\mathbb{A})) \). Then we have

$$\operatorname{tr}\rho(f) = \sum_{\pi} m_{\rho}(\pi) \chi_{\pi}(f).$$

Thus we have the Selberg trace formula

$$\sum_{\{\gamma\}} a(\gamma) O(\gamma, f) = \sum_{\pi} m_{\rho}(\pi) \chi_{\pi}(f).$$

Note that in the right hand side, \( \pi \) extends over the isomorphism classes of irreducible automorphic representations.

Now, we consider the relative trace formula. Let \( H_{1}, H_{2} \subset G \) be connected algebraic subgroups of \( G \). Let \( \theta_{i} : H_{i}(\mathbb{A}) \to \mathbb{C}^{\times} \) be a character which is trivial on \( H_{i}(k) \) for \( i = 1, 2 \). As before, the kernel function \( K_{f}(g_{1}, g_{2}) \) is defined by

$$K_{f}(g_{1}, g_{2}) = \sum_{\gamma \in G(k)} f(g_{1}^{-1}\gamma g_{2})$$

for a test function \( f \in C_{0}^{\infty}(G(\mathbb{A})) \).
Consider the integral
\[ \int_{H_{1}(k) \backslash H_{1}(A)} \int_{H_{2}(k) \backslash H_{2}(A)} K_{f}(h_{1}, h_{2}) \theta_{1}(h_{1}) \overline{\theta_{2}(h_{2})} \, dh_{1} \, dh_{2} \]
\[ = \sum_{\gamma \in H_{1}(k) \backslash G(k) / H_{2}(k)} \int_{H_{1}(A)} \int_{H_{2,\gamma}(k) \backslash H_{2}(A)} f(h_{1}^{-1} \gamma h_{2}) \theta_{1}(h_{1}) \overline{\theta_{2}(h_{2})} \, dh_{1} \, dh_{2}. \]

Here, \( H_{2,\gamma} = \gamma^{-1}H_{1}\gamma \cap H_{2} \). In this sum, \( \gamma \) contributes only when \( \theta_{1}(\gamma h_{2} \gamma^{-1}) = \theta_{2}(h_{2}) \) for any \( h_{2} \in H_{2,\gamma}(A) \), in which case \( \gamma \) is said to be \( (\theta_{1}, \theta_{2}) \)-relevant (or simply "relevant"). Set
\[ a(\gamma) = Vol(H_{2,\gamma}(k) \backslash H_{2,\gamma}(A)), \]
\[ I_{\gamma}(\theta_{1}, \theta_{2}; f) = \int_{H_{1}(A)} \int_{H_{2,\gamma}(A) \backslash H_{2}(A)} f(h_{1}^{-1} \gamma h_{2}) \theta_{1}(h_{1}) \overline{\theta_{2}(h_{2})} \, dh_{1} \, dh_{2}. \]

Then we have
\[ \int_{H_{1}(k) \backslash H_{1}(A)} \int_{H_{2}(k) \backslash H_{2}(A)} K_{f}(h_{1}, h_{2}) \theta_{1}(h_{1}) \overline{\theta_{2}(h_{2})} \, dh_{1} \, dh_{2} \]
\[ = \sum_{\gamma \in H_{1} \backslash G / H_{2} \text{ relevant}} a(\gamma) I_{\gamma}(\theta_{1}, \theta_{2}; f). \]

On the other hand, note that
\[ \rho(f) \varphi_{1}(g_{2}) = \int_{G(k) \backslash G(A)} K_{f}(g_{1}, g_{2}) \varphi_{1}(g_{1}) \, dg_{1} \]
\[ = \sum_{\pi} \sum_{\varphi_{2} \in \pi} \int_{G(k) \backslash G(A)} K_{f}(g_{1}, g_{2}) \varphi_{1}(g_{1}) \varphi_{2}(g_{2}') \, dg_{1} \, dg_{2}' \cdot \overline{\varphi_{2}(g_{2})} \]
\[ = \sum_{\pi} \sum_{\varphi_{2} \in \pi} \langle K_{f}, \overline{\varphi_{1}} \otimes \varphi_{2} \rangle \cdot \overline{\varphi_{2}(g_{2})}. \]

Here, \( \varphi_{2} \) extends over a complete orthonormal system (CONS) for \( \pi \). It follows that
\[ K_{f}(g_{1}, g_{2}) = \sum_{\pi} \sum_{\varphi_{1}, \varphi_{2} \in \pi \text{ CONS}} \langle K_{f}, \overline{\varphi_{1}} \otimes \varphi_{2} \rangle \cdot \overline{\varphi_{1}(g_{1}) \varphi_{2}(g_{2})} \]
\[ = \sum_{\pi} \sum_{\varphi_{1} \in \pi \text{ CONS}} \varphi(g_{1}) \cdot \rho(f) \varphi(g_{2}). \]
Therefore we have
\[
\int_{H_1(k) \backslash H_1(A)} \int_{H_2(k) \backslash H_2(A)} K_f(h_1, h_2) \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \int_{H_1(k) \backslash H_1(A)} \int_{H_2(k) \backslash H_2(A)} \left[ \sum_{\pi} \sum_{\varphi \in \pi} \overline{\varphi(g_1) \cdot \rho(f) \varphi(g_2)} \right] \theta_1(h_1) \overline{\theta_2(h_2)} \, dh_1 \, dh_2
\]
\[
= \sum_{\pi} \sum_{\varphi \in \pi} \mathcal{P}_{H_1, \theta_1}(\varphi) \mathcal{P}_{H_2, \theta_2}(\rho(f) \varphi).
\]
Set
\[
I_\pi(\theta_1, \theta_2; f) = \sum_{\varphi \in \pi_{\text{CONS}}} \mathcal{P}_{H_1, \theta_1}(\varphi) \mathcal{P}_{H_2, \theta_2}(\rho(f) \varphi).
\]
The automorphic representation \( \pi \) is said to be \((\theta_1, \theta_2)\)-distinguished (or simply "distinguished") if it is \((H_1, \theta_1)\)-distinguished and \((H_2, \theta_2)\)-distinguished. Then we have the relative trace formula
\[
\sum_{\gamma \in H_1 \backslash G / H_2 \ \text{relevant}} a(\gamma) I_\gamma(\theta_1, \theta_2; f) = \sum_{\pi: \text{distinguished}} I_\pi(\theta_1, \theta_2; f).
\]
Note that in the right hand side, \( \pi \) extends over some orthogonal decomposition \( \rho = \sum_\pi \pi \). (Therefore different \( \pi \)'s can be isomorphic.)

Remark 3.1. Assume that \( G \) is the product \( G = G' \times G' \). Let \( H_1 \) be the diagonal subgroup \( H_1 = \Delta(G') = \{(g', g') \mid g' \in G'\} \) and \( H_2 \) be the second factor \( H_2 = \{(1, g') \mid g' \in G'\} \). Set \( \theta_1 = \theta_2 = 1 \). Then the double coset \( H_1 \backslash G / H_2 \) can be identified with the conjugacy classes of \( G' \). If \( \gamma \in H_1 \backslash G / H_2 \) correspond to the conjugacy class \( \gamma' \) of \( G' \), then we have
\[
I_\gamma(\theta_1, \theta_2; f) = O(\gamma', f'),
\]
where
\[
f'(g') = \int_{G'(A)} f(g_1', g_1'g') \, dg_1.
\]
Moreover, an irreducible automorphic representation \( \pi = \pi'_1 \boxtimes \pi'_2 \) is \((\theta_1, \theta_2)\)-distinguished if and only if \( \pi'_1 \simeq \tilde{\pi}'_2 \). In this case, we have \( I_\pi(\theta_1, \theta_2; f) = \text{tr} \pi'_2(f') \). Thus the Selberg trace formula can be considered as a special case of the relative trace formula.

Let \( G', H'_1, \theta'_1, H'_2, \) and \( \theta'_2 \) be another set of data. We assume there exists a bijection
\[
\{ \gamma \in H_1 \backslash G / H_2 \mid \gamma: \text{relevant} \} \simeq \{ \gamma' \in H'_1 \backslash G' / H'_2 \mid \gamma': \text{relevant} \}
\]
with the following properties:

(1) (matching) For each test function $f \in C_0(G(\mathbb{A}))$, there exists a test function $f' \in C_0(G'(\mathbb{A}))$ such that $I_\gamma(\theta_1, \theta_2; f) = I_{\gamma'}(\theta_1, \theta_2; f')$.

(2) (fundamental lemma) For almost all unramified $v$, there exists a Hecke algebra homomorphism

$$\mathcal{H}(K_{G,v} \backslash G_v / K_{G,v}) \rightarrow \mathcal{H}(K_{G',v} \backslash G'_v / K_{G',v})$$

which is compatible with the matching.

Then it is expected that there exists a correspondence for the $L$-packets of $G(\mathbb{A})$ and $G'(\mathbb{A})$ such that

$$I_{\Pi}^\kappa(\theta_1, \theta_2; f) = I_{\Pi'}^{\kappa'}(\theta_1', \theta_2'; f').$$

Here, $\Pi$ is an $L$-packet for $G(\mathbb{A})$, and $\kappa$ is certain function on the $L$-packet and

$$I_{\Pi}^\kappa(\theta_1, \theta_2; f) = \sum_{\pi \in \Pi} \kappa(\pi) I_{\pi}(\theta_1, \theta_2; f).$$

In the right hand side, $\Pi'$ is the $L$-packet of $G'(\mathbb{A})$ corresponding to $\Pi$, and $I_{\Pi'}^{\kappa'}(\theta_1', \theta_2'; f')$ is defined in a similar way.

This equation would imply that there exists a certain relation between period integrals for $G(\mathbb{A})$ and $G'(\mathbb{A})$. In this way, it would be possible to reduce a period formula for $G(\mathbb{A})$ to an analogous formulas for $G'(\mathbb{A})$.

Recently, H. Jacquet [16] proposed a program to attack an analogue of the Gross-Prasad type conjecture for the unirary groups.

**REFERENCES**


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