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Hilbert-Jacobi forms of a certain index of $\mathbb{Q}(\sqrt{5})$

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0 Introduction

The purpose of this survey is to give an example of a structure theorem of the space of Hilbert-Jacobi forms of a certain index with concerning to $K = \mathbb{Q}(\sqrt{5})$ (Theorem 1.2). We give also an example of a structure theorem of the space of Jacobi forms of a matrix index (Theorem 1.3). We used theorem 1.3 to show theorem 1.2.

1 Main theorem

In this section, we recall the definition of Hilbert-Jacobi forms, and give an example of a structure theorem of the space of Hilbert-Jacobi forms and of Jacobi forms of a matrix index.

1.1 Notations

Let K be a totally real field with degree n, let $\mathfrak{d}^{-1}$ be the inverse of the different, and let $\mathcal{O}$ be the principal order of K. We denote by $\mathfrak{H}$ the Poincaré upper half plane. For $z = (z_1, ..., z_n) \in \mathbb{C}^n$ we set $e(z) := e^{2\pi i \operatorname{tr}(z)}$, where $\operatorname{tr}(z) = z_1 + \cdots + z_n$. By abuse of language, we set $z^k := \prod_{i=1}^{n} z_i^{k_i}$ for $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $k = (k_1, ..., k_n) \in \mathbb{R}^n$.

1.2 Definition

For $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and for totally positive number $m \in \mathfrak{d}^{-1}$, we define *Hilbert-Jacobi forms* of weight k of index m as follows.

Definition 1. *Let ϕ be a holomorphic function on $\mathfrak{H}^n \times \mathbb{C}^n$. We say ϕ is a Hilbert-Jacobi form of weight k of index m if ϕ satisfies the following three conditions.*

(i) For any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathcal{O})$, *any* $\tau = (\tau_1, \ldots, \tau_n) \in \mathfrak{H}^n$ *and any* $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, ϕ *satisfies*

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e\left(m(c\tau + d)^{-1} c z^2\right) (c\tau + d)^k \phi(\tau, z).$$

(ii) For any λ, $\mu \in \mathcal{O}$,

$$\phi(\tau, z + \lambda\tau + \mu) = e(-m\lambda^2\tau - 2m\lambda z)\phi(\tau, z).$$

(iii) ϕ *has the Fourier expansion :*

$$\phi(\tau, z) = \sum_{u, r \in \mathfrak{d}^{-1}} c(u, r)\, e(u\tau + rz),$$

where in the above summation u *and* r *run over all elements in* $\mathfrak{d}^{-1}$ *such that* $4um - r^2$ *is totally positive or equals to* 0.

When n is larger than 1, then because of Koecher principle the third condition of the definition follows automatically by the first and second conditions.

We denote by $J^K_{k,m}$ the space of Hilbert-Jacobi forms of weight k of index m with respect to $SL(2, \mathcal{O})$.

1.3 Results

We consider the case $K = \mathbb{Q}(\sqrt{5})$, $m = \epsilon/\sqrt{5}$, where $\epsilon = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of the maximal order $\mathcal{O} = \mathbb{Z}[\epsilon]$ of K.

Let $k \in \mathbb{N}$. Now $M^K_{(k_1,k_2)}$ denotes the space of Hilbert modular forms of weight $(k_1, k_2) \in \mathbb{Z}^2$ with respect to $SL(2, \mathcal{O})$. We quote the following structure theorem of the space of Hilbert modular forms obtained by Gundlach [2].

Theorem 1.1 (Gundlach[2]).

$$\bigoplus_{k \in \mathbb{Z}} M^K_{(k,k)} = \mathbb{C}[G_2, G_5, G_6] \oplus G_{15}\mathbb{C}[G_2, G_5, G_6],$$

where G_2, G_5, G_6 *and* G_{15} *are Hilbert modular forms of weight* $2, 5, 6$ *and* 15, *respectively. There exists a polynomial* $P(X_1, X_2, X_3)$ *such that* ${G_{15}}^2 = P(G_2, G_5, G_6)$.

The main theorem of this report is as follows.

Theorem 1.2. *The space* $\bigoplus_{k\in\mathbf{Z}} J^K_{(k,k),m}$ *is a* $\mathbb{C}[G_2, G_5, G_6]$*-module generated by eight forms* $F_k \in J^K_{(k,k),m}$ $(k = 2,4,5,6,7,11,14,15)$*, and the dimension formula is given by*

$$\sum_{k\in\mathbf{Z}} \dim(J^K_{(k,k),m})\, t^k = \frac{t^2+t^4+t^5+t^6+t^7+t^{11}+t^{14}+t^{15}}{(1-t^2)(1-t^5)(1-t^6)}.$$

These eight forms F_k *are obtained explicitly by using Hilbert modular forms* G_2*,* G_5*,* G_6*,* G_{15} *and differential operators (see subsection 2.5).*

To show this theorem we need the following structure theorem of Jacobi forms of matrix index. We denote by $J_{k,1_2}$ the space of Jacobi forms of index $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ (cf. about the definition of Jacobi forms of matrix index, see Ziegler [4] page 193). We put $J_{*,1_2} := \bigoplus_{k\in\mathbf{Z}} J_{k,1_2}$, and $M_* := \bigoplus_{k\in\mathbf{Z}} M_k$, where M_k is the space of elliptic modular forms of weight k with respect to $SL(2,\mathbb{Z})$.

Theorem 1.3. *The space* $J_{*,1_2}$ *is a free* M_**-module with rank* 4 *and* $\{\psi_4, \psi_6, \psi_8, \psi_{10}\}$ *is a basis of* $J_{*,1_2}$*, and the dimension formula is given by*

$$\sum_{k\in\mathbf{N}} \dim(J_{k,1_2})t^k = \frac{t^4+t^6+t^8+t^{10}}{(1-t^4)(1-t^6)},$$

where the forms $\psi_k \in J_{k,1_2}$ $(k = 4, 6, 8.10)$ *are given in subsection 2.4.*

2 Construction of Jacobi forms

In this section, we explain a construction of Hilbert-Jacobi forms from pair of Hilbert modular forms. The original idea of this construction in the case of usual Jacobi forms was given by N.-P.Skoruppa [3]. We shall also explain in this section the idea of the proof for Theorem 1.2.

2.1 Wronskian

In this subsection and the next subsection, we explain a construction of Hilbert-Jacobi forms from pairs of Hilbert modular forms for arbitrary totally real field K and for arbitrary index m.

Let $\phi \in J^K_{k,m}$. We take the theta expansion :

$$\phi(\tau, z) := \sum_{\alpha\in\mathfrak{d}^{-1}/2m\mathcal{O}} f_\alpha(\tau)\vartheta_{m,\alpha}(\tau, z),$$

where $\vartheta_{m,\alpha}(\tau, z) = \sum_{\substack{r\in\mathfrak{d}^{-1} \\ r\equiv\alpha(2m\mathcal{O})}} e(\frac{1}{4m}r^2\tau + rz)$.

Let $l := |\mathfrak{d}^{-1}/2m\mathcal{O}| = N(2m)D_K$, where D_K is the discriminant of K. We put $\theta(\tau, z) := (\vartheta_{m,\alpha_0}(\tau, z), ..., \vartheta_{m,\alpha_{l-1}}(\tau, z))$, where $\tau = (\tau_1, ..., \tau_n) \in \mathfrak{H}^n$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$, and where $(\alpha_0, ..., \alpha_{l-1})$ is a complete set of the representatives of $\mathfrak{d}^{-1}/2m\mathcal{O}$.

For $u = (u_0, ..., u_{l-1}) \in (\mathbb{N}^n)^l$, we set

$$W(\tau) := W_u(\tau) := \begin{pmatrix} \partial_z^{u_0}\theta|_{z=0} \\ \vdots \\ \partial_z^{u_{l-1}}\theta|_{z=0} \end{pmatrix},$$

where we defined $\partial_z^{u_i} := \partial_{z_1}^{u_{i,1}} \cdots \partial_{z_n}^{u_{i,n}}$ for $u_i = (u_{i,1}, ..., u_{i,n}) \in \mathbb{N}^n$, and $\partial_{z_i} := \frac{1}{2\pi i}\frac{\delta}{\delta_{z_i}}$.

If u satisfies the following condition $[Cu]$, then $\det(W)$ is a Hilbert modular form of weight $(l/2, ..., l/2) + \sum_{i=0}^{l-1} u_i$ with a certain character.

[Cu] If $v = (v_1, ..., v_n) \in \mathbb{N}^n$ satisfies $v \leq u_j$, $v \equiv u_j \mod 2$ with a $j \in \{0, ..., l-1\}$, then $v \in \{u_0, ..., u_{l-1}\}$. Here $v \leq u_j$ means $v_i \leq u_{j,i}$ for any $i \in \{1, ..., n\}$.

2.2 Construction of Hilbert Jacobi forms

Let $\phi \in J_{k,m}^K$. We have

$$\phi(\tau, z) = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau)\vartheta_{m,\alpha_i}(\tau, z) = \sum_{\nu\in\mathbb{N}^n} g_\nu(\tau)\frac{(2\pi i)^\nu z^\nu}{\nu!},$$

where $\nu! := \prod_{j=1}^{n} \nu_j!$, $\nu = (\nu_1, ..., \nu_n)$, and $g_\nu(\tau) = \partial_z^\nu\phi|_{z=0} = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau)(\partial_z^\nu\vartheta_{m,\alpha_i})|_{z=0}$.

Thus for $u = (u_0, ..., u_{l-1}) \in (\mathbb{N}^n)^l$ we have

$$^t(g_{u_0}(\tau), ..., g_{u_{l-1}}(\tau)) \;=\; W(\tau)^t(f_{\alpha_0}(\tau), ..., f_{\alpha_{l-1}}(\tau)).$$

Now $(g_{u_0}, ..., g_{u_{l-1}})$ satisfies a certain transformation formula, so there exists a pair of Hilbert modular forms $(G_{u_0}, ..., G_{u_{l-1}}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K$ such that

$$^t(g_{u_0}(\tau), ..., g_{u_{l-1}}(\tau)) = D^t(G_{u_0}(\tau), ..., G_{u_{l-1}}(\tau)),$$

where D is a certain matrix of differential operators depending only on k and u. Hence if $\det(W)$ is not identically zero, then

$$\phi = \theta^t(f_{\alpha_0}, \ldots, f_{\alpha_{l-1}}) = \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{l-1}})).$$

On the other hand, for any pair of Hilbert modular forms $(G_{u_0}, \ldots, G_{u_{l-1}}) \in M^K_{k+u_0} \times \cdots \times M^K_{k+u_{l-1}}$, by using the above identity, we can construct a meromorphic function on $\mathfrak{H}^n \times \mathbb{C}^n$ which satisfies the transformation formula of Hilbert Jacobi forms (conditions (i), (ii) of the definition 1.) We denote this map by $\tilde{\lambda}_k$:

$$\tilde{\lambda}_k : M^K_{k+u_0} \times \cdots \times M^K_{k+u_{l-1}} \to J^{K,mero}_{k,m}$$

via

$$\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{l-1}}) := \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{l-1}})).$$

Thus for constructing Hilbert-Jacobi forms in general, we need to know when $\det(W)$ is not identically zero , and when $\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{l-1}})$ is holomorphic.

2.3 Example $K = \mathbb{Q}(\sqrt{5}), m = (5+\sqrt{5})/10$

We fix $K = \mathbb{Q}\sqrt{5}$, and $m = (5+\sqrt{5})/10$. In this subsection we give explicitly the matrix D and construct Hilbert-Jacobi forms of index m.

By straightforward calculation we obtain $\mathfrak{d}^{-1} = m\mathcal{O}$, $\mathfrak{d}^{-1}/2m\mathcal{O} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $|\mathfrak{d}^{-1}/2m\mathcal{O}| = 4$.

We put $u := (u_0, u_1, u_2, u_3) \in (\mathbb{N}^2)^4$, where $u_0 := (0,0)$, $u_1 := (0,2)$, $u_2 := (2,0)$ and $u_3 := (1,1)$. Then, $\det(W) = c \cdot G_5$ with non zero constant c. Here G_5 is the Hilbert modular form of weight $(5,5)$ denoted in Theorem 1.1.

Let $k = (k_1, k_2) \in \mathbb{N}^2$. For $(G_{u_0}, \ldots, G_{u_3}) \in M^K_{k+u_0} \times \cdots \times M^K_{k+u_{l-1}}$, we put

$$\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_3}) := \phi := \theta W^{-1}(D \cdot^t (G_{u_0}, \ldots, G_{u_3})),$$

where $D := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2m}{k_1}\partial_{\tau_1} & 1 & 0 & 0 \\ \frac{2m'}{k_2}\partial_{\tau_2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and m' is the Galois conjugation of m. Due to the consideration of the previous subsection we have $\phi \in J^{K,mero}_{k,m}$.

We denote by $J_{l,1_2}$ the space of Jacobi forms of weight $l \in \mathbb{N}$ of index $1_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now, for $k = (k_1, k_1) \in \mathbb{N}^2$ we consider the following map

$$\mathbb{D} \ : \ J^K_{k,m} \to J_{2k_1, 1_2}$$

via

$$\mathbb{D}(\phi)(\tau,(z_1,z_2)) := \phi((\tau,\tau),(z_1,z_2)\cdot V),$$

where $\phi \in J^K_{k,m}$, $(z_1,z_2) \in \mathbb{C}^2$, $\tau \in \mathfrak{H}$, $V = \begin{pmatrix} 1 & 1 \\ \epsilon^{-1} & \epsilon'^{-1} \end{pmatrix}$, $\epsilon = (1+\sqrt{5})/2$ and $\epsilon' = (1-\sqrt{5})/2$.

2.4 The space of Jacobi forms of index 1_2

As for the structure of the space of Hilbert modular forms of index 1_2, we have the following theorem.

Theorem 2.1. *For any $k' \in \mathbb{Z}$, we have $J_{k',1_2} \cong M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k'+4}$, where $M_{k'}$ (resp. $S_{k'}$) is the space of elliptic modular forms (resp. cusp forms) of weight k' with respect to $SL_2(\mathbb{Z})$.*

The idea of the proof of the above theorem is as follows. By similar method as in the subsection 2.2, we have a similar map as $\tilde{\lambda}_k$ in the subsection 2.2 for the space of Jacobi forms of index 1_2. We can construct meromorphic Jacobi forms of index 1_2. In this case, by choosing a suitable $u \in (\mathbb{N}^2)^4$, the Wronskian is the Ramanujan-Δ function. Hence we can check when the image of the map $\hat{\nu}_{k'}$, which corresponds to $\tilde{\lambda}_k$ in the case of Hilbert Jacobi forms, is holomorphic. The surjectivity of the map $\hat{\nu}_{k'}$ follows from this fact. Thus we obtain theorem 2.1.

The idea for the proof of Theorem 1.3.

Due to Theorem 2.1, we have the dimension formula for $\bigoplus_{k'\in\mathbb{Z}} J_{k,1_2}$, and we obtain Theorem 1.3 by constructing suitable basis of the space of Jacobi forms of index 1_2 as $\bigoplus_{k\in\mathbb{Z}} M_k$-module.

The basis of $\bigoplus_{k'\in\mathbb{Z}} J_{k',1_2}$ is give by the following four forms : $\psi_4 := \hat{\nu}_4((E_4,0,0,0)) \in J_{4,1_2}$, $\psi_6 := \hat{\nu}_6((E_6,0,0,0)) \in J_{6,1_2}$, $\psi_{10} := \hat{\nu}_{10}((0,0,\Delta,0)) \in J_{10,1_2}$, and $\psi_8 := \hat{\nu}_8((0,0,0,\Delta)) \in J_{8,1_2}$. Here $\hat{\nu}_{k'}$ is the map from $M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k'+4}$ to $J_{k',1_2}$, and $E_{k'}$ are the Eisenstein series of weight k'.

2.5 The space of Hilbert-Jacobi forms of index m

Let $k = (k_1,k_1) \in \mathbb{N}^2$. We put

$$\tilde{J}^K_{k,m} := \tilde{\lambda}_k(M^K_{(k_1,k_1)} \times S^K_{(k_1,k_1+2)} \times S^K_{(k_1+2,k_1)} \times M^K_{(k_1+1,k_1+1)}),$$

where $S^K_{(k_1,k_2)}$ is the space of Hilbert cusp forms of weight (k_1,k_2) with respect to $SL(2.\mathcal{O})$.

As for for the space of Hilbert cusp forms $S^K_{(k_1,k_1+2)}$ the following theorem is known by H.Aoki [1].

Theorem 2.2 (Aoki). *The structure of* $\bigoplus_{k_1\in\mathbf{Z}} S^K_{(k_1,k_1+2)}$ *is given by*

$$\bigoplus_{k_1\in\mathbf{Z}} S^K_{(k_1,k_1+2)} = A_{7,9}B + A_{8,10}B + A_{11,13}B,$$

where $A_{7,9} := [G_2, G_5] := 2G_2(\partial_{\tau_2} G_5) - 5G_5(\partial_{\tau_2} G_2)$, $A_{8,10} := [G_6, G_2]$, $A_{11,13} := [G_5, G_6]$ *and* $B = \mathbb{C}[G_2, G_5, G_6]$. *Here* $A_{7,9}$, $A_{8,10}$ *and* $A_{11,13}$ *satisfy the following Jacobi identity :* $6\,G_6A_{7,9} + 5\,G_5A_{8,10} + 2\,G_2A_{11,13} = 0$. *Except this identity, there are no relation among* $A_{7,9}$, $A_{8,10}$ *and* $A_{11,13}$.

To show theorem 1.2, we need the following proposition.

Proposition 2.3. *Let* $\phi \in J^K_{k,m}$. *Then* $\mathbb{D}(\phi) = 0$ *if and only if* $G_5|\phi$.

Thus we have the following short exact sequence :

$$0 \longrightarrow J^K_{k,m} \longrightarrow \hat{J}^K_{k,m} \longrightarrow J_{2k_1+10,1_2},$$

where the second map is the embedding, and the last map is given via ϕ to $\mathbb{D}(G_5 \cdot \phi)$ for $\phi \in \hat{J}^K_{k,m}$.

By using theorem 1.1 and theorem 2.2 we can calculate the dimension of $\hat{J}^K_{k,m}$, and also we have the dimension of the image of the above last map. Hence, we have the dimension formula for $\dim(J^K_{(k,k),m})$ written in Theorem 1.2. The basis of $\bigoplus_{k\in\mathbf{Z}} J^K_{(k,k),m}$ as $\mathbb{C}[G_2, G_5, G_6]$-module is given as follows :

$$\begin{aligned}
&F_2 := \tilde{\lambda}_2(G_2, 0, 0, 0),\ F_4 := \tilde{\lambda}_4(0, 0, 0, G_5),\\
&F_5 := \tilde{\lambda}_5\left(G_5, 0, 0, \frac{2}{5\sqrt{5}}G_6\right),\ F_6 := \tilde{\lambda}_6(G_6, 0, 0, 0),\\
&F_7 := \tilde{\lambda}_7(0, mA'_{7,9}, -m'A_{7,9}, 0),\ F_{11} := \tilde{\lambda}_{11}(0, mA'_{11,13}, -m'A_{11,13}, 0),\\
&F_{14} := \tilde{\lambda}_{14}\left(0, \frac{\epsilon}{2}A'_{8,10}, \frac{\epsilon'}{2}A_{8,10}, G_{15}\right),\ F_{15} := \tilde{\lambda}_{15}(G_{15}, 0, 0, 0),
\end{aligned}$$

where $A'_{7,9} := 2\,G_2(\partial_{\tau_1} G_5) - 5\,G_5(\partial_{\tau_1} G_2)$, $A'_{8,10} := 6\,G_6(\partial_{\tau_1} G_2) - 2\,G_2(\partial_{\tau_1} G_6)$ and $A'_{11,13} := 5\,G_5(\partial_{\tau_1} G_6) - 6\,G_6(\partial_{\tau_1} G_5)$.

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