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SPECIAL COHOMOLOGY CLASSES FOR THE WEIL REPRESENTATION

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1. Introduction

In this note, I give a survey of my joint work with J. Millson [12, 13] on the Kudla-Millson theory, see e.g. [21]. The theme of this theory is to utilize Riemannian geometry and the theory of dual pairs and theta correspondence to construct modular forms with geometric interpretations. These modular forms, which are realized by theta series, take values in the (co)homology of arithmetic quotients X of symmetric spaces D associated to orthogonal or unitary groups. Furthermore, in this way, one obtains generating series of totally geodesic so-called "special" cycles, which are arithmetic quotients of subsymmetric spaces of D.

The work of Kudla-Millson is mainly concerned with X compact, or if X is non-compact with the cohomology of compact supports of X. One of the main motivations for our work is the problem to extend the Kudla-Millson lift from the cohomology of compact supports to cohomology groups of X which capture the geometry of the boundary of X. This is in particular inspired by Hirzebruch and Zagier [16] who with their groundbreaking work in the case of Hilbert modular surfaces initiated the study of the relationship between cycles in locally symmetric spaces and modular forms. In their work the intersection of the Hirzebruch-Zagier curves with the compactifying divisors of Hirzebruch's non-singular (toroidal) compactification of X plays a crucial role. In an upcoming work [14] we show how the results in [16] can be recovered in terms of differential geometry and theta series.

As mentioned above we mainly discuss [12, 13]. In [12], we extend the Kudla-Millson theory to the case where the cycles have local coefficients for X attached to the real orthogonal group $\mathrm{O}(p,q)$. Now the correspondence involves vector-valued Siegel modular forms. In [13], which actually serves as the main motivation for [12], we restrict the theta series underlying this correspondence to the Borel-Serre boundary of X. We show that the theta series extend to the Borel-Serre compactfication of X. Moreover the restriction is again a theta series as in [12], now for a smaller orthogonal group and a larger coefficient system. As application we establish the cohomological nonvanishing of the special (co)cyles when passing to an appropriate finite cover of X. In particular, the (co)homology groups in question do not vanish.

For simplicity, we restrict ourselves in this note to the case when the cycles have codimension q. This situation is covered by the dual pair $\mathrm{SL}(2) \times \mathrm{O}(p,q)$. We also assume that $p+q$ is even, so that we don't have to deal with covering groups. The general case is $\mathrm{Sp}(g) \times \mathrm{O}(p,q)$ with $g \leq p$ and q arbitrary.

In section 2, we consider the "degenerate" positive definite case when $q=0$ and the symmetric space is a point. In that case our theory with non-trivial coefficients is the theory of theta series with harmonic polynomials. However, our approach using Howe operators and Hermite polynomials is somewhat different than the usual treatment. We hope that this discussion sheds some light on the general constructions. In section 3, we

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introduce the special cycles in question. In section 4, we discuss the basics of the Kudla-Millson theory. In section 5, we equip the cycles with local coefficients and describe the theta lift obtained in [12] for this situation. In section 6, we discuss the restriction formula for these classes, which we apply in section 7 for a non-vanishing result for the special cycles.

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2. Positive definite theta series

Let V be a rational vector space of dimension $m = p + q$ with a non-degenerate positive definite symmetric bilinear form $(\, , \,)$. For simplicity we assume m is even. We let x_α denote the coordinate functions on V with respect to an orthonormal basis $\{e_\alpha\}$ of $V_\mathbb{R}$.

2.1. The classical theta series. We let L be an even lattice of level N, that is, $Q(x) := \frac{1}{2}(x,x) \in \mathbb{Z}$ for $x \in L$ and $Q(L^\#)\mathbb{Z} = \frac{1}{N}\mathbb{Z}$. Here $L^\#$ is the dual lattice. Furthermore, we fix a a vector $h \in L^\#/L$ once and for all and write $\mathcal{L} = L + h$.

It is very well known (e.g. [23]) and an application of the Poisson summation formula that for $\tau \in \mathbb{H}$, the upper half plane, the associated theta series

$$\theta(\tau, \mathcal{L}) = \sum_{x \in \mathcal{L}} e^{\pi i (x,x)\tau} \in M_{\frac{m}{2}}(\Gamma(N)) \tag{2.1}$$

is a modular form for the principal congruence subgroup $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ of weight $\frac{m}{2}$. Furthermore, we let $q = e^{2\pi i \tau}$ and $r_n(\mathcal{L}) = \#\{x \in \mathcal{L};\ Q(x) = n\}$ is the number of ways n can be represented by $\mathcal{L}$. Then $\theta(\tau, \mathcal{L})$ is equal to the generating series of these representation numbers:

$$\theta(\tau, \mathcal{L}) = \sum_{n \in \mathbb{Q}_+}^{\infty} r_n(\mathcal{L}) q^n. \tag{2.2}$$

2.2. The Weil representation and the construction of theta series. We describe a more representation-theoretic approach to the theory of theta series.

We let $\mathcal{S}(V_\mathbb{R})$ be the space of Schwartz functions on $V_\mathbb{R}$. We write $G' = \mathrm{SL}_2(\mathbb{R})$ and let $K' = \mathrm{SO}(2)$ be its standard maximal compact subgroup. We let $G = \mathrm{O}(V_\mathbb{R})$ be the orthogonal group of $V_\mathbb{R}$. Then $G' \times G$ acts on $\mathcal{S}(V_\mathbb{R})$ via the Weil representation ω for the additive character $t \mapsto e^{2\pi i t}$. Note that G' acts naturally on $\mathcal{S}(V_\mathbb{R})$ by $\omega(g)\varphi(x) = \varphi(g^{-1}x)$. The action of G' is given as follows:

$$\omega\left(\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)\right)\varphi(x) = a^{m/2}\varphi(ax) \qquad \text{and} \qquad \omega\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right)\varphi(x) = e^{\pi i b(x,x)}\varphi(x),$$

for $a > 0$. Finally,

$$\omega\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right)\varphi(x) = i^{m/2}\hat{\varphi}(x),$$

where $\hat{\varphi}(y) = \int_{V(\mathbb{R})} \varphi(x) e^{-2\pi i (x,y)} dx$ is the Fourier transform.

We let $\mathbb{C}[\chi_r]$ be the standard one-dimensional character of $K' = \mathrm{SO}(2) \simeq U(1)$ given by $z \mapsto \chi_r(z) = z^r$. Let $\varphi \in \mathcal{S}(V_\mathbb{R})$ be an eigenfunction under the maximal compact subgroup of $\mathrm{SO}(2)$ of weight r, that is, $\omega(k')\varphi = \chi_r(k')\varphi$. Then we can define

$$\varphi(x,\tau) = j(g'_\tau, i)^r \omega(g'_\tau)\varphi(x) = v^{-r/2+m/4}\varphi(\sqrt{v}x)e^{\pi i (x,x)u}, \tag{2.3}$$

where $g'_\tau \in \mathrm{SL}_2(\mathbb{R})$ is any element which moves the base point $i \in \mathbb{H}$ to $\tau = u + iv \in \mathbb{H}$ and $j(g'_\tau, i) = v^{-1/2}$ denotes the usual automorphy factor. Of course, we can take $g'_\tau = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}$. Then by the standard theta machinery, the associated theta series

(2.4) $$\theta(\tau, \varphi, \mathcal{L}) := \sum_{x \in \mathcal{L}} \varphi(x, \tau)$$

is a in general non-holomorphic modular form of level N and weight r. For the theta series above, the Gaussian $\varphi_0(x) := e^{-\pi(x,x)}$ has weight $m/2$ and $\theta(\tau, \mathcal{L}) = \theta(\tau, \varphi_0, \mathcal{L})$.

2.3. **Theta series with harmonic coefficients.** The construction of $\theta(\tau, \mathcal{L})$ can be generalized in the following way. We let $\mathrm{Sym}^{\ell'}(V_\mathbb{C})$ be the space of symmetric ℓ'-tensors on $V_\mathbb{C}$, and we identify $\mathrm{Sym}^{\ell'}(V_\mathbb{C})$ with the space of homogeneous polynomials on V of degree ℓ' in the usual way via the canonical isomorphism between V and V^* given by the inner product. We use the same symbol $\mathrm{Sym}^{\ell'}(V_\mathbb{C})$ for both. Then we let $\mathcal{H}^{\ell'}(V_\mathbb{C})$ be the subspace of harmonic polynomials on V. That is, a polynomial $P \in \mathrm{Sym}^{\ell'}(V_\mathbb{C})$ is harmonic if $\sum_{\alpha=1}^m \frac{\partial^2}{\partial x_\alpha^2} P = 0$. Note that $\mathcal{H}^{\ell'}(V_\mathbb{C})$ is an irreducible representation of G. Moreover, the main theorem of spherical harmonics states

(2.5) $$\mathrm{Sym}^{\ell'}(V_\mathbb{C}) = \mathcal{H}^{\ell'}(V_\mathbb{C}) \oplus r^2 \,\mathrm{Sym}^{\ell'-2}(V_\mathbb{C}),$$

as G-modules. Here $r^2 = \sum_{\alpha=1}^m e_\alpha^2$. Note that this decomposition is orthogonal with respect to the inner product on $\mathrm{Sym}^{\ell'}(V)$ induced by $(\,,\,)$.

For $P \in \mathcal{H}^{\ell'}(V_\mathbb{C})$, the theta series

(2.6) $$\theta(\tau, P, \mathcal{L}) = \sum_{x \in \mathcal{L}} P(x) e^{\pi i (x,x)\tau} \in S_{\frac{m}{2}+\ell'}(\Gamma(N))$$

is a cusp form of weight $\frac{m}{2} + \ell'$ and level N, see e.g. [23]. In terms of Schwartz functions, this corresponds to that $P(x)\varphi_0(x) \in \mathcal{S}(V_\mathbb{R})$ has weight $\frac{m}{2} + \ell'$. Furthermore, we have

Lemma 2.1. *For given $P(x) \in \mathcal{H}^{\ell'}(V_\mathbb{C})$, there exists a coset of a lattice $\mathcal{L}$ such that*
$$\theta(\tau, P, \mathcal{L}) \neq 0.$$

Proof. We give a very simple argument which we learned from E. Freitag and R. Schulze-Pillot. We can assume $V = \mathbb{Q}^m$ with the standard inner product. First find a vector $h \in \frac{1}{N_1}\mathbb{Z}^m$ with $N_1 \in \mathbb{Z}$ such that $P(h) \neq 0$. Then pick a lattice $L = N_1 N_2 \mathbb{Z}^m$ such that $|\sum_{x \in L} P(x) e^{-\pi(x,x)}| < |P(h)|$. Such a $N_2 \in \mathbb{Z}$ exists as $P(x)e^{-\pi(x,x)} \in \mathcal{S}(V_\mathbb{R})$. Then the theta series $\theta(\tau, P, \mathcal{L})$ for $\mathcal{L} = L + h$ does not vanish. □

In summary,

Proposition 2.2. *Given $\mathcal{L} = L + h$ of level N, we have the theta lift*

(2.7) $$\Lambda_{[\ell']}(\mathcal{L}) : \mathcal{H}^{\ell'}(V_\mathbb{C}) \longrightarrow S_{\frac{m}{2}+\ell'}(\Gamma(N))$$

given by $P \mapsto \theta(\tau, P, \mathcal{L})$, and for each P, there exists an $\mathcal{L}$ such that $\theta(\tau, P, \mathcal{L}) \neq 0$.

Remark 2.3. This lift realizes the (global) theta/Howe correspondence between $\mathrm{O}(V)$ and SL_2 for which on the orthogonal side only the representations $\mathcal{H}^{\ell'}(V_\mathbb{C})$ occur. For other representations of $\mathrm{O}(V)$ to occur one would need to consider the dual pair $\mathrm{O}(V) \times \mathrm{Sp}(g)$ for $g > 1$. In that case, the correspondence will involve vector-valued holomorphic Siegel modular forms of degree g, see e.g. [18, 10].

2.4. **Howe operators.** We now describe a somewhat different approach to theta series of spherical harmonics. This approach is a "degenerate" case of our work for V having arbitrary signature. The following discussion is also closely related to [17], chapter III.2.

We define the Howe operators $\mathcal{D}_\alpha$, $\alpha = 1, \dots, m$, a family of commuting operators on $\mathcal{S}(V_{\mathbb{R}})$, by

$$\mathcal{D}_\alpha = x_\alpha - \frac{1}{2\pi}\frac{\partial}{\partial x_\alpha}, \tag{2.8}$$

We then introduce the Schwartz function

$$\varphi_{\ell'} = \frac{1}{2^{\ell'}} \sum_{\alpha_1,\dots,\alpha_{\ell'}=1}^{m} (\mathcal{D}_{\alpha_1} \circ \cdots \circ \mathcal{D}_{\alpha_{\ell'}})(\varphi_0) \otimes e_{\alpha_1} \cdots e_{\alpha_{\ell'}} \in \mathcal{S}(V_{\mathbb{R}}) \otimes \mathrm{Sym}^{\ell'}(V_{\mathbb{C}}), \tag{2.9}$$

which takes values in the symmetric ℓ'-tensors of V. Note

$$\mathcal{D}_\alpha^k(e^{-\pi x_\alpha^2}) = (2\pi)^{-k/2} H_k\left(\sqrt{2\pi} x_\alpha\right) e^{-\pi x_\alpha^2}, \tag{2.10}$$

where $H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} e^{-t^2}$ is the k-th Hermite polynomial. Hence $\varphi_{\ell'}$ is a linear combination of products of Hermite polynomials times the Gaussian φ_0. Note however, that the Hermite polynomials are *not* homogeneous. In [12], section 6, we show that

$$\omega(k')\varphi_{\ell'} = \chi_{\frac{m}{2}+\ell'}(k')\varphi_{\ell'}. \tag{2.11}$$

This follows directly from the fact that $H_k\left(\sqrt{2\pi} x_\alpha\right) e^{-\pi x_\alpha^2}$ is an eigenfunction under the Fourier transform. Summarizing, we have

Proposition 2.4. *For the forms $\varphi_{\ell'}$, we have*

$$\varphi_{\ell'} \in \left[\mathbb{C}[\chi_{-\frac{m}{2}-\ell'}] \otimes \mathcal{S}(V_{\mathbb{R}}) \otimes \mathrm{Sym}^{\ell'}(V_{\mathbb{C}})\right]^{K' \times G}.$$

Here K' acts diagonally on the first two tensor factors, while G acts diagonally on the last two factors. Note that the K'-invariance is a different way of stating (2.11).

Hence the associated theta series $\theta(\tau, \varphi_{\ell'}, \mathcal{L})$, see (2.4)*, gives rise to a non-holomorphic modular form with values in* $\mathrm{Sym}_{\ell'}(V_{\mathbb{C}})$*:*

$$\theta(\tau, \varphi_{\ell'}, \mathcal{L}) \in \mathrm{NHolM}_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes \mathrm{Sym}_{\ell'}(V_{\mathbb{C}}).$$

Projecting φ_ℓ in the $\mathrm{Sym}^{\ell'}(V_{\mathbb{C}})$-factor onto $\mathcal{H}^{\ell'}(V_{\mathbb{C}})$, see (2.5), we obtain the form

$$\varphi_{[\ell']} \in \left[\mathbb{C}[\chi_{-\frac{m}{2}-\ell'}] \otimes \mathcal{S}(V_{\mathbb{R}}) \otimes \mathcal{H}_{\ell'}(V_{\mathbb{C}})\right]^{K' \times G}. \tag{2.12}$$

Then $\varphi_{[\ell']}$ becomes tautological in the following sense:

Lemma 2.5. *We have*

$$\varphi_{[\ell']}(x) = \sum_P P(x)\varphi_0(x) \otimes P,$$

where the sum extends over an orthonormal basis of $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$. In the second tensor factor we again identified the harmonic tensors with the harmonic polynomials. In particular,

$$\theta(\tau, \varphi_{[\ell']}, \mathcal{L}) \in S_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes \mathcal{H}_{\ell'}(V_{\mathbb{C}}).$$

Proof. By construction we easily see

$$\varphi_{\ell'}(x) = \sum_{\alpha_1,\dots,\alpha_{\ell'}=1}^{m} (x_{\alpha_1}\cdots x_{\alpha_{\ell'}} + \text{lower order terms})\varphi_0(x)\otimes e_{\alpha_1}\cdots e_{\alpha_{\ell'}}$$

Under projection the highest order terms of degree ℓ' give $\sum_P P(x)\varphi_0(x)\otimes P$. On the other hand, Theorem 6.11 in [12] (with $q=0$, $\ell=\ell'$) implies that $\theta(\tau,\varphi_{[\ell']},\mathcal{L})$ is a priori a holomorphic cusp form. But this is only possible if under the projection $\sum_{\alpha_1,\dots,\alpha_{\ell'}=1}^{m}(\text{lower order terms})\varphi_0\otimes e_{\alpha_1}\cdots e_{\alpha_{\ell'}}$ vanishes. Otherwise, in (2.3) a power of v would still be present, and the theta series $\theta(\tau,\varphi_{[\ell']},\mathcal{L})$ could not be holomorphic. □

Proposition 2.6. *The theta series $\theta(\tau,\varphi_{[\ell']},\mathcal{L})$ is a kernel function for the lift $\Lambda_{[\ell']}(\mathcal{L})$. Namely, for $P\in\mathcal{H}_{\ell'}(V_\mathbb{C})$, we have*

$$\Lambda_{[\ell']}(\mathcal{L})(P) = \big(\theta(\tau,\varphi_{[\ell']},\mathcal{L}),P\big) = \theta(\tau,P,\mathcal{L}).$$

Here the inner product $(\,,\,)$ in the middle term is the one on $\mathcal{H}_{\ell'}(V_\mathbb{C})$ induced by the inner product on V.

Remark 2.7. Via the orthogonal decomposition (2.5), we have

$$\big(\theta(\tau,\varphi_{[\ell']},\mathcal{L}),P\big) = \big(\theta(\tau,\varphi_{\ell'},\mathcal{L}),P\big)$$

for $P\in\mathcal{H}_{\ell'}(V_\mathbb{C})$. Hence we could also use $\theta(\tau,\varphi_{\ell'},\mathcal{L})$ as kernel function for $\Lambda_{[\ell']}(\mathcal{L})$. In this context note that there exists a nonzero constant $c_{m,\ell'}$ such that for $P\in\operatorname{Sym}_{\ell'-2}(V_\mathbb{C})$, we have

$$\big(\theta(\tau,\varphi_{\ell'},\mathcal{L}),r^2P\big) = c_{m,\ell'}R_{\frac{m}{2}+\ell'-2}\big(\theta(\tau,\varphi_{\ell'-2},\mathcal{L}),P\big)\,. \tag{2.13}$$

Here $R_{\frac{m}{2}+\ell'-2} = 2i\frac{\partial}{\partial\tau}+(\frac{m}{2}+\ell'-2)v^{-1}$ is the Maass raising operator which maps forms of weight $\frac{m}{2}+\ell'-2$ to weight $\frac{m}{2}+\ell'$. (2.13) follows from the adjointness of multiplication with r^2 and the Laplace operator Δ and an explicit calculation in the Weil representation giving $c_{m,\ell'}R_{\frac{m}{2}+\ell'-2}\varphi_{\ell'-2}(x,\tau) = \Delta\varphi_{\ell'}(x,\tau)$. This provides a complete description of the lift induced by $\theta(\tau,\varphi_{\ell'},\mathcal{L})$.

Example 2.8. Let $m=\ell'=2$. Then $\operatorname{Sym}^2(V_\mathbb{C})$ is spanned by $\{e_1^2,e_1e_2,e_2^2\}$, while $\mathcal{H}^2(V_\mathbb{C})$ is spanned by $\{e_1^2-e_2^2,e_1e_2\}$ (or in terms of polynomials by $\{x_1^2-x_2^2,x_1x_2\}$). Then

$$\begin{aligned}\varphi_2(x,\tau) &= v^{-1}(vx_1^2-\tfrac{1}{4\pi})e^{\pi i(x,x)\tau}\otimes e_1^2 + v^{-1}(vx_2^2-\tfrac{1}{4\pi})e^{\pi i(x,x)\tau}\otimes e_2^2 + x_1x_2e^{\pi i(x,x)\tau}\otimes e_1e_2\\ &= \frac{1}{2}(x_1^2-x_2^2)e^{\pi i(x,x)\tau}\otimes(e_1^2-e_2^2) + x_1x_2e^{\pi i(x,x)\tau}\otimes e_1e_2\\ &\quad+\frac{1}{2}(x_1^2+x_2^2-\frac{1}{2\pi v})e^{\pi i(x,x)\tau}\otimes(e_1^2+e_2^2).\end{aligned}$$

The two terms in the middle line give rise to the holomorphic theta series associated to the harmonic polynomials $x_1^2-x_2^2$ and x_1x_2, while the term in the third line coming from the trivial representation of G inside $\operatorname{Sym}^2(V_\mathbb{C})$ gives $-\frac{1}{4\pi}R_1\theta(\tau,\varphi_0,\mathcal{L})$.

3. Locally symmetric spaces of orthogonal type

3.1. **The symmetric space.** Let V be a rational vector space of even dimension $m=p+q$ with a non-degenerate symmetric bilinear form $(\,,\,)$ of signature (p,q). We let $\underline{G}=\mathrm{SO}(V)$. We let $G=\underline{G}_0(\mathbb{R})\simeq\mathrm{SO}_0(p,q)$ be the connected component of the identity of the real points of $\underline{G}$. It is most convenient to identify the associated symmetric space $D = D_V$ with the space of negative q-planes in $V(\mathbb{R})$ on which the bilinear form $(\,,\,)$ is negative definite:

$$D = \{z \subset V_{\mathbb{R}};\ \dim z = q \text{ and } (\,,\,)|_z < 0\}. \tag{3.1}$$

We pick an orthogonal basis $\{e_\alpha, e_\mu\}$ of $V_{\mathbb{R}}$ with $(e_\alpha, e_\alpha) = 1$ and $(e_\mu, e_\mu) = -1$. Here and throughout the paper $1 \leq \alpha \leq p$ and $p+1 \leq \mu \leq m$. We denote the coordinates of a vector x with respect to this basis by $\{x_\alpha, x_\mu\}$. We pick as base point of D the q-plane $z_0 = \operatorname{span}\{e_\mu\}$, and we let $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$ be the maximal compact subgroup of G stabilizing z_0. Thus

$$D \simeq G/K. \tag{3.2}$$

Recall that we can also identify D with the space of minimal majorants of $(\,,\,)$ on $V_{\mathbb{R}}$. More precisely, for $z \in D$, the associated majorant $(\,,\,)_z$ is given by

$$(x,x)_z = \begin{cases} (x,x) & \text{if } x \in z^\perp, \\ -(x,x) & \text{if } x \in z. \end{cases} \tag{3.3}$$

We let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ associated to K. We identify $\mathfrak{g} \simeq \bigwedge^2 V_{\mathbb{R}}$ as usual via $(v_1 \wedge v_2)(v) = (v_1, v)v_2 - (v_2, v)v_1$. We write $X_{ij} = e_i \wedge e_j \in \mathfrak{g}$ and note that $\mathfrak{p}$ is spanned by $X_{\alpha\mu}$ with α "positive" and μ "negative". We write $\omega_{\alpha\mu}$ for their dual in $\mathfrak{p}^*$. Note that the tangent space $T_{z_0}(D)$ at the base point z_0 is naturally isomorphic to $\mathfrak{p}$.

Again, we let L be an even lattice in V of level N and set $\mathcal{L} = L + h$ with $h \in L^\#$. We let $\Gamma \subseteq \operatorname{Stab} L$ be a subgroup of finite index of the stabilizer of $\mathcal{L}$ in G. We assume that Γ is neat, in particular torsion free. We then let

$$X = \Gamma \backslash D, \tag{3.4}$$

which defines a (typically non compact) manifold of real dimension pq.

3.2. **Special Cycles.** For $x \in V$ such that $(x,x) > 0$, we define

$$D_x = \{z \in D;\ z \perp x\}. \tag{3.5}$$

Then D_x is a subsymmetric space arising from an orthogonal subgroup of signature $(p-1, q)$ in G. We let $\Gamma_x \subset \Gamma$ be the stabilizer of x and set

$$C_x = \Gamma_x \backslash D_x. \tag{3.6}$$

By [21] Lemma 2.1, C_x embeds into X, and by slight abuse we identify C_x with its image in X. These cycles are totally geodesic and are also known as generalized modular symbols.

The special cycles are typically non-compact, so they define relative cohomology classes for X as the image of the fundamental class $[C_x, \partial C_x]$ in $H_{(p-1)q}(X, \partial X, \mathbb{Z}) \simeq H^q(X, \mathbb{Z})$.

For a positive rational number n, we write $L_n = \{x \in \mathcal{L};\ \frac{1}{2}(x,x) = n\}$. Then the composite cycles C_x are given by

$$C_n = \sum_{x \in \Gamma \backslash L_n} C_x \in H^q(X, \mathbb{Z}). \tag{3.7}$$

If we want to emphasize that we consider the cycles in (co)homology, we also write $[C_n]$.

In the general case for $\mathrm{O}(p,q) \times \mathrm{Sp}(g)$, the cycles arise from embedded $\mathrm{O}(p-g, q)$, and the composite cycles are parameterized by positive (semi)-definite $g \times g$ matrices.

4. Work of Kudla-Millson

In the previous section, we saw that positive integers (if $h = 0$) naturally parametrize totally geodesic cycles in orthogonal locally symmetric spaces. On the other hand, the Fourier expansion of modular forms is also parametrized by positive integers. The main point of the work of Kudla and Millson throughout the 1980's following Hirzebruch-Zagier [16] is that this not coincidental!

4.1. **Statement.** Kudla-Millson established the following result in much greater generality for cycles of higher codimension and also for unitary groups $\mathrm{U}(p,q)$.

Theorem 4.1. [21] *The generating series of the special cycles defines a holomorphic modular form of weight $\frac{m}{2}$ and level N with values in $H^q(X,\mathbb{Z})$:*

$$P_{q,0}(\tau) := \delta_{0h} e_q + \sum_{n>0} [C_n] q^n \in M_{\frac{m}{2}}(\Gamma(N)) \otimes H^q(X,\mathbb{Z}).$$

Here δ_{h0} is the Kronecker delta, and e_q is for q even the (suitably) normalized Euler form on D and $e_q = 0$ if q is odd. That is, for η a closed compactly supported differential $(p-1)q$-form on X, the generating series

$$\Lambda_{q,0}(\tau,\eta,\mathcal{L}) := \delta_{h0}\left(\int_X \eta \wedge e_q\right) + \sum_{n>0}\left(\int_{C_n} \eta\right) q^n$$

is a holomorphic modular form of weight $\frac{m}{2}$ and level N. This map factors through the (deRham) cohomology with compact supports $H_c^{(p-1)q}(X)$, and we obtain a lift

$$\Lambda_{q,0}(\mathcal{L}) : H_c^{(p-1)q}(X) \longrightarrow M_{\frac{m}{2}}(\Gamma(N)).$$

4.2. **Special Schwartz forms.** We give a very rough sketch of the basic ideas underlying the above result. First note that the formulas and constructions for Schwartz functions and the Weil representation from the positive definite case are still valid *mutatis mutandis.* Kudla and Millson construct a Schwartz form $\varphi_{q,0}$ on $V_\mathbb{R}$ taking values in $\mathcal{A}^q(D)$, the differential q-forms on D. Explicitly, at the base point z_0 it is given by

$$\varphi_{q,0} = \frac{1}{2^{q/2}} \sum_{\alpha_1,\ldots,\alpha_q=1}^{p} (\mathcal{D}_{\alpha_1} \circ \cdots \circ \mathcal{D}_{\alpha_q})(\varphi_0) \otimes \omega_{\alpha_1 p+1} \cdots \omega_{\alpha_q m}. \tag{4.1}$$

(Recall $T_{z_0}(D)^* \simeq \mathfrak{p}^*$). Here $\varphi_0 = e^{-\pi(x,x)_{z_0}} = e^{-\pi(\sum_\alpha x_\alpha^2 + \sum_\mu x_\mu^2)} \in \mathcal{S}(V_\mathbb{R})^K$ is the Gaussian at z_0. By [20] Theorem 3.1, $\varphi_{q,0}$ is an eigenfunction of K' of weight $\frac{m}{2}$. Thus

$$\varphi_{q,0} \in [\mathbb{C}[\chi_{-\frac{m}{2}}] \otimes \mathcal{S}(V_\mathbb{R}) \otimes \mathcal{A}^q(D)]^{K' \times G} \simeq [\mathbb{C}[\chi_{-\frac{m}{2}}] \otimes \mathcal{S}(V_\mathbb{R}) \otimes \textstyle\bigwedge^q(\mathfrak{p}^*)]^{K' \times K}, \tag{4.2}$$

where the isomorphism is given by evaluation at the base point of D. Furthermore, $\varphi_{q,0}(x)$ is a closed form on D for all x. We can form the theta series for $\varphi_{q,0}$, see (2.4) which is easily seen to be a Γ-invariant differential form on D. Hence it descends to a form on X. Thus

$$\theta(\tau,\varphi_{q,0},\mathcal{L}) \in \mathrm{NHolM}_{\frac{m}{2}}(\Gamma(N)) \otimes \mathcal{A}^q(X)$$

is a non-holomorphic modular form of weight $m/2$ with values in the closed differential q-forms of X. The key fact is now: For $n > 0$,

The nth-Fourier coefficient of $\theta(\tau,\varphi_{q,0},\mathcal{L})$ is a Poincaré dual form for the cycle C_n.

This reduces to that for $x \in V$ with $(x,x) > 0$ and for η a closed compactly supported $(p-1)q$ form on the "tube" $\Gamma_x \backslash D$, one has the Thom Lemma,

$$\int_{\Gamma\backslash D} \eta \wedge \sum_{\gamma \in \Gamma_x \backslash \Gamma} \gamma^* \varphi_{q,0}(x) = \int_{\Gamma_x \backslash D} \eta \wedge \varphi_{q,0}(x) = \left(\int_{C_x} \eta \right) e^{-\pi(x,x)}. \tag{4.3}$$

For $n < 0$, the n-Fourier coefficient of $\theta(\tau, \varphi_{q,0}, \mathcal{L})$ turns out to be exact, while for $n = 0$ one obtains up to exact terms, the invariant q-form e_q. Hence, while the differential form $\theta(\tau, \varphi_{q,0}, \mathcal{L})$ is non-holomorphic in $\tau \in \mathbb{H}$, the cohomology class $[\theta(\tau, \varphi_{q,0}, \mathcal{L})]$ defines a holomorphic modular form and

$$[\theta(\tau, \varphi_{q,0}, \mathcal{L})] = P_{q,0}(\tau). \tag{4.4}$$

Summarizing, Kudla-Millson showed

Theorem 4.2. *The theta series $\theta(\tau, \varphi_{q,0}, \mathcal{L})$ is a kernel function for the lift $\Lambda_{q,0}(\mathcal{L})$: For η closed compactly supported differential $(p-1)q$-form on X, one has*

$$\int_X \eta \wedge \theta(\tau, \varphi_{q,0}, \mathcal{L}) = \delta_{h0} \left(\int_X \eta \wedge e_q \right) + \sum_{n>0} \left(\int_{C_n} \eta \right) q^n.$$

4.3. **Examples.**

- For signature $(2,1)$, we have $D \simeq \mathbb{H}$, and the cycles C_x are geodesics inside arithmetic quotients of $\mathbb{H}$ and are the classical modular symbols. The relationship to modular forms was first established in the classical paper of Shintani [27].
- For signature $(1,2)$, we again have $D \simeq \mathbb{H}$, but now the cycles are 0-cycles, more precisely, quadratic irrationalities in $\mathbb{H}$ and give rise to Heegner divisors (or CM points) inside a modular or Shimura curve. In [11, 8] we consider the Kudla-Millson lift with modular functions as input generalizing work of Zagier [31, 33].
- For signature $(2,2)$, X is a Hilbert modular surface, that is a quotient of $\mathbb{H} \times \mathbb{H}$ (if the $\mathbb{Q}$-rank of G is one), and the special cycles are embedded modular/Shimura curves. These are the famous Hirzebruch-Zagier curves [16] which have been the starting point for a great part of the study of totally geodesic cycles inside locally symmetric spaces and their relationship to modular forms. In [14], we show how the results in [16] can be obtained via our theory.
- For signature $(3,2)$, one has $D \simeq \mathbb{H}_2$, the Siegel upper half space of degree 2, and the cycles are the Humbert surfaces. The modularity of the generating series of theses cycles is discussed in [30], chapter IX.
- In general, for signature $(p,2)$, D is Hermitian, and X has the structure of a quasi-projective algebraic variety. The cycles are now divisors. These divisors play a crucial role in Borcherds' theory of singular theta lifts, see [2, 6]. A duality statement of the Kudla-Millson lift with the Borcherds' lift was shown in joint work with Bruinier in [7]. As an application we establish in [9] the surjectivity of the Borcherds lift.
- For signature $(2,q)$, the cycles are quotients of hyperbolic q-space. The relationship to modular forms was first established by Oda [24].

5. Cycles with coefficients

We first describe how one can equip the special cycles with coefficients. We let U be a (finite-dimensional) representation of G. Hence U is self-dual, i.e., $U \simeq U^*$, and we won't distinguish between U and U^*. We write $\widetilde{U}$ for the associated local system of U.

Lemma 5.1. *Let $C_x = \Gamma_X \backslash D_x$ be a special cycle, and let $v \in U^{\Gamma_x}$, a Γ_x-invariant vector in U. Then the pair (C_x, v) defines a class in $H_{(p-1)q}(X, \partial X, \widetilde{U}) \simeq H^q(X, \widetilde{U})$, which we denote by $C_x \otimes v$.*

Proof. Since D_x is simply connected, we have

$$U^{\Gamma_x} \simeq H^0(\Gamma_x, U) \simeq H^0(C_x, \widetilde{U}) \simeq H_{(p-1)q}(C_x, \partial C_x, \widetilde{U}). \tag{5.1}$$

So $v \in U^{\Gamma_x}$ gives rise to an element in $H_{(p-1)q}(C_x, \partial C_x, \widetilde{U})$ which can be pushed over to define an element in $H_{(p-1)q}(X, \partial X, \widetilde{U}) \simeq H^q(X, \widetilde{U})$. □

In [12], we give a more detailed summary of simplicial homology with non-trivial coefficients.

We now take $U = \mathcal{H}_{\ell'}(V_{\mathbb{C}}) \subset \mathrm{Sym}^{\ell'}(V_{\mathbb{C}})$, the harmonic symmetric tensors (with respect to the indefinite Laplacian defined by $(\, ,\,)$). We let $\pi_{\ell'} : \mathrm{Sym}^{\ell'}(V_{\mathbb{C}}) \to \mathcal{H}_{\ell'}(V_{\mathbb{C}})$ be the orthogonal projection. Then $\pi_{\ell'}(x^{\ell'}) \in \mathcal{H}_{\ell'}(V_{\mathbb{C}})^{\Gamma_x}$, and we set

$$C_{x,[\ell']} := C_x \otimes \pi_{\ell'}(x^{\ell'}) \in H_{(p-1)q}(C_x, \partial C_x, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})}) \simeq H^q(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})}) \tag{5.2}$$

We then define for $n > 0$ the composite cycle $C_{n,[\ell']} = \sum_{x \in \Gamma \backslash L_n} C_{x,[\ell']}$ as before.

We let $\langle\, ,\, \rangle$ be the pairing between $H_{(p-1)q}(C_x, \partial C_x, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$ and $H_c^{(p-1)q}(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$. Then in [12] we explain that for η a closed compactly supported differential $(p-1)q$-form on X with values in $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$ representing a class $[\eta]$ in $H_c^{(p-1)q}(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$, one has

$$\langle C_{x,[\ell']}, [\eta] \rangle = \int_{C_x} \left(\eta, \pi_{\ell'}(x^{\ell'}) \right). \tag{5.3}$$

Here $(\eta, \pi_{\ell'}(x^{\ell'}))$ is the scalar-valued differential form obtained by taking the pairing $(\, ,\,)$ in the fiber.

The generalization of Theorem 4.1 is

Theorem 5.2. [12] *The generating series of the special cycles with coefficients in $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$ defines a holomorphic cusp form of weight $\frac{m}{2} + \ell$ and level N with values in $H^q(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$:*

$$P_{q,[\ell']}(\tau) := \sum_{n>0} [C_{n,[\ell']}] q^n \in S_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes H^q(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})}).$$

That is, for a class $[\eta]$ in $H_c^{(p-1)q}(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$, the generating series

$$\Lambda_{q,[\ell']}(\tau, [\eta], \mathcal{L}) := \sum_{n>0} \langle C_{n,[\ell']}, [\eta] \rangle \, q^n$$

is a holomorphic cusp form of weight $\frac{m}{2} + \ell'$ and level N. We obtain a lift

$$\Lambda_{q,[\ell']}(\mathcal{L}) : H_c^{(p-1)q}(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})}) \longrightarrow S_{\frac{m}{2}+\ell'}(\Gamma(N)).$$

Remark 5.3. Using the definition of the pairing $\langle C_{n,[\ell']}, \eta \rangle$ and (the analog of) (2.5), we easily see that for η a closed compactly supported $(p-1)q$-form on X with values in $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$ representing a class $[\eta]$ in $H_c^{(p-1)q}(X, \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})})$, we have

$$\Lambda_{q,[\ell']}(\tau, [\eta], \mathcal{L}) = \sum_{\substack{x \in \Gamma \backslash \mathcal{L} \\ (x,x)>0}} \left(\int_{C_x} (\eta, x^{\ell'}) \right) e^{\pi i (x,x) \tau}.$$

Several (sporadic) cases for generating series for periods over cycles with nontrivial coefficients as elliptic modular forms were already known (in a somewhat different setting). For signature $(2,1)$ by Shintani [27], signature $(2,2)$ by Tong [29] and Zagier [32], and for signature $(2,q)$ by Oda [24] and Rallis and Schiffmann [25].

Remark 5.4. For the general case with $g > 1$, the special with coefficients are homomorphisms from a representation space ρ of $\mathrm{GL}_g(\mathbb{C}^n)$ with highest weight λ to the (co)homology of X. Then the generating series of these cycles gives a vector-valued holomorphic Siegel modular form of type ρ.

5.1. **Special Schwartz forms with non-trivial coefficients.** We again realize the lift $\Lambda_{q,[\ell']}$ via a theta function. We first construct a Schwartz form $\varphi_{q,\ell'}$ with $\mathrm{Sym}^{\ell'}$-coefficients by applying the operator defined by (2.9) on $\varphi_{q,0}$. Then $\varphi_{q,\ell'}$ has weight $\frac{m}{2}+\ell'$ under K', see [12], section 6. We then project onto $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$ in the coefficients to obtain $\varphi_{q,[\ell']}$. That is,

$$\varphi_{q,[\ell']} = \frac{1}{2^{\ell'}} \sum_{\alpha_1,\dots,\alpha_{\ell'}=1}^{p} (\mathcal{D}_{\alpha_1} \circ \cdots \circ \mathcal{D}_{\alpha_{\ell'}})(\varphi_{q,0}) \otimes \pi_{\ell'}\left(e_{\alpha_1}\cdots e_{\alpha_{\ell'}}\right). \tag{5.4}$$

Thus

$$\varphi_{q,[\ell']} \in [\mathbb{C}[\chi_{-\frac{m}{2}-\ell'}] \otimes \mathcal{S}(V_{\mathbb{R}}) \otimes \textstyle\bigwedge^{q}(\mathfrak{p}^*) \otimes \mathcal{H}_{\ell'}(V_{\mathbb{C}})]^{K'\times K}, \tag{5.5}$$

where K acts now diagonally on the last three factors. We then form the theta series $\theta(\tau,\varphi_{q,[\ell']},\mathcal{L})$ as before. The main result of [12] is then

Theorem 5.5. *The theta series $\theta(\tau,\varphi_{q,[\ell']},\mathcal{L})$ is a kernel function for the lift $\Lambda_{q,[\ell']}(\mathcal{L})$: For η closed compactly supported $(p-1)q$-form on X with values in $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$, one has*

$$\int_X \eta \wedge \theta(\tau,\varphi_{q,[\ell']},\mathcal{L}) = \sum_{n>0} \langle C_{n,[\ell']},[\eta]\rangle\, q^n.$$

Remark 5.6. The key point in obtaining the Fourier coefficient formula in Theorem 5.5 is to establish

$$[\varphi_{q,[\ell']}(x)] = [\varphi_{q,0}(x)\otimes \pi_{\ell'}(x^{\ell'})],$$

which is Theorem 5.9 in [12]. Then the analog for coefficients of (4.3) follows from (4.3) and from the formula (5.3) for $\langle C_{x,[\ell']},[\eta]\rangle$ and

$$\int_{\Gamma_x\backslash D} \eta\wedge\varphi_{q,[\ell']}(x) = \int_{\Gamma_x\backslash D} \eta\wedge(\varphi_{q,0}(x)\otimes\pi_{\ell'}(x^{\ell'})) = \int_{\Gamma_x\backslash D} (\eta,\pi_{\ell'}(x^{\ell'}))\wedge\varphi_{q,0}(x).$$

Remark 5.7. As in the positive definite case, we can also use the theta series $\theta(\tau,\varphi_{q,\ell'},\mathcal{L})$ as the kernel function for $\Lambda_{q,[\ell']}$ on forms with coefficients in $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$, see Remark 2.7. Then analogously to V positive definite, for η taking values in $\mathrm{Sym}_{\ell'-2}(V_{\mathbb{C}})$, we can consider $r^2\eta$ with values in $\mathrm{Sym}_{\ell'}(V_{\mathbb{C}})$, where $r^2 = \sum_\alpha e_\alpha^2 - \sum_\mu e_\mu^2$. Then there exists a nonzero constant $c_{q,\ell'}$ independent of η such that

$$\int_X (r^2\eta)\wedge\theta(\tau,\varphi_{q,\ell'},\mathcal{L}) = c_{q,\ell'} R_{\frac{m}{2}+\ell'-2}\int_X \eta\wedge\theta(\tau,\varphi_{q,\ell'-2},\mathcal{L}).$$

6. Restriction to the boundary

We let $\overline{X}$ be the Borel-Serre compactification of the locally symmetric space X, see [4, 3]. Then $\overline{X}$ is a manifold with corners. If $\underline{P}_1, \dots, \underline{P}_k$ is a set of representatives of Γ-conjugacy classes of rational parabolic subgroups of $\underline{G}$, then (as sets)

$$\overline{X} = X \cup \coprod_{i=1}^{k} e(P_i'). \tag{6.1}$$

Here $e'(P_i) = \Gamma_{P_i} \backslash e(P_i)$ is the face of the Borel-Serre compactification associated to $P = \underline{P}_i(\mathbb{R})_0$ with $\Gamma_{P_i} = \Gamma \cap P_i$.

We now describe the faces $e(P)$. The rational parabolic subgroup $\underline{P}$ in $\underline{G}$ stabilizes a flag $\mathbf{F}$ of totally isotropic rational subspaces in V. We can choose $\mathbf{F}$ in such a way such that the θ-stable subgroup of $\underline{P}$ forms a Levi subgroup. We let $P = NAM$ be the associated (rational) Langlands decomposition, and we let $\mathfrak{m}$ and $\mathfrak{n}$ be the Lie algebras of M and N respectively. We set $\mathfrak{p}_M = \mathfrak{p} \cap \mathfrak{m}$. Let E be the largest element in the isotropic flag $\mathbf{F}$ with dimension ℓ. We let $W = E^{\perp}/E$, which is naturally a quadratic space of signature $(p-\ell, q-\ell)$, and we realize W as a subspace of V such that the Cartan involution θ for $\mathrm{O}(V)$ restricts to one for $\mathrm{O}(W)$. We obtain the Witt decomposition

$$V = E \oplus W \oplus E', \tag{6.2}$$

where E' is a complementary totally isotropic subspace of dimension ℓ. Then M splits naturally into the product of $\mathrm{SO}_0(W(\mathbb{R}))$ with a product of special linear groups of subquotients of $E(\mathbb{R})$, and consequently we have a projection map from the symmetric space D_M for M to D_W, the symmetric space for $\mathrm{SO}_0(W_{\mathbb{R}})$. We let $e(P) = NM/K_P$ where $K_P = M \cap K$.

It is well-known that the cohomology of $e'(P)$ decomposes as

$$H^k(e'(P), \widetilde{U}) \simeq \bigoplus_{a,b,a+b=k} H^a(X_M, H^b(\mathfrak{n}, U)), \tag{6.3}$$

see [15, 26]. Here $H^\bullet(\mathfrak{n}, U))$ denotes the Lie algebra cohomology of $\mathfrak{n}$, which was determined by Kostant [19] as an M-module. In [13], we explicitly construct a map

$$\iota_P' : H^{q-\ell}(X_W, \widetilde{\mathcal{H}_{\ell+\ell'}(W_{\mathbb{C}})}) \longrightarrow H^q(e'(P), \widetilde{\mathcal{H}_{\ell'}(V_{\mathbb{C}})}). \tag{6.4}$$

This map ι_P' arises by the pullback of D_W to D_M, and the explicit construction of an embedding of M-modules

$$\tau_{\ell,\ell'} : \mathcal{H}_{\ell+\ell'}(W_{\mathbb{C}}) \hookrightarrow H^{\ell}(\mathfrak{n}, \mathcal{H}_{\ell'}(V_{\mathbb{C}})) \tag{6.5}$$

to obtain a map from $H^{q-\ell}(X_W, \mathcal{H}_{\ell+\ell'}(W_{\mathbb{C}}))$ to $H^{q-\ell}(X_M, H^{\ell}(\mathfrak{n}, \mathcal{H}_{\ell'}(V_{\mathbb{C}})))$. Combining this with (6.3) gives ι_P'.

The main result of [13] (in greater generality) is

Theorem 6.1. *The differential form $\theta(\tau, \varphi^V_{q,[\ell]}, \mathcal{L}_V)$ on X extends to the Borel-Serre compactification $\overline{X}$. Furthermore, let $r'(P)$ be the restriction to the boundary face $e'(P)$ of $\overline{X}$. Then there exists a theta distribution $\mathcal{L}_W$ for W such that in cohomology we have*

$$\begin{aligned}[r'(P)\theta(\tau, \varphi^V_{q,[\ell']}, \mathcal{L}_V)] &= \iota_P'[\theta(\tau, \varphi^W_{q-\ell,[\ell+\ell']}, \mathcal{L}_W)] \\ &\in S_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes \iota_P' \left(H^{q-\ell}(X_W, \widetilde{\mathcal{H}_{\ell+\ell'}(V_{\mathbb{C}})}) \right).\end{aligned}$$

Loosely speaking the theorem can be summarized that the restriction of our theta series for V with $\mathcal{H}_{\ell'}(V_{\mathbb{C}})$-coefficients to a face of the Borel-Serre compactification "is" the theta series for W of the same type corresponding to the enlarged coefficient system $\mathcal{H}_{\ell'+\ell}(W_{\mathbb{C}})$.

Remark 6.2. The theorem is rather delicate since the theta series $\theta(\tau, \varphi^V_{q,[\ell']}, \mathcal{L}_V)$ is termwise moderately increasing. We obtain the restriction by switching to a mixed model of the Weil representation adopted to the Witt decomposition (6.2) of V.

Remark 6.3. We also have a formula for the restriction of $\theta(\tau, \varphi^V_{q,\ell'}, \mathcal{L}_V)$ with values in $\mathrm{Sym}^{\ell'}(V_{\mathbb{C}})$ on the level of differential forms. One obtains $\theta(\tau, \varphi^W_{q-\ell,\ell+\ell'}, \mathcal{L}_W)$. In fact, one derives Theorem 6.1 from this.

Remark 6.4. The theta distribution $\mathcal{L}_W$ is explicitly given as follows. For the coset $\mathcal{L}_V = \mathcal{L} = L + h$ of the lattice L in V, we can write

$$\mathcal{L} \cap E^{\perp} = \coprod_k (L_{E,k} + h_{E,k}) \oplus (L_{W,k} + h_{W,k}).$$

for lattices $L_{E,k} \subset E$ and $L_{W,k} \subset W$ respectively and vectors $h_{E,k} \in (L^{\#}_{E,k})$ etc. Then

$$\mathcal{L}_W = \sum_k \det(L_{E,k})^{-1} (L_{W,k} + h_{W,k}).$$

7. Nonvanishing

In the situation of the previous section, we consider a parabolic P such that $\dim E = \ell = q$, hence W is positive definite and X_W is a point. Then by Theorem 6.1

$$\begin{aligned} [r'(P)\theta(\tau, \varphi^V_{q,[\ell']}, \mathcal{L}_V)] &= \iota'_P[\theta(\tau, \varphi^W_{0,[q+\ell']}, \mathcal{L}_W)] \\ &\in S_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes \iota'_P \left(H^0(X_W, \mathcal{H}_{q+\ell'}(V_{\mathbb{C}}))\right) \\ &\simeq S_{\frac{m}{2}+\ell'}(\Gamma(N)) \otimes \tau_{q,\ell'} \left(\mathcal{H}_{q+\ell'}(V_{\mathbb{C}})\right). \end{aligned} \tag{7.1}$$

So in this case ι'_P is an embedding. Hence the restriction to $e'(P)$ vanishes if and only if the positive definite theta series $\theta(\tau, \varphi^W_{0,[q+\ell']}, \mathcal{L}_W)$ vanishes. Using Lemma 2.1 we then easily conclude

Theorem 7.1. *Assume that the $\mathbb{Q}$-rank and the $\mathbb{R}$-rank of G coincide and that $p - q \geq 2$. Then there exists a finite cover X' of X such that*

$$[\theta(\varphi^V_{q,[\ell']})] \neq 0.$$

On the level of special cycles this means

$$[C_{n,[\ell']}] \neq 0$$

for infinitely many n. In particular, $H^q(X', \mathcal{H}_{\ell'}(V_{\mathbb{C}})) \neq 0$.

Proof. The condition on the rank of G is necessary to be able to restrict to a face such that the associated W is positive definite. We need $\dim W = p - q \geq 2$ to have a theory of harmonic polynomials for W. By Lemma 2.1 one can find sublattice of $\mathcal{L}_W$ such that the theta series associated to the coset of this lattice does not vanish. This then gives rise to a sublattice L' of L in V, and we set $\Gamma' = \Gamma \cap \mathrm{Stab}\, L'$. Then one can take $X' = \Gamma' \backslash D$. □

Remark 7.2. Consider the Hirzebruch-Zagier [16] case when X is a Hilbert modular surface. In that situation, the restriction of the classes $\theta(\varphi_{2,0}^V)$ to any boundary face is exact. This shows that the hypotheses of Theorem 7.1 are indeed necessary in general.

In a different paper [14], we will explicitly construct a primitive for $r_{e'(P)}(\theta(\varphi_{2,0}^V))$. In this way, we will provide a full explanation of the celebrated result of Hirzebruch-Zagier in terms of the theory described in this paper.

Remark 7.3. Bergeron [1] in the compact case established nonvanishing of the Kudla-Millson classes by considering the analogous classes in $\mathrm{U}(p,q)$ and using the nonvanishing result in the unitary case by Kazhdan, see e.g.[5].

Li [22] also used the theta correspondence to establish nonvanishing for the cohomology of orthogonal groups, again in the compact (or L^2)-case (without giving a geometric interpretation of the classes).

Speh and Venkataramana [28] gave a general criterion for the non-vanishing of certain modular symbols on locally symmetric spaces in terms of the compact dual. In particular, they obtain a nonvanishing result for some subgroups of the unitary group $\mathrm{U}(p,q)$.

One feature of our method to establish non-vanishing is that we retain some control over the cover X', since this reduces to the very concrete question of non-vanishing of positive definite theta series. An easy example for this is the following.

Example 7.4. Consider the integral quadratic form given by

$$y_1y_1' + \cdots + y_qy_q' + 2x_1^2 + \cdots + 2x_k^2$$

with $y_i, y_i', x_j \in \mathbb{Z}$. So $L = \mathbb{Z}^m$ with $m = 2q + k$. Assume $k \geq q$. Note $L^\# \subset \frac{1}{4}\mathbb{Z}^m$. We let Γ be the subgroup in $\mathrm{Stab}(L)$ which stabilizes $L^\#/L$. Then

$$H^q(\Gamma, \mathbb{Z}) \neq 0.$$

We leave the details to the reader. In view of the above results, it follows from the non-vanishing of the theta series $\sum_{\mathbf{x}\in\mathbf{Z}^k+(\frac{1}{4},\dots,\frac{1}{4})} x_1\cdots x_q e^{4\pi i(\sum x_i^2)\tau}$.

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