Title: Hilbert-Jacobi forms of a certain index of $\mathbb{Q} (\sqrt{5})$

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Hilbert-Jacobi forms of a certain index of $\mathbb{Q}(\sqrt{5})$

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0 Introduction

The purpose of this survey is to give an example of a structure theorem of the space of Hilbert-Jacobi forms of a certain index with concerning to $K = \mathbb{Q}(\sqrt{5})$ (Theorem 1.2). We give also an example of a structure theorem of the space of Jacobi forms of a matrix index (Theorem 1.3). We used theorem 1.3 to show theorem 1.2.

1 Main theorem

In this section, we recall the definition of Hilbert-Jacobi forms, and give an example of a structure theorem of the space of Hilbert-Jacobi forms and of Jacobi forms of a matrix index.

1.1 Notations

Let $K$ be a totally real field with degree $n$, let $\mathfrak{d}^{-1}$ be the inverse of the different, and let $\mathcal{O}$ be the principal order of $K$. We denote by $\mathfrak{H}$ the Poincaré upper half plane. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $e(z) := e^{2\pi i \text{tr}(z)}$, where $\text{tr}(z) = z_1 + \cdots + z_n$. By abuse of language, we set $z^k := \prod_{i=1}^{n} z_i^{k_i}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$.

1.2 Definition

For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and for totally positive number $m \in \mathfrak{d}^{-1}$, we define Hilbert-Jacobi forms of weight $k$ of index $m$ as follows.

\textbf{Definition 1.} Let $\phi$ be a holomorphic function on $\mathfrak{H}^n \times \mathbb{C}^n$. We say $\phi$ is a Hilbert-Jacobi form of weight $k$ of index $m$ if $\phi$ satisfies the following three conditions.
(i) For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{O})$, any $\tau = (\tau_1, ..., \tau_n) \in \hslash^n$ and any $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $\phi$ satisfies

$$\phi\left(\frac{a\tau + b}{ct + d}, \frac{z}{ct + d}\right) = e\left(m(ct + d)^{-1}cz^2\right)(ct + d)^k\phi(\tau, z).$$

(ii) For any $\lambda, \mu \in \mathcal{O}$,

$$\phi(\tau, z + \lambda\tau + \mu) = e(-m\lambda^2\tau - 2m\lambda z)\phi(\tau, z).$$

(iii) $\phi$ has the Fourier expansion:

$$\phi(\tau, z) = \sum_{u, r \in \mathfrak{y}^{-1}} c(u, r)e(u\tau + rz),$$

where in the above summation $u$ and $r$ run over all elements in $\mathfrak{y}^{-1}$ such that $4um - r^2$ is totally positive or equals to 0.

When $n$ is larger than 1, then because of Koecher principle the third condition of the definition follows automatically by the first and second conditions.

We denote by $J_{k,m}^K$ the space of Hilbert-Jacobi forms of weight $k$ of index $m$ with respect to $SL(2, \mathcal{O})$.

1.3 Results

We consider the case $K = \mathbb{Q}(\sqrt{5})$, $m = \epsilon/\sqrt{5}$, where $\epsilon = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of the maximal order $\mathcal{O} = \mathbb{Z}[\epsilon]$ of $K$.

Let $k \in \mathbb{N}$. Now $M_{(k_1, k_2)}^K$ denotes the space of Hilbert modular forms of weight $(k_1, k_2) \in \mathbb{Z}^2$ with respect to $SL(2, \mathcal{O})$. We quote the following structure theorem of the space of Hilbert modular forms obtained by Gundlach [2].

**Theorem 1.1** (Gundlach[2]).

$$\bigoplus_{k \in \mathbb{Z}} M_{(k,k)}^K = \mathbb{C}[G_2, G_5, G_6] \oplus G_{15}\mathbb{C}[G_2, G_5, G_6],$$

where $G_2$, $G_5$, $G_6$ and $G_{15}$ are Hilbert modular forms of weight 2, 5, 6 and 15, respectively. There exists a polynomial $P(X_1, X_2, X_3)$ such that $G_{15}^2 = P(G_2, G_5, G_6)$.

The main theorem of this report is as follows.
Theorem 1.2. The space $\bigoplus_{k \in \mathbb{Z}} J_{(k,k),m}^{K}$ is a $\mathbb{C}[G_{2}, G_{5}, G_{6}]$-module generated by eight forms $F_k \in J_{(k,k),m}^{K}$ ($k = 2, 4, 5, 6, 7, 11, 14, 15$), and the dimension formula is given by
\[ \sum_{k \in \mathbb{Z}} \dim(J_{(k,k),m}^{K}) t^k = \frac{t^2 + t^4 + t^6 + t^7 + t^{11} + t^{14} + t^{15}}{(1-t^2)(1-t^5)(1-t^6)}. \]

These eight forms $F_k$ are obtained explicitly by using Hilbert modular forms $G_2, G_5, G_6, G_{15}$ and differential operators (see subsection 2.5).

To show this theorem we need the following structure theorem of Jacobi forms of matrix index. We denote by $J_{k,12}$ the space of Jacobi forms of index $(\frac{1}{2}, \frac{1}{2})$ (cf. about the definition of Jacobi forms of matrix index, see Ziegler [4] page 193). We put $J_{*,12} := \bigoplus_{k \in \mathbb{Z}} J_{k,12}$, and $M_* := \bigoplus_{k \in \mathbb{Z}} M_k$, where $M_k$ is the space of elliptic modular forms of weight $k$ with respect to $SL(2, \mathbb{Z})$.

Theorem 1.3. The space $J_{*,12}$ is a free $M_*$-module with rank 4 and $\{\psi_4, \psi_6, \psi_8, \psi_{10}\}$ is a basis of $J_{*,12}$, and the dimension formula is given by
\[ \sum_{k \in \mathbb{N}} \dim(J_{k,12}) t^k = \frac{t^4 + t^6 + t^8 + t^{10}}{(1-t^4)(1-t^6)}, \]
where the forms $\psi_k \in J_{k,12}$ ($k = 4, 6, 8, 10$) are given in subsection 2.4.

2 Construction of Jacobi forms

In this section, we explain a construction of Hilbert-Jacobi forms from pair of Hilbert modular forms. The original idea of this construction in the case of usual Jacobi forms was given by N.-P.Skoruppa [3]. We shall also explain in this section the idea of the proof for Theorem 1.2.

2.1 Wronskian

In this subsection and the next subsection, we explain a construction of Hilbert-Jacobi forms from pairs of Hilbert modular forms for arbitrary totally real field $K$ and for arbitrary index $m$.

Let $\phi \in J_{k,m}^{K}$. We take the theta expansion:
\[ \phi(\tau, z) := \sum_{\alpha \in \Theta^{-1}/2mO} f_{\alpha}(\tau) \theta_{m,\alpha}(\tau, z), \]
where
\[
\theta_{m_{r}\alpha}(	au, z) = r \equiv \alpha(2mO) \sum_{r \in \theta^{-1}} e\left(\frac{1}{4m}r^{2}\tau + rz\right).
\]

Let
\[
l := |\mathfrak{d}^{-1}/2m\mathcal{O}| = N(2m)D_{K}
\]
where \(D_{K}\) is the discriminant of \(K\). We put
\[
\theta(	au, z) := (\theta_{m_{0}\alpha_{0}}(\tau, z), \ldots, \theta_{m_{l-1}\alpha_{l-1}}(\tau, z)),
\]
where \(\tau = (\tau_{1}, \ldots, \tau_{n}) \in \mathbb{H}^{n}, z = (z_{1}, \ldots, z_{n}) \in \mathbb{C}^{n}\), and where \((\alpha_{0}, \ldots, \alpha_{l-1})\) is a complete set of the representatives of \(\mathfrak{d}^{-1}/2m\mathcal{O}\).

For \(u = (u_{0}, \ldots, u_{l-1}) \in (\mathbb{N}^{n})^{l}\), we set
\[
W(\tau) := W_{u}(\tau) := \left(\begin{array}{l}
\partial_{z^{O}}^{u_{0}}\theta|_{z=0} \\
\vdots \\
\partial_{z^{u_{l-1}}}\theta|_{z=0}
\end{array}\right),
\]
where we defined \(\partial_{z}^{u} := \partial_{z_{1}}^{u_{1}} \cdots \partial_{z_{n}}^{u_{n}}\) for \(u_{i} = (u_{i,1}, \ldots, u_{i,n}) \in \mathbb{N}^{n}\), and \(\partial_{z_{i}} := \frac{1}{2\pi i} \delta_{i}\).

If \(u\) satisfies the following condition \([Cu]\), then \(\det(W)\) is a Hilbert modular form of weight \((l/2, \ldots, l/2) + \sum_{i=0}^{l-1} u_{i}\) with a certain character.

\([Cu]\) If \(v = (v_{1}, \ldots, v_{n}) \in \mathbb{N}^{n}\) satisfies \(v \leq u_{j}, v \equiv u_{j}\) mod 2 with a \(j \in \{0, \ldots, l-1\}\), then \(v \in \{u_{0}, \ldots, u_{l-1}\}\). Here \(v \leq u_{j}\) means \(v_{i} \leq u_{j,i}\) for any \(i \in \{1, \ldots, n\}\).

2.2 Construction of Hilbert Jacobi forms

Let \(\phi \in J_{k,m}^{K}\). We have
\[
\phi(\tau, z) = \sum_{i=0}^{l-1} f_{\alpha_{i}}(\tau) \theta_{m_{\alpha_{i}}}(\tau, z) = \sum_{\nu \in \mathbb{N}^{n}} g_{\nu}(\tau) \frac{(2\pi i)^{\nu}z^{\nu}}{\nu!},
\]
where \(\nu! := \prod_{j=1}^{n} \nu_{j}!\), \(\nu = (\nu_{1}, \ldots, \nu_{n})\), and \(g_{\nu}(\tau) = \partial_{z}^{\nu}\phi|_{z=0} = \sum_{i=0}^{l-1} f_{\alpha_{i}}(\tau) (\partial_{z}^{\nu}\theta_{m_{\alpha_{i}}})|_{z=0}\).

Thus for \(u = (u_{0}, \ldots, u_{l-1}) \in (\mathbb{N}^{n})^{l}\) we have
\[
^{t}(g_{u_{0}}(\tau), \ldots, g_{u_{l-1}}(\tau)) = W(\tau)^{t}(f_{\alpha_{0}}(\tau), \ldots, f_{\alpha_{l-1}}(\tau)).
\]

Now \((g_{u_{0}}, \ldots, g_{u_{l-1}})\) satisfies a certain transformation formula, so there exists a pair of Hilbert modular forms \((G_{u_{0}}, \ldots, G_{u_{l-1}}) \in M_{k+u_{0}}^{K} \times \cdots \times M_{k+u_{l-1}}^{K}\) such that
\[
^{t}(g_{u_{0}}(\tau), \ldots, g_{u_{l-1}}(\tau)) = D^{t}(G_{u_{0}}(\tau), \ldots, G_{u_{l-1}}(\tau)),
\]
where $D$ is a certain matrix of differential operators depending only on $k$ and $u$. Hence if $\det(W)$ is not identically zero, then
\[
\phi = \theta^t(f_{\alpha_0}, \ldots, f_{\alpha_{i-1}}) = \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{i-1}})).
\]
On the other hand, for any pair of Hilbert modular forms $(G_{u_0}, \ldots, G_{u_{i-1}}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{i-1}}^K$, by using the above identity, we can construct a meromorphic function on $\mathbb{H}^n \times \mathbb{C}^n$ which satisfies the transformation formula of Hilbert Jacobi forms (conditions (i), (ii) of the definition 1.) We denote this map by $\tilde{\lambda}_k :$
\[
\tilde{\lambda}_k : M_{k+u_0}^K \times \cdots \times M_{k+u_{i-1}}^K \rightarrow J_{k,m,\text{mero}}
\]
via
\[
\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{i-1}}) := \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_{i-1}})).
\]
Thus for constructing Hilbert-Jacobi forms in general, we need to know when $\det(W)$ is not identically zero, and when $\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_{i-1}})$ is holomorphic.

2.3 Example $K = \mathbb{Q}(\sqrt{5}), m = (5 + \sqrt{5})/10$

We fix $K = \mathbb{Q}(\sqrt{5})$, and $m = (5 + \sqrt{5})/10$. In this subsection we give explicitly the matrix $D$ and construct Hilbert-Jacobi forms of index $m$.

By straightforward calculation we obtain $\mathfrak{d}^{-1} = m\mathcal{O}$, $\mathfrak{d}^{-1}/2m\mathcal{O} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $|\mathfrak{d}^{-1}/2m\mathcal{O}| = 4$.

We put $u := (u_0, u_1, u_2, u_3) \in (\mathbb{N}^2)^4$, where $u_0 := (0, 0), u_1 := (0, 2), u_2 := (2, 0)$ and $u_3 := (1, 1)$. Then, $\det(W) = c \cdot G_5$ with non zero constant $c$. Here $G_5$ is the Hilbert modular form of weight $(5, 5)$ denoted in Theorem 1.1.

Let $k = (k_1, k_2) \in \mathbb{N}^2$. For $(G_{u_0}, \ldots, G_{u_3}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{i-1}}^K$, we put
\[
\tilde{\lambda}_k(G_{u_0}, \ldots, G_{u_3}) := \phi := \theta W^{-1}(D^t(G_{u_0}, \ldots, G_{u_3})),
\]
where $D := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$, and $m'$ is the Galois conjugation of $m$. Due to the consideration of the previous subsection we have $\phi \in J_{k,m,\text{mero}}^K$.

We denote by $J_{l,12}$ the space of Jacobi forms of weight $l \in \mathbb{N}$ of index $12 = (1, 0)$. Now, for $k = (k_1, k_1) \in \mathbb{N}^2$ we consider the following map
\[
\mathbb{D} : J_{k,m}^K \rightarrow J_{2k_1,12}
\]
\[
\mathbb{D}(\phi)(\tau, (z_1, z_2)) := \phi((\tau, \tau), (z_1, z_2) \cdot V),
\]
where \( \phi \in J_{k,m}^{K} \), \((z_1, z_2) \in \mathbb{C}^2, \tau \in \mathfrak{H}, V = (\epsilon_1^{-1}, \epsilon_1^{-1})\), \(\epsilon = (1 + \sqrt{5})/2\), and \(\epsilon' = (1 - \sqrt{5})/2\).

### 2.4 The space of Jacobi forms of index 1_2

As for the structure of the space of Hilbert modular forms of index 1_2, we have the following theorem.

**Theorem 2.1.** For any \(k' \in \mathbb{Z}\), we have \(J_{k',1_2} \cong M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k'+4}\), where \(M_{k'}\) (resp. \(S_{k'}\)) is the space of elliptic modular forms (resp. cusp forms) of weight \(k'\) with respect to \(SL_2(\mathbb{Z})\).

The idea of the proof of the above theorem is as follows. By similar method as in the subsection 2.2, we have a similar map as \(\tilde{\lambda}_k\) in the subsection 2.2 for the space of Jacobi forms of index 1_2. We can construct meromorphic Jacobi forms of index 1_2.

The idea of the proof of Theorem 1.3.

Due to Theorem 2.1, we have the dimension formula for \(\bigoplus_{k' \in \mathbb{Z}} J_{k',1_2}\), and we obtain Theorem 1.3 by constructing suitable basis of the space of Jacobi forms of index 1_2 as \(\bigoplus_{k \in \mathbb{Z}} M_k\)-module.

The basis of \(\bigoplus_{k' \in \mathbb{Z}} J_{k',1_2}\) is given by the following four forms: 
\[
\psi_4 := \tilde{\nu}_4((E_{4},0,0,0)) \in J_{4,1_2}, \quad \psi_6 := \tilde{\nu}_6((E_{6},0,0,0)) \in J_{6,1_2}, \quad \psi_{10} := \tilde{\nu}_{10}((0,0,\Delta,0)) \in J_{10,1_2}, \quad \text{and} \quad \psi_8 := \tilde{\nu}_8((0,0,0,\Delta)) \in J_{8,1_2}.
\]
Here \(\tilde{\nu}_{k'}\) is the map from \(M_{k'} \times S_{k'+2} \times S_{k'+2} \times S_{k'+4}\) to \(J_{k',1_2}\), and \(E_{k'}\) are the Eisenstein series of weight \(k'\).

### 2.5 The space of Hilbert-Jacobi forms of index \(m\)

Let \(k = (k_1, k_1) \in \mathbb{N}^2\). We put 
\[
\tilde{J}_{k,m}^{K} := \tilde{\lambda}_k(M_{(k_1,k_1)}^{K} \times S_{(k_1,k_1+2)}^{K} \times S_{(k_1+2,k_1)}^{K} \times M_{(k_1+1,k_1+1)}^{K}),
\]
where \(S_{(k_1,k_2)}^{K}\) is the space of Hilbert cusp forms of weight \((k_1, k_2)\) with respect to \(SL(2,\mathcal{O})\).
As for the space of Hilbert cusp forms $S^K_{(k_1,k_1+2)}$ the following theorem is known by H. Aoki [1].

**Theorem 2.2 (Aoki).** The structure of $\bigoplus_{k_1 \in \mathbb{Z}} S^K_{(k_1,k_1+2)}$ is given by

\[ \bigoplus_{k_1 \in \mathbb{Z}} S^K_{(k_1,k_1+2)} = A_{7,9}B + A_{8,10}B + A_{11,13}B, \]

where $A_{7,9} := [G_2, G_5] := 2G_2(\partial_\tau G_5) - 5G_5(\partial_\tau G_2)$, $A_{8,10} := [G_6, G_2]$, $A_{11,13} := [G_5, G_6]$ and $B = \mathbb{C}[G_2, G_5, G_6]$. Here $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$ satisfy the following Jacobi identity:

\[ 6 \, G_6 A_{7,9} + 5 \, G_5 A_{8,10} + 2 \, G_2 A_{11,13} = 0. \]

Except this identity, there are no relation among $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$.

To show theorem 1.2, we need the following proposition.

**Proposition 2.3.** Let $\phi \in J^K_{k,m}$. Then $D(\phi) = 0$ if and only if $G_5 | \phi$.

Thus we have the following short exact sequence:

\[ 0 \rightarrow J^K_{k,m} \rightarrow \hat{J}^K_{k,m} \rightarrow J_{2k_1+10,12}, \]

where the second map is the embedding, and the last map is given via $\phi$ to $D(G_5 \cdot \phi)$ for $\phi \in \hat{J}^K_{k,m}$.

By using theorem 1.1 and theorem 2.2 we can calculate the dimension of $\hat{J}^K_{k,m}$, and also we have the dimension of the image of the above last map. Hence, we have the dimension formula for $\dim(J^K_{(k,k),m})$ written in Theorem 1.2. The basis of $\bigoplus_{k \in \mathbb{Z}} J^K_{(k,k),m}$ as $\mathbb{C}[G_2, G_5, G_6]$-module is given as follows:

\[
\begin{align*}
F_2 &:= \tilde{\lambda}_2(G_2, 0, 0, 0), \\
F_4 &:= \tilde{\lambda}_4(0, 0, 0, G_5), \\
F_5 &:= \tilde{\lambda}_5 \left( G_5, 0, 0, \frac{2}{5 \sqrt{5}} G_6 \right), \\
F_6 &:= \tilde{\lambda}_6(G_6, 0, 0, 0), \\
F_7 &:= \tilde{\lambda}_7(0, mA'_{7,9}, -m' A_{7,9}, 0), \\
F_{11} &:= \tilde{\lambda}_{11}(0, mA'_{11,13}, -m' A_{11,13}, 0), \\
F_{14} &:= \tilde{\lambda}_{14} \left( 0, \frac{\epsilon}{2} A'_{8,10}, \frac{\epsilon'}{2} A_{8,10}, G_{15} \right), \\
F_{15} &:= \tilde{\lambda}_{15}(G_{15}, 0, 0, 0),
\end{align*}
\]

where $A'_{7,9} := 2G_2(\partial_\tau G_5) - 5G_5(\partial_\tau G_2)$, $A'_{8,10} := 6G_6(\partial_\tau G_2) - 2G_2(\partial_\tau G_6)$ and $A'_{11,13} := 5G_5(\partial_\tau G_6) - 6G_6(\partial_\tau G_5)$. 
References


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