<table>
<thead>
<tr>
<th>Title</th>
<th>Hilbert-Jacobi forms of a certain index of $\mathbb{Q} (\sqrt{5})$ (Automorphic Representations, Automorphic Forms, L-functions, and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Hayashida, Shuichi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2008, 1617: 98-105</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2008-10</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140172">http://hdl.handle.net/2433/140172</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Hilbert-Jacobi forms of a certain index of $\mathbb{Q}(\sqrt{5})$

S.Hayashida (Universität Siegen)
(joint work with N.-P.Skoruppa (Universität Siegen))

0 Introduction

The purpose of this survey is to give an example of a structure theorem of the space of Hilbert-Jacobi forms of a certain index with concerning to $K = \mathbb{Q}(\sqrt{5})$ (Theorem 1.2). We give also an example of a structure theorem of the space of Jacobi forms of a matrix index (Theorem 1.3). We used theorem 1.3 to show theorem 1.2.

1 Main theorem

In this section, we recall the definition of Hilbert-Jacobi forms, and give an example of a structure theorem of the space of Hilbert-Jacobi forms and of Jacobi forms of a matrix index.

1.1 Notations

Let $K$ be a totally real field with degree $n$, let $\mathfrak{o}^{-1}$ be the inverse of the different, and let $\mathcal{O}$ be the principal order of $K$. We denote by $\mathcal{H}$ the Poincaré upper half plane. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $e(z) := e^{2\pi i \text{tr}(z)}$, where $\text{tr}(z) = z_1 + \cdots + z_n$. By abuse of language, we set $z^k := \prod_{i=1}^{n} z_i^{k_i}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$.

1.2 Definition

For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and for totally positive number $m \in \mathfrak{o}^{-1}$, we define Hilbert-Jacobi forms of weight $k$ of index $m$ as follows.

Definition 1. Let $\phi$ be a holomorphic function on $\mathcal{H}^n \times \mathbb{C}^n$. We say $\phi$ is a Hilbert-Jacobi form of weight $k$ of index $m$ if $\phi$ satisfies the following three conditions.
For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{O})$, any $\tau = (\tau_1, ..., \tau_n) \in \mathfrak{h}^n$ and any $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $\phi$ satisfies

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e(m(c\tau + d)^{-1}cz^2)(c\tau + d)^k\phi(\tau, z).$$

(ii) For any $\lambda, \mu \in \mathcal{O}$,

$$\phi(\tau, z + \lambda\tau + \mu) = e(-m\lambda^2\tau - 2m\lambda z)\phi(\tau, z).$$

(iii) $\phi$ has the Fourier expansion:

$$\phi(\tau, z) = \sum_{u, r \in \mathfrak{y}^{-1}} c(u, r) e(u\tau + rz),$$

where in the above summation $u$ and $r$ run over all elements in $\mathfrak{y}^{-1}$ such that $4um - r^2$ is totally positive or equals to 0.

When $n$ is larger than 1, then because of Koecher principle the third condition of the definition follows automatically by the first and second conditions.

We denote by $J_{k,m}^K$ the space of Hilbert-Jacobi forms of weight $k$ of index $m$ with respect to $SL(2, \mathcal{O})$.

1.3 Results

We consider the case $K = \mathbb{Q}(\sqrt{5})$, $m = \epsilon/\sqrt{5}$, where $\epsilon = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of the maximal order $\mathcal{O} = \mathbb{Z}[\epsilon]$ of $K$.

Let $k \in \mathbb{N}$. Now $M_{(k_1, k_2)}^K$ denotes the space of Hilbert modular forms of weight $(k_1, k_2) \in \mathbb{Z}^2$ with respect to $SL(2, \mathcal{O})$. We quote the following structure theorem of the space of Hilbert modular forms obtained by Gundlach [2].

Theorem 1.1 (Gundlach[2]).

$$\bigoplus_{k \in \mathbb{Z}} M_{(k,k)}^K = \mathbb{C}[G_2, G_5, G_6] \oplus G_{15}\mathbb{C}[G_2, G_5, G_6],$$

where $G_2, G_5, G_6$ and $G_{15}$ are Hilbert modular forms of weight 2, 5, 6 and 15, respectively. There exists a polynomial $P(X_1, X_2, X_3)$ such that $G_{15}^2 = P(G_2, G_5, G_6)$.

The main theorem of this report is as follows.
Theorem 1.2. The space $\bigoplus_{k \in \mathbb{Z}} J_{(k,k),m}^{K}$ is a $\mathbb{C}[G_{2}, G_{5}, G_{6}]$-module generated by eight forms $F_k \in J_{(k,k),m}^{K}$ ($k = 2, 4, 5, 6, 7, 11, 14, 15$), and the dimension formula is given by
\[
\sum_{k \in \mathbb{Z}} \dim(J_{(k,k),m}^{K}) t^k = \frac{t^2 + t^4 + t^6 + t^7 + t^{11} + t^{14} + t^{15}}{(1-t^2)(1-t^5)(1-t^6)}.
\]
These eight forms $F_k$ are obtained explicitly by using Hilbert modular forms $G_{2}$, $G_{5}$, $G_{6}$, $G_{15}$ and differential operators (see subsection 2.5).

To show this theorem we need the following structure theorem of Jacobi forms of matrix index. We denote by $J_{k,12}$ the space of Jacobi forms of index $(\frac{1}{2}, \frac{1}{2})$ (cf. about the definition of Jacobi forms of matrix index, see Ziegler [4] page 193). We put $J_{*,12} := \bigoplus_{k \in \mathbb{Z}} J_{k,12}$, and $M_{*} := \bigoplus_{k \in \mathbb{Z}} M_{k}$, where $M_{k}$ is the space of elliptic modular forms of weight $k$ with respect to $SL(2, \mathbb{Z})$.

Theorem 1.3. The space $J_{*,12}$ is a free $M_{*}$-module with rank 4 and $\{\psi_{4}, \psi_{6}, \psi_{8}, \psi_{10}\}$ is a basis of $J_{*,12}$, and the dimension formula is given by
\[
\sum_{k \in \mathbb{N}} \dim(J_{k,12}) t^k = \frac{t^4 + t^6 + t^8 + t^{10}}{(1-t^4)(1-t^6)},
\]
where the forms $\psi_k \in J_{k,12}$ ($k = 4, 6, 8, 10$) are given in subsection 2.4.

2 Construction of Jacobi forms

In this section, we explain a construction of Hilbert-Jacobi forms from pair of Hilbert modular forms. The original idea of this construction in the case of usual Jacobi forms was given by N.-P.Skoruppa [3]. We shall also explain in this section the idea of the proof for Theorem 1.2.

2.1 Wronskian

In this subsection and the next subsection, we explain a construction of Hilbert-Jacobi forms from pairs of Hilbert modular forms for arbitrary totally real field $K$ and for arbitrary index $m$.

Let $\phi \in J_{k,m}^{K}$. We take the theta expansion:
\[
\phi(\tau, z) := \sum_{\alpha \in \Theta^{-1}/2mO} f_{\alpha}(\tau) \vartheta_{m,\alpha}(\tau, z),
\]
where \( \vartheta_{m,\alpha}(\tau, z) = \sum_{r \equiv \alpha(2mO)} e^{r^2 \tau + rz} \).

Let \( l := |\mathfrak{d}^{-1}/2m\mathcal{O}| = N(2m)D_K \), where \( D_K \) is the discriminant of \( K \). We put \( \theta(\tau, z) := (\vartheta_{m,\alpha_0}(\tau, z), \ldots, \vartheta_{m,\alpha_{t-1}}(\tau, z)) \), where \( \tau = (\tau_1, \ldots, \tau_n) \in \mathbb{H}^n \), \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), and where \( (\alpha_0, \ldots, \alpha_{l-1}) \) is a complete set of the representatives of \( \mathfrak{d}^{-1}/2m\mathcal{O} \).

For \( u = (u_0, \ldots, u_{l-1}) \in (\mathbb{N}^n)^l \), we set

\[
W(\tau) := W_u(\tau) := \begin{pmatrix}
\partial_{z_0}^{u_0} \theta|_{z=0} \\
\vdots \\
\partial_{z_{n-1}}^{u_{l-1}} \theta|_{z=0}
\end{pmatrix},
\]

where we defined \( \partial_{z_i}^{u_i} := \partial_{z_1}^{u_{i,1}} \cdots \partial_{z_n}^{u_{i,n}} \) for \( u_i = (u_{i,1}, \ldots, u_{i,n}) \in \mathbb{N}^n \), and \( \partial_{z_i} := \frac{1}{2\pi i} \frac{\delta}{\delta i} \).

If \( u \) satisfies the following condition \([Cu]\), then \( \det(W) \) is a Hilbert modular form of weight \((l/2, \ldots, l/2) + \sum_{i=0}^{l-1} u_i \) with a certain character.

If \( v = (v_1, \ldots, v_n) \in \mathbb{N}^n \) satisfies \( v \leq u_j, v \equiv u_j \mod 2 \) with a \( j \in \{0, \ldots, l-1\} \), then \( v \in \{u_0, \ldots, u_{l-1}\} \). Here \( v \leq u_j \) means \( v_i \leq u_{j,i} \) for any \( i \in \{1, \ldots, n\} \).

### 2.2 Construction of Hilbert Jacobi forms

Let \( \phi \in J_{k,m}^K \). We have

\[
\phi(\tau, z) = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau) \vartheta_{m,\alpha_i}(\tau, z) = \sum_{\nu \in \mathbb{N}^n} g_\nu(\tau) \frac{(2\pi i)^\nu z^\nu}{\nu!},
\]

where \( \nu! := \prod_{j=1}^n \nu_j! \), \( \nu = (\nu_1, \ldots, \nu_n) \), and \( g_\nu(\tau) = \partial_{z_0}^\nu \phi|_{z=0} = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau) (\partial_{z_0}^\nu \vartheta_{m,\alpha_i})|_{z=0} \).

Thus for \( u = (u_0, \ldots, u_{l-1}) \in (\mathbb{N}^n)^l \) we have

\[
t^t(g_{u_0}(\tau), \ldots, g_{u_{l-1}}(\tau)) = W(\tau)^t(f_{\alpha_0}(\tau), \ldots, f_{\alpha_{l-1}}(\tau)).
\]

Now \( (g_{u_0}, \ldots, g_{u_{l-1}}) \) satisfies a certain transformation formula, so there exists a pair of Hilbert modular forms \((G_{u_0}, \ldots, G_{u_{l-1}}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K \) such that

\[
t^t(g_{u_0}(\tau), \ldots, g_{u_{l-1}}(\tau)) = D^t(G_{u_0}(\tau), \ldots, G_{u_{l-1}}(\tau)),
\]
where $D$ is a certain matrix of differential operators depending only on $k$ and $u$. Hence if $\det(W)$ is not identically zero, then

$$\phi = \theta^t(f_{a_0}, \ldots, f_{a_{l-1}}) = \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{l-1}})).$$

On the other hand, for any pair of Hilbert modular forms $(G_{u_0}, \ldots, G_{u_{l-1}}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K$, by using the above identity, we can construct a meromorphic function on $\mathbb{H}^n \times \mathbb{C}^n$ which satisfies the transformation formula of Hilbert Jacobi forms (conditions $(i)$, $(ii)$ of the definition 1.) We denote this map by $\tilde{\lambda}_k$:

$$\tilde{\lambda}_k : M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K \rightarrow J_{k,m}^{K,\text{mero}}$$

via

$$\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{l-1}}) := \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{l-1}})).$$

Thus for constructing Hilbert-Jacobi forms in general, we need to know when $\det(W)$ is not identically zero, and when $\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{l-1}})$ is holomorphic.

### 2.3 Example $K = \mathbb{Q}(\sqrt{5}), m = (5 + \sqrt{5})/10$

We fix $K = \mathbb{Q}(\sqrt{5})$, and $m = (5 + \sqrt{5})/10$. In this subsection we give explicitly the matrix $D$ and construct Hilbert-Jacobi forms of index $m$.

By straightforward calculation we obtain $d^{-1} = mO$, $d^{-1}/2mO \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $|d^{-1}/2mO| = 4$.

We put $u := (u_0, u_1, u_2, u_3) \in (\mathbb{N}^2)^4$, where $u_0 := (0, 0)$, $u_1 := (0, 2)$, $u_2 := (2, 0)$ and $u_3 := (1, 1)$. Then, $\det(W) = c \cdot G_5$ with non zero constant $c$. Here $G_5$ is the Hilbert modular form of weight $(5, 5)$ denoted in Theorem 1.1.

Let $k = (k_1, k_2) \in \mathbb{N}^2$. For $(G_{u_0}, \ldots, G_{u_3}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K$, we put

$$\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{l-1}}) := \phi := \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{3}})),$$

where $D := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2m}{k_1} \partial_{\tau_1} & 0 & 0 & 0 \\ \frac{2m'}{k_2} \partial_{\tau_2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $m'$ is the Galois conjugation of $m$. Due to the consideration of the previous subsection we have $\phi \in J_{k,m}^{K,\text{mero}}$.

We denote by $J_{l,12}$ the space of Jacobi forms of weight $l \in \mathbb{N}$ of index $1_2 = (1 \ 0 \ 0 \ 1)$.

Now, for $k = (k_1, k_1) \in \mathbb{N}^2$ we consider the following map

$$\mathcal{D} : J_{k,m}^K \rightarrow J_{2k_1,12}$$
via
$$D(\phi)(\tau, (z_1, z_2)) := \phi((\tau, \tau), (z_1, z_2) \cdot V),$$
where \(\phi \in J_{k,m}^{K}(z_1, z_2) \in \mathbb{C}^2, \tau \in \mathfrak{H}, V = (1, 1 - 1), \epsilon = (1 + \sqrt{5})/2\) and \(\epsilon' = (1 - \sqrt{5})/2\).

2.4 The space of Jacobi forms of index 12

As for the structure of the space of Hilbert modular forms of index 12, we have the following theorem.

**Theorem 2.1.** For any \(k' \in \mathbb{Z}\), we have \(J_{k',12} \cong M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k' + 4}\), where \(M_{k'} \) (resp. \(S_{k'} \)) is the space of elliptic modular forms (resp. cusp forms) of weight \(k'\) with respect to \(SL_2(\mathbb{Z})\).

The idea of the proof of the above theorem is as follows. By similar method as in the subsection 2.2, we have a similar map as \(\tilde{\lambda}_k\) in the subsection 2.2 for the space of Jacobi forms of index 12. We can construct meromorphic Jacobi forms of index 12. In this case, by choosing a suitable \(u \in (\mathbb{N}^2)^4\), the Wronskian is the Ramanujan-\(\Delta\) function. Hence we can check when the image of the map \(\hat{\nu}_{k'}\), which corresponds to \(\tilde{\lambda}_k\) in the case of Hilbert Jacobi forms, is holomorphic. The surjectivity of the map \(\hat{\nu}_{k'}\) follows from this fact. Thus we obtain theorem 2.1.

The idea for the proof of Theorem 1.3.

Due to Theorem 2.1, we have the dimension formula for \(\bigoplus_{k' \in \mathbb{Z}} J_{k,12}\), and we obtain Theorem 1.3 by constructing suitable basis of the space of Jacobi forms of index 12 as \(\bigoplus_{k \in \mathbb{Z}} M_k\)-module.

The basis of \(\bigoplus_{k' \in \mathbb{Z}} J_{k',12}\) is give by the following four forms: \(\psi_4 := \hat{\nu}_4((E_4, 0, 0, 0)) \in J_{4,12}\), \(\psi_6 := \hat{\nu}_6((E_6, 0, 0, 0)) \in J_{6,12}\), \(\psi_{10} := \hat{\nu}_{10}((0, 0, \Delta, 0)) \in J_{10,12}\), and \(\psi_8 := \hat{\nu}_8((0, 0, 0, \Delta)) \in J_{8,12}\). Here \(\hat{\nu}_{k'}\) is the map from \(M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k' + 4}\) to \(J_{k',12}\), and \(E_{k'}\) are the Eisenstein series of weight \(k'\).

2.5 The space of Hilbert-Jacobi forms of index \(m\)

Let \(k = (k_1, k_1) \in \mathbb{N}^2\). We put
$$\tilde{J}_{k,m}^{K} := \tilde{\lambda}_k(M_{(k_1,k_1)}^{K} \times S_{(k_1,k_1+2)}^{K} \times S_{(k_1+2,k_1)}^{K} \times M_{(k_1+1,k_1+1)}^{K}),$$
where \(S_{(k_1,k_2)}^{K}\) is the space of Hilbert cusp forms of weight \((k_1, k_2)\) with respect to \(SL(2,\mathcal{O})\).
As for the space of Hilbert cusp forms $S^K_{(k_1, k_1+2)}$ the following theorem is known by H. Aoki [1].

**Theorem 2.2 (Aoki).** The structure of $\bigoplus_{k_1 \in \mathbb{Z}} S^K_{(k_1, k_1+2)}$ is given by

$$\bigoplus_{k_1 \in \mathbb{Z}} S^K_{(k_1, k_1+2)} = A_{7,9}B + A_{8,10}B + A_{11,13}B,$$

where $A_{7,9} := [G_2, G_5] := 2G_2(\tau G_5) - 5G_5(\tau G_2)$, $A_{8,10} := [G_6, G_2]$, $A_{11,13} := [G_5, G_6]$ and $B = \mathbb{C}[G_2, G_5, G_6]$. Here $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$ satisfy the following Jacobi identity: $6G_6A_{7,9} + 5G_5A_{8,10} + 2G_2A_{11,13} = 0$. Except this identity, there are no relation among $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$.

To show theorem 1.2, we need the following proposition.

**Proposition 2.3.** Let $\phi \in J^K_{k,m}$. Then $D(\phi) = 0$ if and only if $G_5|\phi$.

Thus we have the following short exact sequence:

$$0 \rightarrow J^K_{k,m} \rightarrow \hat{J}^K_{k,m} \rightarrow J_{2k_1+10,12},$$

where the second map is the embedding, and the last map is given via $\phi$ to $D(G_5 \cdot \phi)$ for $\phi \in \hat{J}^K_{k,m}$.

By using theorem 1.1 and theorem 2.2 we can calculate the dimension of $\hat{J}^K_{k,m}$, and also we have the dimension of the image of the above last map. Hence, we have the dimension formula for $\dim(J^K_{(k_1,k),m})$ written in Theorem 1.2. The basis of $\bigoplus_{k_1 \in \mathbb{Z}} J^K_{(k_1,k),m}$ as $\mathbb{C}[G_2, G_5, G_6]$-module is given as follows:

$$F_2 := \tilde{\lambda}_2(G_2, 0, 0, 0), \quad F_4 := \tilde{\lambda}_4(0, 0, 0, G_5),$$
$$F_5 := \tilde{\lambda}_5 \left(G_5, 0, 0, \frac{2}{5\sqrt{5}}G_6\right), \quad F_6 := \tilde{\lambda}_6(G_6, 0, 0, 0),$$
$$F_7 := \tilde{\lambda}_7(0, mA'_{7,9}, -m'A_{7,9}, 0), \quad F_{11} := \tilde{\lambda}_{11}(0, mA'_{11,13}, -m'A_{11,13}, 0),$$
$$F_{14} := \tilde{\lambda}_{14} \left(0, \frac{\epsilon}{2}A'_{8,10}, \frac{\epsilon'}{2}A_{8,10}, G_{15}\right), \quad F_{15} := \tilde{\lambda}_{15}(G_{15}, 0, 0, 0),$$

where $A'_{7,9} := 2G_2(\tau G_5) - 5G_5(\tau G_2)$, $A'_{8,10} := 6G_6(\tau G_2) - 2G_2(\tau G_6)$ and $A'_{11,13} := 5G_5(\tau G_6) - 6G_6(\tau G_5)$. 
References


Fachbereich 6 Mathematik, Universität Siegen,
Walter-Flex-Str. 3, 57068 Siegen, Germany.
e-mail hayashida@mathematik.uni-siegen.de