Hilbert-Jacobi forms of a certain index of $\mathbb{Q} (\sqrt{5})$

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0 Introduction

The purpose of this survey is to give an example of a structure theorem of the space of Hilbert-Jacobi forms of a certain index with concerning to $K = \mathbb{Q} (\sqrt{5})$ (Theorem 1.2). We give also an example of a structure theorem of the space of Jacobi forms of a matrix index (Theorem 1.3). We used theorem 1.3 to show theorem 1.2.

1 Main theorem

In this section, we recall the definition of Hilbert-Jacobi forms, and give an example of a structure theorem of the space of Hilbert-Jacobi forms and of Jacobi forms of a matrix index.

1.1 Notations

Let $K$ be a totally real field with degree $n$, let $\mathfrak{d}^{-1}$ be the inverse of the different, and let $\mathcal{O}$ be the principal order of $K$. We denote by $\mathcal{F}$ the Poincaré upper half plane.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $e(z) := e^{2\pi i \text{tr}(z)}$, where $\text{tr}(z) = z_1 + \cdots + z_n$. By abuse of language, we set $z^k := \prod_{i=1}^{n} z_i^{k_i}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$.

1.2 Definition

For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and for totally positive number $m \in \mathfrak{d}^{-1}$, we define Hilbert-Jacobi forms of weight $k$ of index $m$ as follows.

**Definition 1.** Let $\phi$ be a holomorphic function on $\mathcal{F}^n \times \mathbb{C}^n$. We say $\phi$ is a Hilbert-Jacobi form of weight $k$ of index $m$ if $\phi$ satisfies the following three conditions.
(i) For any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathcal{O})$, any $\tau = (\tau_1, ..., \tau_n) \in \mathfrak{h}^n$ and any $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $\phi$ satisfies
\[
\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e(m(c\tau + d)^{-1}cz^2)(c\tau + d)^k\phi(\tau, z).
\]

(ii) For any $\lambda, \mu \in \mathcal{O}$,
\[
\phi(\tau, z + \lambda\tau + \mu) = e(-m\lambda^2\tau - 2m\lambda z)\phi(\tau, z).
\]

(iii) $\phi$ has the Fourier expansion:
\[
\phi(\tau, z) = \sum_{u, r \in \mathfrak{o}^{-1}} c(u, r) e(ut + rz),
\]
where in the above summation $u$ and $r$ run over all elements in $\mathfrak{o}^{-1}$ such that $4um - r^2$ is totally positive or equals to 0.

When $n$ is larger than 1, then because of Koecher principle the third condition of the definition follows automatically by the first and second conditions.

We denote by $J_{k, m}^K$ the space of Hilbert-Jacobi forms of weight $k$ of index $m$ with respect to $SL(2, \mathcal{O})$.

1.3 Results

We consider the case $K = \mathbb{Q}(\sqrt{5})$, $m = \epsilon/\sqrt{5}$, where $\epsilon = \frac{1 + \sqrt{5}}{2}$ is the fundamental unit of the maximal order $\mathcal{O} = \mathbb{Z}[\epsilon]$ of $K$.

Let $k \in \mathbb{N}$. Now $M_{(k_1, k_2)}^K$ denotes the space of Hilbert modular forms of weight $(k_1, k_2) \in \mathbb{Z}^2$ with respect to $SL(2, \mathcal{O})$. We quote the following structure theorem of the space of Hilbert modular forms obtained by Gundlach [2].

Theorem 1.1 (Gundlach[2]).
\[
\bigoplus_{k \in \mathbb{Z}} M_{(k,k)}^K = C[G_2, G_5, G_6] \oplus G_{15}C[G_2, G_5, G_6],
\]
where $G_2$, $G_5$, $G_6$ and $G_{15}$ are Hilbert modular forms of weight 2, 5, 6 and 15, respectively. There exists a polynomial $P(X_1, X_2, X_3)$ such that $G_{15}^2 = P(G_2, G_5, G_6)$.

The main theorem of this report is as follows.
Theorem 1.2. The space $\bigoplus_{k\in\mathbb{Z}} J_{(k,k),m}^{K}$ is a $\mathbb{C}[G_{2}, G_{5}, G_{6}]$-module generated by eight forms $F_{k} \in J_{(k,k),m}^{K}$ ($k = 2, 4, 5, 6, 7, 11, 14, 15$), and the dimension formula is given by

$$\sum_{k\in\mathbb{Z}} \dim(J_{(k,k),m}^{K}) t^{k} = \frac{t^{2} + t^{4} + t^{5} + t^{6} + t^{7} + t^{11} + t^{14} + t^{15}}{(1-t^{2})(1-t^{5})(1-t^{6})}.$$ 

These eight forms $F_{k}$ are obtained explicitly by using Hilbert modular forms $G_{2}, G_{5}, G_{6}, G_{15}$ and differential operators (see subsection 2.5).

To show this theorem we need the following structure theorem of Jacobi forms of matrix index. We denote by $J_{k,12}$ the space of Jacobi forms of index $\left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ (cf. about the definition of Jacobi forms of matrix index, see Ziegler [4] page 193). We put $J_{*,12} := \bigoplus_{k\in\mathbb{Z}} J_{k,12}$, and $M_{*} := \bigoplus_{k\in\mathbb{Z}} M_{k}$, where $M_{k}$ is the space of elliptic modular forms of weight $k$ with respect to $SL(2, \mathbb{Z})$.

Theorem 1.3. The space $J_{*,12}$ is a free $M_{*}$-module with rank 4 and $\{\psi_{4}, \psi_{6}, \psi_{8}, \psi_{10}\}$ is a basis of $J_{*,12}$, and the dimension formula is given by

$$\sum_{k\in\mathbb{N}} \dim(J_{k,12}) t^{k} = \frac{t^{4} + t^{6} + t^{8} + t^{10}}{(1-t^{4})(1-t^{6})},$$

where the forms $\psi_{k} \in J_{k,12}$ ($k = 4, 6, 8, 10$) are given in subsection 2.4.

2 Construction of Jacobi forms

In this section, we explain a construction of Hilbert-Jacobi forms from pair of Hilbert modular forms. The original idea of this construction in the case of usual Jacobi forms was given by N.-P.Skoruppa [3]. We shall also explain in this section the idea of the proof for Theorem 1.2.

2.1 Wronskian

In this subsection and the next subsection, we explain a construction of Hilbert-Jacobi forms from pairs of Hilbert modular forms for arbitrary totally real field $K$ and for arbitrary index $m$.

Let $\phi \in J_{k,m}^{K}$. We take the theta expansion:

$$\phi(\tau, z) := \sum_{\alpha \in \Theta^{-1}/2mO} f_{\alpha}(\tau) \vartheta_{m,\alpha}(\tau, z),$$
where \( \vartheta_{m, \alpha}(\tau, z) = \sum_{r \equiv \alpha(2m) \mod{2m\mathcal{O}}} e\left(\frac{1}{4m}r^2 \tau + rz\right) \).

Let \( l := |\mathfrak{d}^{-1}/2m\mathcal{O}| = N(2m)D_K \), where \( D_K \) is the discriminant of \( K \). We put \( \theta(\tau, z) := (\vartheta_{m, \alpha_0}(\tau, z), \ldots, \vartheta_{m, \alpha_{l-1}}(\tau, z)) \), where \( \tau = (\tau_1, \ldots, \tau_n) \in \mathcal{H}^n \), \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), and where \( (\alpha_0, \ldots, \alpha_{l-1}) \) is a complete set of the representatives of \( \mathfrak{d}^{-1}/2m\mathcal{O} \).

For \( u = (u_0, \ldots, u_{l-1}) \in (\mathbb{N}^n)^l \), we set
\[
W(\tau) := W_u(\tau) := \left( \begin{array}{c}
\partial_{z^0} u \theta|_{z=0} \\
\vdots \\
\partial_{z^{l-1}} u \theta|_{z=0}
\end{array} \right),
\]
where we defined \( \partial_{z^i} := \partial_{z_{i,1}} \cdots \partial_{z_{i,n}} \) for \( u_i = (u_{i,1}, \ldots, u_{i,n}) \in \mathbb{N}^n \), and \( \partial_{z_i} := \frac{1}{2\pi i} \delta_{z_i} \).

If \( u \) satisfies the following condition \([Cu]\), then \( \det(W) \) is a Hilbert modular form of weight \((l/2, \ldots, l/2) + \sum_{i=0}^{l-1} u_i \) with a certain character.

\([Cu\) If \( v = (v_1, \ldots, v_n) \in \mathbb{N}^n \) satisfies \( v \leq u_j, v \equiv u_j \mod{2} \) with a \( j \in \{0, \ldots, l-1\} \), then \( v \in \{u_0, \ldots, u_{l-1}\} \). Here \( v \leq u_j \) means \( v_i \leq u_{j,i} \) for any \( i \in \{1, \ldots, n\} \).

### 2.2 Construction of Hilbert Jacobi forms

Let \( \phi \in J_{k, m}^{K} \). We have
\[
\phi(\tau, z) = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau) \vartheta_{m, \alpha_i}(\tau, z) = \sum_{\nu \in \mathbb{N}^n} g_{\nu}(\tau) \frac{(2\pi i)^\nu z^\nu}{\nu!},
\]
where \( \nu! := \prod_{j=1}^n \nu_j! \), \( \nu = (\nu_1, \ldots, \nu_n) \), and \( g_{\nu}(\tau) = \partial_{z}^{\nu} \phi|_{z=0} = \sum_{i=0}^{l-1} f_{\alpha_i}(\tau)(\partial_{z}^{\nu} \vartheta_{m, \alpha_i})|_{z=0} \).

Thus for \( u = (u_0, \ldots, u_{l-1}) \in (\mathbb{N}^n)^l \) we have
\[
\iota^t(g_{u_0}(\tau), \ldots, g_{u_{l-1}}(\tau)) = W(\tau)^t(f_{\alpha_0}(\tau), \ldots, f_{\alpha_{l-1}}(\tau)).
\]

Now \( (g_{u_0}, \ldots, g_{u_{l-1}}) \) satisfies a certain transformation formula, so there exists a pair of Hilbert modular forms \( (G_{u_0}, \ldots, G_{u_{l-1}}) \in M_{k+u_0}^{K} \times \ldots \times M_{k+u_{l-1}}^{K} \) such that
\[
\iota^t(g_{u_0}(\tau), \ldots, g_{u_{l-1}}(\tau)) = D^t(G_{u_0}(\tau), \ldots, G_{u_{l-1}}(\tau)),
\]
where $D$ is a certain matrix of differential operators depending only on $k$ and $u$. Hence if $\det(W)$ is not identically zero, then

$$\phi = \theta^t(f_{a_0}, ..., f_{a_{l-1}}) = \theta W^{-1}(D^t(G_{u_0}, ..., G_{u_{l-1}})).$$

On the other hand, for any pair of Hilbert modular forms $(G_{u_0}, ..., G_{u_{l-1}}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K$, by using the above identity, we can construct a meromorphic function on $\mathfrak{H}^n \times \mathbb{C}^n$ which satisfies the transformation formula of Hilbert Jacobi forms (conditions (i), (ii) of the definition 1.) We denote this map by $\tilde{\lambda}_k$:

$$\tilde{\lambda}_k : M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K \rightarrow J_{k,m}^{K,mero}$$

via

$$\tilde{\lambda}_k(G_{u_0}, ..., G_{u_{l-1}}) := \theta W^{-1}(D^t(G_{u_0}, ..., G_{u_{l-1}})).$$

Thus for constructing Hilbert-Jacobi forms in general, we need to know when $\det(W)$ is not identically zero, and when $\tilde{\lambda}_k(G_{u_0}, ..., G_{u_{l-1}})$ is holomorphic.

2.3 Example $K = \mathbb{Q}(\sqrt{5}), m = (5 + \sqrt{5})/10$

We fix $K = \mathbb{Q}(\sqrt{5})$, and $m = (5 + \sqrt{5})/10$. In this subsection we give explicitly the matrix $D$ and construct Hilbert-Jacobi forms of index $m$.

By straightforward calculation we obtain $\mathfrak{d}^{-1} = m\mathcal{O}, \mathfrak{d}^{-1}/2m\mathcal{O} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $|\mathfrak{d}^{-1}/2m\mathcal{O}| = 4$.

We put $u := (u_0, u_1, u_2, u_3) \in (\mathbb{N}^2)^4$, where $u_0 := (0, 0), u_1 := (0, 2), u_2 := (2, 0)$ and $u_3 := (1, 1)$. Then, $\det(W) = c \cdot G_5$ with non zero constant $c$. Here $G_5$ is the Hilbert modular form of weight $(5, 5)$ denoted in Theorem 1.1.

Let $k = (k_1, k_2) \in \mathbb{N}^2$. For $(G_{u_0}, ..., G_{u_3}) \in M_{k+u_0}^K \times \cdots \times M_{k+u_{l-1}}^K$, we put

$$\tilde{\lambda}_k(G_{u_0}, ..., G_{u_{3}}) := \phi := \theta W^{-1}(D^t(G_{u_0}, ..., G_{u_{3}})),$$

where $D := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2m_{k_1} \partial_{\tau_2} & 1 & 0 & 0 \\ 2m'_{k_2} \partial_{\tau_1} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and $m'$ is the Galois conjugation of $m$. Due to the consideration of the previous subsection we have $\phi \in J_{k,m}^{K,mero}$.

We denote by $J_{l,12}$ the space of Jacobi forms of weight $l \in \mathbb{N}$ of index $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Now, for $k = (k_1, k_1) \in \mathbb{N}^2$ we consider the following map

$$\mathbb{D} : J_{k,m}^{K} \rightarrow J_{2k,12}$$
\[ \mathbb{D}(\phi) \tau, (z_1, z_2) := \phi(\tau, \tau, (z_1, z_2) \cdot V), \]

where \( \phi \in J_{k,m}^K \), \((z_1, z_2) \in \mathbb{C}^2\), \( \tau \in \mathfrak{H} \), \( V = (\epsilon \epsilon^{-1}, 1) \), \( \epsilon = (1 + \sqrt{5})/2 \) and \( \epsilon' = (1 - \sqrt{5})/2 \).

### 2.4 The space of Jacobi forms of index 1_2

As for the structure of the space of Hilbert modular forms of index 1_2, we have the following theorem.

**Theorem 2.1.** For any \( k' \in \mathbb{Z} \), we have \( J_{k',1_2} \cong M_{k'} \times S_{k'+2} \times S_{k'+4} \), where \( M_{k'} \) (resp. \( S_{k'} \)) is the space of elliptic modular forms (resp. cusp forms) of weight \( k' \) with respect to \( SL_2(\mathbb{Z}) \).

The idea of the proof of the above theorem is as follows. By similar method as in the subsection 2.2, we have a similar map as \( \tilde{\lambda}_k \) in the subsection 2.2 for the space of Jacobi forms of index 1_2. We can construct meromorphic Jacobi forms of index 1_2. In this case, by choosing a suitable \( u \in (\mathbb{N}^2)^4 \), the Wronskian is the Ramanujan-\( \Delta \) function. Hence we can check when the image of the map \( \hat{\nu}_{k'} \), which corresponds to \( \tilde{\lambda}_k \) in the case of Hilbert Jacobi forms, is holomorphic. The surjectivity of the map \( \hat{\nu}_{k'} \) follows from this fact. Thus we obtain theorem 2.1.

**The idea for the proof of Theorem 1.3.**

Due to Theorem 2.1, we have the dimension formula for \( \bigoplus_{k' \in \mathbb{Z}} J_{k',1_2} \), and we obtain Theorem 1.3 by constructing suitable basis of the space of Jacobi forms of index 1_2 as \( \bigoplus_{k \in \mathbb{Z}} M_k \)-module.

The basis of \( \bigoplus_{k' \in \mathbb{Z}} J_{k',1_2} \) is give by the following four forms: \( \psi_4 := \hat{\nu}_4((E_4, 0, 0, 0)) \in J_{4,1_2}, \psi_6 := \hat{\nu}_6((E_6, 0, 0, 0)) \in J_{6,1_2}, \psi_{10} := \hat{\nu}_{10}((0, 0, \Delta, 0)) \in J_{10,1_2}, \) and \( \psi_8 := \hat{\nu}_8((0, 0, 0, \Delta)) \in J_{8,1_2} \). Here \( \hat{\nu}_{k'} \) is the map from \( M_{k'} \times S_{k'+2} \times S_{k'+4} \) to \( J_{k',1_2} \), and \( E_{k'} \) are the Eisenstein series of weight \( k' \).

### 2.5 The space of Hilbert-Jacobi forms of index m

Let \( k = (k_1, k_1) \in \mathbb{N}^2 \). We put

\[ \tilde{J}^K_{k,m} := \tilde{x}_k(M^K_{(k_1,k_1)} \times S^K_{(k_1,k_1+2)} \times S^K_{(k_1+2,k_1)} \times M^K_{(k_1+1,k_1+1)}), \]

where \( S^K_{(k_1,k_2)} \) is the space of Hilbert cusp forms of weight \( (k_1, k_2) \) with respect to \( SL(2, \mathcal{O}) \).
As for the space of Hilbert cusp forms $S_{(k_{1}, k_{1}+2)}^{K}$ the following theorem is known by H.Aoki [1].

**Theorem 2.2 (Aoki).** The structure of $\bigoplus_{k_{1} \in \mathbb{Z}} S_{(k_{1}, k_{1}+2)}^{K}$ is given by

$$\bigoplus_{k_{1} \in \mathbb{Z}} S_{(k_{1}, k_{1}+2)}^{K} = A_{7,9}B + A_{8,10}B + A_{11,13}B,$$

where $A_{7,9} := [G_{2}, G_{5}] := 2G_{2}(\partial_{\tau}G_{5}) - 5G_{5}(\partial_{\tau_{2}}G_{2})$, $A_{8,10} := [G_{6}, G_{2}]$, $A_{11,13} := [G_{5}, G_{6}]$ and $B = \mathbb{C}[G_{2}, G_{5}, G_{6}]$. Here $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$ satisfy the following Jacobi identity: $6G_{6}A_{7,9} + 5G_{5}A_{8,10} + 2G_{2}A_{11,13} = 0$. Except this identity, there are no relation among $A_{7,9}$, $A_{8,10}$ and $A_{11,13}$.

To show theorem 1.2, we need the following proposition.

**Proposition 2.3.** Let $\phi \in J_{k,m}^{K}$. Then $D(\phi) = 0$ if and only if $G_{5}|\phi$.

Thus we have the following short exact sequence:

$$0 \rightarrow J_{k,m}^{K} \rightarrow \hat{J}_{k,m}^{K} \rightarrow J_{2k_{1}+10,12}^{K},$$

where the second map is the embedding, and the last map is given via $\phi$ to $D(G_{5} \cdot \phi)$ for $\phi \in \hat{J}_{k,m}^{K}$.

By using theorem 1.1 and theorem 2.2 we can calculate the dimension of $\hat{J}_{k,m}^{K}$, and also we have the dimension of the image of the above last map. Hence, we have the dimension formula for $\dim(J_{(k_{2}, k)_{J}m}^{K})$ written in Theorem 1.2. The basis of $\bigoplus_{k \in \mathbb{Z}} J_{(k_{2}, k),m}^{K}$ as $\mathbb{C}[G_{2}, G_{5}, G_{6}]$-module is given as follows:

$$F_{2} := \tilde{\lambda}_{2}(G_{2}, 0, 0, 0), \quad F_{4} := \tilde{\lambda}_{4}(0, 0, 0, G_{5}),$$
$$F_{5} := \tilde{\lambda}_{5}(G_{5}, 0, 0, \frac{2}{5\sqrt{5}}G_{6}), \quad F_{6} := \tilde{\lambda}_{6}(G_{6}, 0, 0, 0),$$
$$F_{7} := \tilde{\lambda}_{7}(0, mA'_{7,9}, -m'A_{7,9}, 0), \quad F_{11} := \tilde{\lambda}_{11}(0, mA'_{11,13}, -m'A_{11,13}, 0),$$
$$F_{14} := \tilde{\lambda}_{14}(0, \frac{\epsilon'}{2}A'_{8,10}, \frac{\epsilon}{2}A_{8,10}, G_{15}), \quad F_{15} := \tilde{\lambda}_{15}(G_{15}, 0, 0, 0),$$

where $A'_{7,9} := 2G_{2}(\partial_{\tau}G_{5}) - 5G_{5}(\partial_{\tau_{2}}G_{2})$, $A'_{8,10} := 6G_{6}(\partial_{\tau}G_{2}) - 2G_{2}(\partial_{\tau}G_{6})$ and $A'_{11,13} := 5G_{5}(\partial_{\tau}G_{6}) - 6G_{6}(\partial_{\tau}G_{5})$. 
References


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