Title: The Kodaira dimension of subvarieties of Siegel modular varieties (Automorphic Representations, Automorphic Forms, L-functions, and Related Topics)

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1 Introduction

Let $t$ be a positive integer, and $A_{g,t}$ the moduli space of $g$-dimensional abelian varieties with polarizations of type $T = (1, \ldots, 1, t)$. We write $\tilde{A}_{g,t}$ for a smooth compactification of $A_{g,t}$. It is known that $\tilde{A}_{g,t}$ is of general type in the following cases:

1. (Tai, Freitag, Mumford) $t = 1$, $g \geq 7$,
2. (Tai) $t \neq 1, 2$, $g \geq 16$.

We have the same result for subvarieties in $A_{g,t}$. To be more precise, Freitag, Weissauer and Tsuyumine showed that in the case where $g \geq 10$, $t = 1$, all subvarieties of codimension one in $A_{g,t}$ are of general type. Here they adopted the weakened form of the notion "general type". The purpose of the present paper is to report a similar result for the case where $t \neq 1$. Our main result is the following:

**Theorem (Theorem 7)** Assume $g \geq 13$. If $\tilde{A}_{g,t}$ is of general type, then any irreducible variety in $A_{g,t}$ is of general type.

Throughout this article, we assume $g \geq 13$.

2 Siegel modular varieties

Let $H_g = \{ Z \in M_g(\mathbb{C}) \mid \text{Im} Z > 0 \}$ denote the Siegel upper half space. The symplectic group $Sp_{2g}(\mathbb{R})$ acts on $H_n$ by the usual symplectic substitution

$$\gamma \rightarrow \gamma Z = (AZ + B)(CZ + D)^{-1}, \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R}).$$

For a positive integer $t$, let

$$\Delta_t = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \quad \Lambda_t = \begin{pmatrix} O & \Delta_t \\ -\Delta_t & O \end{pmatrix}.$$

We now define some kinds of modular groups. Define

$$\tilde{\Gamma}_t = \{ \gamma \in GL(2g, \mathbb{Z}) \mid \gamma \Lambda_t \gamma^t = \Lambda_t \}.$$
Using $R = \begin{pmatrix} B & O \\ 0 & A_t \end{pmatrix}$, put $\Gamma_t = R^{-1}\Gamma_t R$. For $L = \mathbb{Z}^2 \subset \mathbb{C}^2$, $( , ) : L \times L \to \mathbb{C}$ is defined by $(x, y) \mapsto x\Lambda_t^t y$. We denote by $L'$ the dual lattice of $L$ with respect to $( , )$. Put
\[
\Gamma_t^{\text{lev}} = \{ \gamma \in \Gamma_T \mid M|_{L' \vee / L} = \text{id}|_{L' \vee / L} \}.
\]
We call $\Gamma_t$ and $\Gamma_t^{\text{lev}}$ Siegel modular groups. When $t = 1$, we write $\Gamma_t = \Gamma_t$. Concerning $\Gamma_t^{\text{lev}}$, it is well known that $\Gamma_t^{\text{lev}}$ is a subgroup of $Sp_{2g}(\mathbb{Z})$, and that $\Gamma_t^{\text{lev}}$ is a normal subgroup of finite index in $\Gamma_t$.

Let $\Gamma$ be a Siegel modular group occurring in the above. For a function $f$ on $H_g$ and a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, the slash-operator is defined by
\[
 f|_k\gamma(Z) = \det(CZ + D)^{-k}f(\gamma Z).
\]

A holomorphic function $f$ on $H_g$ is a $\Gamma$-modular form of weight $k$ on $H_g$ if for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,
\[
f|_k\gamma(Z) = f(Z).
\]

Define $A_{g,t} = \Gamma_t \setminus H_g$, $A_{g,t}^{\text{lev}} = \Gamma_t^{\text{lev}} \setminus H_g$. In particular, write $A_g = A_{g,1}$. The quotient spaces $A_{g,t}$ and $A_{g,t}^{\text{lev}}$ are the moduli spaces of $g$-dimensional abelian varieties of the polarization of type $T$ without or with canonical structure, respectively. Let $A$ be a one of them. From the theory of the toroidal compactification, we can construct a projective variety $\overline{A}$ such that $\overline{A} - A$ has normal crossing and $\overline{A}$ has only finite quotient singularities. Resolving these singularities, we obtain a projective nonsingular variety $\overline{A}$. We call $\overline{A}$ and $\overline{A}$ Siegel modular varieties. These varieties are central objects in this paper.

Freitag defined in [1] the following weakened form of the notion “general type”.

**Definition 1** A nonsingular compact irreducible algebraic variety $X$ is of type $G$ (of general type) if there exist $n = \dim X$ algebraically independent rational functions $f_1, \ldots, f_n$ and a non-zero holomorphic tensor $T \in \Omega^\otimes d(X)$ ($d > 0$) such that tensors $f_1 T, \ldots, f_n T$ are holomorphic on $X$.

We adopt this notion for subvarieties of Siegel modular varieties.

## 3 Construction of certain differential forms

Let $Z = (z_{ij})$. Define
\[
 e_{ij} = \begin{cases} 1 & (i \neq j) \\ 2 & (i = j) \end{cases}.
\]

Using it, put
\[
 \omega_{ij} = (-1)^{i+j} e_{ij} dz_{11} \wedge \cdots \wedge \overline{dz_{ij}} \wedge \cdots \wedge dz_{gg} \quad (1 \leq i \leq j \leq g),
\]
where $\overline{dz_{ij}}$ means that $dz_{ij}$ is omitted. Let $\omega = (\omega_{ij})$. Then for a non-negative integer $r$, $\omega^\otimes r$ stands for the tensor power of $\omega$. The tensor power $\omega^\otimes r$ satisfies
\[
 \gamma \cdot \omega^\otimes r = \det(CZ + D)^{-r(g+1)}(CZ + D)^\otimes r \omega^\otimes r \cdot t(CZ + D)^\otimes r
\]
for $\gamma = (A \ B) \in Sp_{2g}(\mathbb{R})$.

Now let $A = (a_{ij})$ be a matrix of size $g$, and $I$, $J$ the ordered sets of $r$ integers in \{1, \ldots, g\}, where a repeated choice is allowed.

For $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_r\}$, define

$$A^{(I,J)} = a_{i_1,j_1} \cdots a_{i_r,j_r}.$$ 

Then the $(k, l)$-entry of $A^\otimes r$ is $A^{(I,J)}$ if

$$k = 1 + \sum_{s=1}^{r} (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1)g^{s-1} \quad (1 \leq k, l \leq g^r).$$

Put $\text{sgn}(I) = \prod_{i \in I} (-1)^i$. Suppose $m \geq 2(g - 1)$. Let $\eta$ be a complex $m \times (g - 1)$ matrix such that $^t\eta \eta = 0$ and $\text{rank} \, \eta = g - 1$. Denote by $\eta_i$ ($1 \leq i \leq g$) the $(g - 1) \times g$ matrix such that

$$\eta_i = \begin{pmatrix} 1 & \cdots & 0 & 1 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 1 \end{pmatrix}.$$ 

We take a fixed positive symmetric matrix $F$ of size $m$ with rational coefficients. Define a theta series associated to $F$ by

$$\theta_{F}^{(I,J)} \begin{pmatrix} u \\ v \end{pmatrix}(Z) = \text{sgn}(I)\text{sgn}(J) \sum_{G} \prod_{i \in I} \det(\eta_i^t (G + u)F^{1/2}\eta)$$

$$\times \prod_{j \in J} \det(\eta_j^t (G + u)F^{1/2}\eta) e^\left[ \text{tr} \left( \frac{1}{2}ZF^t(G + u) + ^t(G + u)v \right) \right],$$

where $G$ runs through all $m \times g$ integral matrices, and $u, v$ are $m \times g$ matrices with rational coefficients.

Let $\Psi_{F,r} \begin{pmatrix} u \\ v \end{pmatrix}(Z)$ be the square matrix of size $g^r$ whose $(k, l)$-entry is $\theta_{F}^{(I,J)} \begin{pmatrix} u \\ v \end{pmatrix}(Z)$, where

$$k = 1 + \sum_{s=1}^{r} (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1)g^{s-1}$$

when $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_r\}$. Tsuyumine [9] shows that there exist $l, r' \in \mathbb{N}$ such that for any $\gamma = (A \ B) \in \Gamma_g(l)$,

$$\left( \Psi_{F,r} \begin{pmatrix} u \\ v \end{pmatrix}(\gamma Z) \right)^{\otimes r'}$$

$$= \det(CZ + D)^{(m/2 + 2r)r'} \left( ^t(CZ + D)^{-1} \right)^{\otimes r'} \left( \Psi_{F,r} \begin{pmatrix} u \\ v \end{pmatrix}(Z) \right)^{\otimes r'} \left( ((CZ + D)^{-1})^{r'} \right)^{\otimes r'}. $$
Let \( \{\gamma_j\} \) be a system of representatives of \( \Gamma_g \) modulo \( \Gamma_g(l) \). When \( \gamma_j = \left( \begin{array}{ll} A_j & B_j \\ C_j & D_j \end{array} \right) \), put

\[
\Psi(Z) = \sum_j \det(C_jZ+D_j)^{-\left(\frac{m}{2}+2r\right)r'} \cdot \left( C_jZ+D_j \right)^{\otimes rr'} \left( \Psi_{F,r} \left( \begin{array}{l} u \\ v \end{array} \right) (\gamma_jZ) \right)^{\otimes r'} \left( (C_jZ+D_j)^{-1} \right)^{\otimes rr'}
\]

Then for \( \gamma = \left( \begin{array}{ll} A & B \\ C & D \end{array} \right) \in \Gamma_g \),

\[
\Psi(\gamma Z) = \det(CZ+D)^{\left(\frac{m}{2}+2r\right)r'} \left( (CZ+D)^{-1} \right)^{\otimes rr'} \Psi(Z) \left( (CZ+D)^{-1} \right)^{\otimes rr'}
\]

Moreover we construct the symmetrization of \( \Psi(Z) \). Let \( \{\delta_i\} \) be a system of representatives of \( \Gamma_t \) modulo \( \Gamma_t^{\text{lev}} \). When \( \delta_i = \left( \begin{array}{ll} A_t & B_i' \\ C_i' & D_i \end{array} \right) \), put

\[
\Phi(Z) = \sum_i \det(C_i'Z+D_i')^{-\left(\frac{m}{2}+2r\right)r'} \cdot \left( C_i'Z+D_i' \right)^{\otimes rr'} \Psi(\delta_iZ) \left( (C_i'Z+D_i')^{-1} \right)^{\otimes rr'}
\]

Then we have

**Proposition 2** For any \( \gamma = \left( \begin{array}{ll} A & B \\ C & D \end{array} \right) \in \Gamma_t \), we have

\[
\Phi(\gamma Z) = \det(CZ+D)^{\left(\frac{m}{2}+2r\right)r'} \left( (CZ+D)^{-1} \right)^{\otimes rr'} \Phi(Z) \left( (CZ+D)^{-1} \right)^{\otimes rr'}
\]

**Proposition 3** Let \( Z_0 \) be any fixed point of \( H_g \). Take any non-zero complex symmetric matrix \( W \) of size \( g \). Let \( m \) be an integer with \( m \geq 2(g-1) \). Then for infinitely many \( r \) and \( r' \), there exists a symmetric matrix \( \Phi(Z) \) occurring in the last proposition such that \( \text{tr}(\Phi(Z_0)W^{\otimes rr'}) \neq 0 \).

Put

\[
\lambda_{m,r,r'} = \text{tr} \left( \Phi(Z)\omega^{\otimes rr'} \right).
\]

Combining Proposition 1 with Proposition 2, we conclude

**Theorem 4** Let \( Z_0 \) be any fixed point in \( H_g \), and \( m \) an integer with \( m \geq 2(g-1) \). Then for infinitely many \( r \) and \( r' \), there exists \( \lambda_{m,r,r'} \) such that

1. \( \lambda_{m,r,r'} \) does not vanish at \( Z = Z_0 \).
2. for all \( \gamma = \left( \begin{array}{ll} A & B \\ C & D \end{array} \right) \in \Gamma_t \),

\[
\gamma \cdot \lambda_{m,r,r'} = \det(CZ+D)^{(m/2-r(g-1))r'} \lambda_{m,r,r'}.
\]

### 4 Extensibility of certain differential forms

For a rational boundary component \( F \), let \( P(F) \subset Sp_{2g}(\mathbb{R}) \) denote the stabilizer of \( F \), \( P'(F) \) the center of the unipotent radical of \( P(F) \), and \( C(F) \) the self-adjoint cone corresponding to \( F \).
Definition 5 Let $\Gamma$ be a Siegel modular group. A $\Gamma$-modular form $f$ vanishes on the rational boundary component $F$ of order at least $l$ if the following condition are satisfied. If we consider the Fourier-Jacobi expansion of $f$ at $F$ 
\begin{equation*}
  f(Z) = \sum_{x \in (P'(F))^{\vee}} a_{x}^{F}(u, t)e[\langle x, z \rangle],
\end{equation*}
then $a_{x}^{F} \neq 0$ implies $\min_{y \in P'(F) \cap \overline{C(F)} - \{0\}} (x, y) \geq l$.

Assume that $r(g - 1) > m/2$. For any $\Gamma_{t}$-modular form $f$ of weight $(r(g - 1) - m/2)r'$, by Theorem 4, $f\lambda_{m, r, r'}$ is a $\Gamma_{t}$-invariant form in $(\Omega_{H_{g}}^{N-1})^{\otimes rr'}$.

Proposition 6 If a $\Gamma_{t}$-modular form $f$ vanishes on all rational corank 1 boundary components of order at least $rr'/C_{t}$, then $f\lambda_{m, r, r'}$ extends to $\tilde{A}_{g,t}$. Here $C_{t}$ stands for $\min\{1, \sqrt{3}/\sqrt[9]{t^{g-1}}\}$.

5 The main result

Theorem 7 If $\tilde{A}_{g,t}$ is of general type, then any irreducible subvariety of codimension one in $A_{g,t}$ is of general type.

Let us give an outline of the proof to the above theorem. Let $D$ be any irreducible subvariety in $A_{g,t}$ of codimension 1, and $\pi : H_{g} \rightarrow A_{g,t}$ the canonical map. It should be noted that we can construct a $\Gamma_{g}$-modular form $f$ whose restriction to $\pi^{-1}(D)$ does not vanish ([11], [10]). For such a weight $k$ modular form $f$, its symmetrization $\text{Sym}(f) = \prod_{\gamma \in \Gamma_{t}/\Gamma_{t}^{1\cdot v}} f|_{k}\gamma$ is a $\Gamma_{t}$-modular form that does not vanish on $\pi^{-1}(D)$. The following diagram is helpful:

\begin{equation*}
  \begin{array}{ccc}
  \mathcal{A}_{g}^{\text{lev}} & \swarrow' & \mathcal{A}_{g,t} \\
    \mathcal{A}_{g} & & \mathcal{A}_{g,t} \\
    \mathcal{A}_{g,t}
  \end{array}
\end{equation*}

If $f$ is a $\Gamma_{g}$-modular form, then we have

\begin{equation*}
  \frac{(g - 1)\text{ord}(\text{Sym}(f))}{\text{weight}(\text{Sym}(f))} = \frac{(g - 1)\text{ord}(f)}{\text{weight}(f)}.
\end{equation*}

Here $\text{ord}(\text{Sym}(f))$ and $\text{ord}(f)$ are vanishing orders at rational corank 1 boundary components.

If $f$ is a non-trivial $\Gamma_{t}$-modular form such that $(g - 1)\text{ord}(f)/\text{weight}(f) > 1$, then for certain integers $a, b$, each modular form $f'$ in $f^{ak}M_{bk}(\Gamma_{t})$ ($k \geq 1$) has enough vanishing order at the cusp. If $\text{weight}(f') = (r(g - 1) - m/2)r'$, then $f'\lambda_{m, r, r'}$ extends to a section of $(\Omega_{H_{g}}^{N-1})^{\otimes rr'}$, where $N = g(g + 1)/2$.

Generalizing Lemma 2.2 in [2], it is possible to find generators for $\Gamma_{t}$. Using them, we see that $[\Gamma_{t}, \Gamma_{t}]$, the commutator subgroup of $\Gamma_{t}$, is $\Gamma_{t}$ itself. Moreover, $\Gamma_{t}$ satisfies $b_{1}(\Gamma_{t}) = 0$, $b_{2}(\Gamma_{t}) = 1$. Hence by Theorem 1' in [8], any effective divisor on $A_{g,t}$ is
defined by some $\Gamma_t$-modular form. Furthermore, the ring $\bigoplus_{k \geq 0} M_k(\Gamma_t)$ of $\Gamma_t$-modular forms is factorial.

There exists a $\Gamma_t$-modular form $h$ such that its divisor $(h)$ on $H_g$ is $\pi^{-1}(D)$. By Theorem 4, for infinitely many $r, r'$, $\lambda_{m,r,r'}\mid \pi^{-1}(D) \neq 0$. We can take such $r, r'$ from the set of multiples of any fixed integer. For a suitable integer $k$, there exist weight $k$ $\Gamma_t$-modular forms $g_1, \ldots, g_N$ such that they do not vanish on $D$, that each $g_i \lambda_{m,r,r'}$ extends to $\tilde{A}_{g,t}$, and that $g_2/g_1, \ldots, g_N/g_1$ are algebraically independent. Therefore $D$ is of general type.

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References


