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Kyoto University
The Kodaira dimension of subvarieties of Siegel modular varieties

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1 Introduction

Let $t$ be a positive integer, and $A_{g,t}$ the moduli space of $g$-dimensional abelian varieties with polarizations of type $T = (1, \ldots, 1, t)$. We write $\tilde{A}_{g,t}$ for a smooth compactification of $A_{g,t}$. It is known that $\tilde{A}_{g,t}$ is of general type in the following cases:

1. (Tai, Freitag, Mumford) $t = 1$, $g \geq 7$,
2. (Gritsenko) $t = 2$, $g \geq 13$,
3. (Tai) $t \neq 1, 2$, $g \geq 16$.

We have the same result for subvarieties in $A_{g,t}$. To be more precise, Freitag, Weissauer and Tsuyumine showed that in the case where $g \geq 10$, $t = 1$, all subvarieties of codimension one in $A_{g,t}$ are of general type. Here they adopted the weakened form of the notion “general type”. The purpose of the present paper is to report a similar result for the case where $t \neq 1$. Our main result is the following:

Theorem (Theorem 7) Assume $g \geq 13$. If $\tilde{A}_{g,t}$ is of general type, then any irreducible variety in $A_{g,t}$ is of general type.

Throughout this article, we assume $g \geq 13$.

2 Siegel modular varieties

Let $H_g = \{ Z \in M_g(\mathbb{C}) \mid ^t \bar{Z} = Z, \text{Im} \ Z > 0 \}$ denote the Siegel upper half space. The symplectic group $Sp_{2g}(\mathbb{R})$ acts on $H_n$ by the usual symplectic substitution

$$ \gamma \to \gamma Z = (AZ + B)(CZ + D)^{-1}, \quad \gamma = (\begin{array}{cc} A & B \\ C & D \end{array}) \in Sp_{2g}(\mathbb{R}). $$

For a positive integer $t$, let

$$ \Delta_t = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & \cdots \\ & & & 1 \\ & & & t \end{pmatrix}, \quad \Lambda_t = \begin{pmatrix} O & \Delta_t \\ -\Delta_t & O \end{pmatrix}. $$

We now define some kinds of modular groups. Define

$$ \tilde{\Gamma}_t = \{ \gamma \in GL(2g, \mathbb{Z}) \mid \gamma \Lambda_t \gamma^t = \Lambda_t \}. $$
Using $R = (\delta_{ij})_{t}$, put $\Gamma_{t} = R^{-1}T_{t}R$. For $L = \mathbb{Z}^{2g} \subset \mathbb{C}^{g}$, $(,) : L \times L \to \mathbb{C}$ is defined by $(x, y) \mapsto xA_{t}y$. We denote by $L'$ the dual lattice of $L$ with respect to $(,)$. Put

$$
\Gamma_{t}^{\text{lev}} = \{ \gamma \in \Gamma_{T} | M|_{L'/L} = \text{id}|_{L'/L} \}.
$$

We call $\Gamma_{t}$ and $\Gamma_{t}^{\text{lev}}$ Siegel modular groups. When $t = 1$, we write $\Gamma_{g} = \Gamma_{t}$. Concerning $\Gamma_{t}^{\text{lev}}$, it is well known that $\Gamma_{t}^{\text{lev}}$ is a subgroup of $Sp_{2g}(\mathbb{Z})$, and that $\Gamma_{t}^{\text{lev}}$ is a normal subgroup of finite index in $\Gamma_{t}$.

Let $\Gamma$ be a Siegel modular group occurring in the above. For a function $f$ on $H_{g}$ and a matrix $\gamma = (A,B,C,D)$, the slash-operator is defined by

$$
f|k\gamma(Z) = \det(CZ + D)^{-k}f(\gamma Z).
$$

A holomorphic function $f$ on $H_{g}$ is a $\Gamma$-modular form of weight $k$ on $H_{g}$ if for all $\gamma = (A,B,C,D) \in \Gamma$,

$$
f|k\gamma(Z) = f(Z).
$$

Define $A_{g,t} = \Gamma_{t} \backslash H_{g}$, $A_{g,t}^{\text{lev}} = \Gamma_{t}^{\text{lev}} \backslash H_{g}$. In particular, write $A_{g} = A_{g,1}$. The quotient spaces $A_{g,t}$ and $A_{g,t}^{\text{lev}}$ are the moduli spaces of $g$-dimensional abelian varieties of the polarization of type $T$ without or with canonical structure, respectively. Let $\mathcal{A}$ be a one of them. From the theory of the toroidal compactification, we can construct a projective variety $\overline{\mathcal{A}}$ such that $\overline{\mathcal{A}} - \mathcal{A}$ has normal crossing and $\overline{\mathcal{A}}$ has only finite quotient singularities. Resolving these singularities, we obtain a projective nonsingular variety $\overline{\mathcal{A}}$. We call $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ Siegel modular varieties. These varieties are central objects in this paper.

Freitag defined in [1] the following weakened form of the notion "general type".

**Definition 1** A nonsingular compact irreducible algebraic variety $X$ is of type $G$ (of general type) if there exist $n = \dim X$ algebraically independent rational functions $f_{1}, \ldots, f_{n}$ and a non-zero holomorphic tensor $T \in \Omega_{d}^{\otimes d}(X)$ ($d > 0$) such that tensors $f_{1}T, \ldots, f_{n}T$ are holomorphic on $X$.

We adopt this notion for subvarieties of Siegel modular varieties.

## 3 Construction of certain differential forms

Let $Z = (z_{ij})$. Define

$$
e_{ij} = \begin{cases} 
1 & (i \neq j) \\
2 & (i = j)
\end{cases}.
$$

Using it, put

$$
\omega_{ij} = (-1)^{i+j}e_{ij}dz_{11} \wedge dz_{12} \wedge \cdots \wedge \overline{dz_{ij}} \wedge \cdots \wedge dz_{gg} \quad (1 \leq i \leq j \leq g),
$$

where $\overline{dz_{ij}}$ means that $dz_{ij}$ is omitted. Let $\omega = (\omega_{ij})$. Then for a non-negative integer $r$, $\omega^{\otimes r}$ stands for the tensor power of $\omega$. The tensor power $\omega^{\otimes r}$ satisfies

$$
\gamma \cdot \omega^{\otimes r} = \det(CZ + D)^{-r(g+1)}(CZ + D)^{\otimes r}\omega^{\otimes r} \cdot t(CZ + D)^{\otimes r}
$$
for $\gamma = (A B) \in Sp_{2g}(\mathbb{R})$.

Now let $A = (a_{ij})$ be a matrix of size $g$, and $I, J$ the ordered sets of $r$ integers in \{1, \ldots, g\}, where a repeated choice is allowed.

For $I = \{i_1, \ldots, i_r\}, J = \{j_1, \ldots, j_r\}$, define

$$A^{(I,J)} = a_{i_1,j_1} \cdots a_{i_r,j_r}.$$ 

Then the $(k, l)$-entry of $A^{\otimes r}$ is $A^{(I,J)}$ if

$$k = 1 + \sum_{s=1}^{r} (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1)g^{s-1} \quad (1 \leq k, l \leq g^r).$$

Put $sgn(I) = \prod_{i \in I} (-1)^i$. Suppose $m \geq 2(g - 1)$. Let $\eta$ be a complex $m \times (g - 1)$ matrix such that $^t\eta \eta = 0$ and rank $\eta = g - 1$. Denote by $\eta_i$ ($1 \leq i \leq g$) the $(g - 1) \times g$ matrix such that $\eta_i = (1 \cdots 1 0 1 \cdots 1)$. We take a fixed positive symmetric matrix $F$ of size $m$ with rational coefficients. Define a theta series associated to $F$ by

$$\theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix}(Z) = sgn(I)sgn(J) \prod_{i \in I} \det(\eta_i^t(G + u)F^{1/2}\eta) \prod_{j \in J} \det(\eta_j^t(G + u)F^{1/2}\eta)e \left[ \text{tr} \left( \frac{1}{2}ZF[G + u] + ^t(G + u)v \right) \right],$$

where $G$ runs through all $m \times g$ integral matrices, and $u, v$ are $m \times g$ matrices with rational coefficients.

Let $\Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix}(Z)$ be the square matrix of size $g^r$ whose $(k, l)$-entry is $\theta_F^{(I,J)} \begin{bmatrix} u \\ v \end{bmatrix}(Z)$, where

$$k = 1 + \sum_{s=1}^{r} (i_s - 1)g^{s-1}, \quad l = 1 + \sum_{s=1}^{r} (j_s - 1)g^{s-1}$$

when $I = \{i_1, \ldots, i_r\}, J = \{j_1, \ldots, j_r\}$. Tsuyumine [9] shows that there exist $l, r' \in \mathbb{N}$ such that for any $\gamma = (A B) \in \Gamma_g(l)$,

$$\left( \Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix}(\gamma Z) \right)^{\otimes r'} = \det(CZ+D)^{(m/2+2r)r'} \left( ^t(CZ + D)^{-1} \right)^{\otimes rr'} \left( \Psi_{F,r} \begin{bmatrix} u \\ v \end{bmatrix}(Z) \right)^{\otimes r'} \left( (CZ + D)^{-1} \right)^{\otimes rr'}.$$
Let \( \{\gamma_j\} \) be a system of representatives of \( \Gamma_g \) modulo \( \Gamma_g(l) \). When \( \gamma_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix} \), put

\[
\Psi(Z) = \sum_j \det(C_jZ + D_j)^{-\frac{m}{2} + 2r} \cdot (C_jZ + D_j)^{\otimes rr'} \cdot \left( \Psi_F(r) \begin{pmatrix} u \\ v \end{pmatrix} \gamma_jZ \right)^{\otimes r'} ((C_jZ + D_j)^{-1})^{\otimes rr'}
\]

Then for \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \)

\[
\Psi(\gamma Z) = \det(CZ + D)^{-\frac{m}{2} + 2r} \cdot (CZ + D)^{\otimes rr'} \Psi(Z) ((CZ + D)^{-1})^{\otimes rr'}
\]

Moreover we construct the symmetrization of \( \Psi(Z) \). Let \( \{\delta_i\} \) be a system of representatives of \( \Gamma_t \) modulo \( \Gamma_t^{\text{lev}} \). When \( \delta_i = \begin{pmatrix} A_i & B_i' \\ C_i' & D_i \end{pmatrix} \), put

\[
\Phi(Z) = \sum_i \det(C_i'Z + D_i')^{-\frac{m}{2} + 2r} \cdot (C_i'Z + D_i')^{\otimes rr'} \Psi(\delta_iZ) ((C_i'Z + D_i'))^{\otimes rr'}
\]

Then we have

**Proposition 2** For any \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t \), we have

\[
\Phi(\gamma Z) = \det(CZ + D)^{\frac{m}{2} + 2r} \cdot (CZ + D)^{-\frac{m}{2} + 2r} \cdot (CZ + D)^{-1})^{\otimes rr'} \Phi(Z) ((CZ + D)^{-1})^{\otimes rr'}
\]

**Proposition 3** Let \( Z_0 \) be any fixed point of \( H_g \). Take any non-zero complex symmetric matrix \( W \) of size \( g \). Let \( m \) be an integer with \( m \geq 2(g - 1) \). Then for infinitely many \( r \) and \( r' \), there exists a symmetric matrix \( \Phi(Z) \) occurring in the last proposition such that \( \text{tr}(\Phi(Z_0)W^{\otimes rr'}) \neq 0 \).

Put

\[
\lambda_{m,r,r'} = \text{tr} \left( \Phi(Z)\omega^{\otimes rr'} \right).
\]

Combining Proposition 1 with Proposition 2, we conclude

**Theorem 4** Let \( Z_0 \) be any fixed point in \( H_g \), and \( m \) an integer with \( m \geq 2(g - 1) \). Then for infinitely many \( r \) and \( r' \), there exists \( \lambda_{m,r,r'} \) such that

1. \( \lambda_{m,r,r'} \) does not vanish at \( Z = Z_0 \).
2. for all \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_t \),

\[
\gamma \cdot \lambda_{m,r,r'} = \det(CZ + D)^{\frac{m}{2} - r(g - 1)r'} \lambda_{m,r,r'}.
\]

### 4 Extensibility of certain differential forms

For a rational boundary component \( F \), let \( P(F) \subset Sp_{2g}(\mathbb{R}) \) denote the stabilizer of \( F \), \( P'(F) \) the center of the unipotent radical of \( P(F) \), and \( C(F) \) the self-adjoint cone corresponding to \( F \).
**Definition 5** Let $\Gamma$ be a Siegel modular group. A $\Gamma$-modular form $f$ vanishes on the rational boundary component $F$ of order at least $l$ if the following condition are satisfied. If we consider the Fourier-Jacobi expansion of $f$ at $F$

$$f(Z) = \sum_{\gamma \in (P'(F))^{\gamma}} a^F_\gamma(u, t)e[\langle x, z \rangle],$$

then $a^F_\gamma \neq 0$ implies $\min_{y \in P'(F) \cap \overline{C(F)} - \{0\}} (x, y) \geq l$.

Assume that $r(g - 1) > m/2$. For any $\Gamma_t$-modular form $f$ of weight $(r(g - 1) - m/2)r'$, by Theorem 4, $f\lambda_{m,r,r'}$ is a $\Gamma_t$-invariant form in $(\Omega^{N-1}_{H_g})^{\otimes rr'}$.

**Proposition 6** If a $\Gamma_t$-modular form $f$ vanishes on all rational corank 1 boundary components of order at least $rr'/(\pi^r)$, then $f\lambda_{m,r,r'}$ extends to $\tilde{A}_{g,t}$. Here $C_i$ stands for $\min\{1, \sqrt{3}/\sqrt[3]{i^2 - 1}\}$.

## 5 The main result

**Theorem 7** If $\tilde{A}_{g,t}$ is of general type, then any irreducible subvariety of codimension one in $A_{g,t}$ is of general type.

Let us give an outline of the proof to the above theorem. Let $D$ be any irreducible subvariety in $A_{g,t}$ of codimension 1, and $\pi : H_g \rightarrow A_{g,t}$ the canonical map. It should be noted that we can construct a $\Gamma_g$-modular form $f$ whose restriction to $\pi^{-1}(D)$ does not vanish ([11], [10]). For such a weight $k$ modular form $f$, its symmetrization $\text{Sym}(f) = \prod_{\gamma \in \Gamma_t/\Gamma_t^{ev}} f|_{k}\gamma$ is a $\Gamma_t$-modular form that does not vanish on $\pi^{-1}(D)$. The following diagram is helpful:

$\begin{array}{ccc}
\tilde{A}_{g,t}^{\text{lev}} & \leftarrow & \tilde{A}_{g,t} \\
A_g & \searrow & A_{g,t}
\end{array}$

If $f$ is a $\Gamma_g$-modular form, then we have

$$\frac{(g - 1)\text{ord}(\text{Sym}(f))}{\text{weight}(\text{Sym}(f))} = \frac{(g - 1)\text{ord}(f)}{\text{weight}(f)}.$$  

Here $\text{ord}(\text{Sym}(f))$ and $\text{ord}(f)$ are vanishing orders at rational corank 1 boundary components.

If $f$ is a non-trivial $\Gamma_t$-modular form such that $(g - 1)\text{ord}(f)/\text{weight}(f) > 1$, then for certain integers $a, b$, each modular form $f'$ in $f^{ak} M_{bk}(\Gamma_t)$ ($k \geq 1$) has enough vanishing order at the cusp. If $\text{weight}(f') = (r(g - 1) - m/2)r'$, then $f'\lambda_{m,r,r'}$ extends to a section of $(\Omega^{N-1}_{H_g})^{\otimes rr'}$, where $N = g(g + 1)/2$.

Generalizing Lemma 2.2 in [2], it is possible to find generators for $\Gamma_t$. Using them, we see that $[\Gamma_t, \Gamma_t]$, the commutator subgroup of $\Gamma_t$, is $\Gamma_t$ itself. Moreover, $\Gamma_t$ satisfies $b_1(\Gamma_t) = 0, b_2(\Gamma_t) = 1$. Hence by Theorem 1' in [8], any effective divisor on $A_{g,t}$ is
defined by some $\Gamma_t$-modular form. Furthermore, the ring $\bigoplus_{k \geq 0} M_k(\Gamma_t)$ of $\Gamma_t$-modular forms is factorial.

There exists a $\Gamma_t$-modular form $h$ such that its divisor $(h)$ on $H_g$ is $\pi^{-1}(D)$. By Theorem 4, for infinitely many $r, r'$, $\lambda_{m,r,r'}|\pi^{-1}(D) \neq 0$. We can take such $r, r'$ from the set of multiples of any fixed integer. For a suitable integer $k$, there exist weight $k$ $\Gamma_t$-modular forms $g_1, \ldots, g_N$ such that they do not vanish on $D$, that each $g_i\lambda_{m,r,r'}$ extends to $\tilde{A}_{g,t}$, and that $g_2/g_1, \ldots, g_N/g_1$ are algebraically independent. Therefore $D$ is of general type.

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References


