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Representations of Clifford algebras and quartic polynomials with local functional equations

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Introduction

Let P and P^* be homogeneous polynomials in n variables of degree d with real coefficients. It is an interesting problem both in Analysis and in Number theory to find a condition on P and P^* under which they satisfy a functional equation, roughly speaking, of the form

$$\text{the Fourier transform of } |P(x)|^s = \text{Gamma factor} \times |P^*(y)|^{-n/d-s}. \tag{1}$$

A beautiful answer to this problem is given by the theory of prehomogeneous vector spaces due to Mikio Sato. Namely, if P and P^* are relative invariants of a regular prehomogeneous vector space and its dual, respectively, and if the characters χ and χ^* corresponding to P and P^*, respectively, satisfy the relation $\chi\chi^* = 1$, then, P and P^* satisfy a functional equation (see [8], [9], [5]). The theory works quite satisfactorily and it might give an impression that prehomogeneous vector spaces are the final answer to the problem.

Meanwhile, in [4], Faraut and Koranyi developed a method of constructing polynomials with the property (1), starting from representations of Euclidean (formally real) Jordan algebras. What is remarkable in their result is that, from representations of simple Jordan algebras of rank 2, one can obtain a series of polynomials satisfying (1), which are not covered by the theory of prehomogeneous vector spaces. Their result was later generalized by Clerc [3]. Thus we got to know that the class of polynomials with the property (1) is broader than the class of relative invariants of regular prehomogeneous vector spaces.

In this note, first we give a new construction of polynomials with the property (1), which includes the result of Faraut, Koranyi and Clerc as a special case.

The result may be outlined as follows: Suppose that we are given homogeneous polynomials P and P^* on a real vector spaces V and its dual V^*, respectively, satisfying a functional equation of the form (1). Further suppose that there exists a nondegenerate quadratic mapping Q (resp. Q^*) of another real vector space W (resp. W^*) to V (resp. V^*), and Q and Q^* are dual. Then, the polynomials $\tilde{P} = P \circ Q$ and $\tilde{P}^* = P^* \circ Q^*$ inherit the property (1) from P and P^* and the gamma factors for the new functional equation have an explicit expression in term of those for P and P^*. A precise formulation of this result will be given in Section 1. For the proof we refer to [6].

It is natural to ask whether global zeta functions with functional equations can be associated with polynomials $\tilde{P}$ and $\tilde{P}^*$ given in our result. For polynomials obtained from the theory of Faraut and Koranyi, this problem was solved by Achab in [1] and [2]. The problem is open in our general setting.

In Section 2, we apply the general result in Section 1 to the case where $V = V^* = \mathbb{R}^n$, and P and P^* are nondegenerate quadratic forms on V and V^* that are dual to each other. Then we can prove that non-degenerate dual quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$ correspond to representations of the tensor product of two Clifford algebras and, starting from representations of Clifford algebras, we can construct quartic polynomials satisfying functional equations of the form (1). Among these polynomials we find several new examples of polynomials satisfying functional equations that do not come from prehomogenous vector spaces. In this quartic case the explicit form of the functional equation for $\tilde{P} = P \circ Q$ and $\tilde{P}^* = P^* \circ Q^*$ can be obtained automatically from the 1-dimensional case (= the Iawasawa-Tate local functional equation over $\mathbb{R}$) by repeated applications of the general result above. The non-prehomogeneous polynomials with the property (1) appearing in the work of Faraut, Koranyi and Clerc is a special case where the signature of the quadratic forms P and P^* is $(1, n-1)$.

1 Quadratic mappings and functional equations

1.1 Local functional equations

Let V be a real vector space of dimension n and V^* the vector space dual to V. Let $P_1, \dots, P_r$ (resp. $P_1^*, \dots, P_r^*$) be $\mathbb{R}$-irreducible homogeneous polynomials on V (resp. V^*). We put

$$\Omega = \{v \in V \mid P_1(v) \cdots P_r(v) \neq 0\} \quad \text{and} \quad \Omega^* = \{v* \in V^* \mid P_1^*(v^*) \cdots P_r^*(v^*) \neq 0\}.$$

We assume that

(A.1) there exists a biregular rational mapping $\phi : \Omega \to \Omega^*$ defined over $\mathbb{R}$.

Let

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_\nu, \quad \Omega^* = \Omega_1^* \cup \cdots \cup \Omega_\nu^*$$

be the decomposition into connected components of Ω and Ω^*. Note that (A.1) implies that the numbers of connected components of Ω and Ω^* are the same and we may assume that

$$\Omega_j^* = \phi(\Omega_j) \quad (j = 1, \dots, \nu).$$

For an $s = (s_1, \dots, s_r) \in \mathbb{C}^r$ with $\Re(s_1), \dots, \Re(s_r) > 0$, we define a continuous function $|P(v)|_j^s$ on V by

$$|P(v)|_j^s = \begin{cases} \prod_{i=1}^r |P_i(v)|^{s_i}, & v \in \Omega_j, \\ 0, & v \notin \Omega_j. \end{cases}$$

The function $|P(v)|_j^s$ can be extended to a tempered distribution depending on s in $\mathbb{C}^r$ meromorphically. For an $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, we put

$$P^{\mathbf{m}}(v) = \prod_{i=1}^r P_i(v)^{m_i}.$$

Sometime we use the symbol $P^{\mathbf{m}}(v)$ for non-integral $\mathbf{m}$ (see the second identity in Lemma 1), which we may regard either as a symbolic expression on which differential operators operate in a usual manner or as a function on the universal covering space of Ω. Similarly we define $|P^*(v^*)|_j^s$ ($s \in \mathbb{C}^r$) and $P^{*\mathbf{m}}(v^*)$ ($\mathbf{m} \in \mathbb{Z}^r$).

We denote by $d(\mathbf{m})$ (resp. $d^*(\mathbf{m})$) the homogeneous degree of $P^{\mathbf{m}}$ (resp. $P^{*\mathbf{m}}$). We put

$$\epsilon_i(\mathbf{m}) = P^{\mathbf{m}}(v)/\,|P^{\mathbf{m}}(v)| \quad (v \in \Omega_i), \quad \epsilon_j^*(\mathbf{m}) = P^{*\mathbf{m}}(v^*)/\,|P^{*\mathbf{m}}(v^*)| \quad (v^* \in \Omega_j^*).$$

Since Ω_i and Ω_j^* are assumed to be connected, $\epsilon_i(\mathbf{m})$ and $\epsilon_j^*(\mathbf{m})$ do not depend on the choice of v and v^*.

We denote by $\mathcal{S}(V)$ and $\mathcal{S}(V^*)$ the spaces of rapidly decreasing functions on the real vector spaces V and V^*, respectively. For $\Phi \in \mathcal{S}(V)$ and $\Phi^* \in \mathcal{S}(V^*)$, we define the local zeta functions by setting

$$\zeta_i(s, \Phi) = \int_V |P(v)|_i^s \, \Phi(v)\, dv, \quad \zeta_i^*(s, \Phi^*) = \int_{V^*} |P^*(v^*)|_i^s \, \Phi^*(v^*)\, dv^* \quad (i = 1, \dots, \nu).$$

It is well-known that the local zeta functions $\zeta_i(s, \Phi)$, $\zeta_i^*(s, \Phi^*)$ are absolutely convergent for $\Re(s_1), \dots, \Re(s_r) > 0$ and have analytic continuations to meromorphic functions of s in $\mathbb{C}^r$. We assume the following:

(A.2) There exist an $A \in GL_r(\mathbb{Z})$ and a $\lambda \in \mathbb{C}^r$ such that a functional equation of the form

$$\zeta_i^*((s + \lambda)A, \hat{\Phi}) = \sum_{j=1}^{\nu} \Gamma_{ij}(s) \zeta_j(s, \Phi) \quad (i = 1, \dots, \nu) \tag{2}$$

holds for every $\Phi \in \mathcal{S}(V)$, where $\Gamma_{ij}(s)$ are meromorphic functions on $\mathbb{C}^r$ not depending on Φ with $\det(\Gamma_{ij}(s)) \neq 0$ and

$$\hat{\Phi}(v^*) = \int_V \Phi(v) \exp(-2\pi\sqrt{-1}\langle v, v^* \rangle)\, dv,$$

the Fourier transform of Φ.

A lot of examples of $\{P_1, \ldots, P_r\}$ and $\{P_1^*, \ldots, P_r^*\}$ satisfying (A.1) and (A.2) can be obtained from relative invariants of regular prehomogeneous vector spaces (see [8], [9], [5]). However, we do not assume here the existence of group action that relates the polynomials to prehomogeneous vector spaces.

Lemma 1 *Assume that the assumption* (A.2) *is satisfied. For* $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ *with* $m_1, \ldots, m_r \geq 0$, *denote by* $P^{*\mathbf{m}}(\partial_v)$ *the linear partial differential operator with constant coefficients satisfying*

$$P^{*\mathbf{m}}(\partial_v)\exp(\langle v, v^*\rangle) = P^{*\mathbf{m}}(v^*)\exp(\langle v, v^*\rangle).$$

Then, there exists a polynomial $b_{\mathbf{m}}(s)$ *of* $s_1, \ldots, s_r$ *such that*

$$P^{*\mathbf{m}}(\partial_v)P^s(v) = b_{\mathbf{m}}(s)P^{s+\mathbf{m}'}(v), \quad \mathbf{m}' = \mathbf{m}A^{-1}.$$

Moreover, the polynomial $b_{\mathbf{m}}(s)$ *is expressed in terms of* $\Gamma_{ij}(s)$ *as follows:*

$$b_{\mathbf{m}}(s) = (-2\pi\sqrt{-1})^{d^*(\mathbf{m})}\epsilon_j(\mathbf{m}')\epsilon_i^*(\mathbf{m}) \cdot \frac{\Gamma_{ij}(s+\mathbf{m}')}{\Gamma_{ij}(s)}.$$

We call $b_{\mathbf{m}}(s)$ the b-functions of $\{P_1, \ldots, P_r\}$. By the last identity in the lemma, we can define $b_{\mathbf{m}}(s)$ for any $\mathbf{m} \in \mathbb{Z}^r$. The b-functions satisfy the cocycle property

$$b_{\mathbf{m}+\mathbf{n}}(s) = b_{\mathbf{m}}(s)b_{\mathbf{n}}(s+\mathbf{m}') \quad (\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r). \tag{3}$$

The lemma says that the existence of b-functions is a necessary condition for local functional equations.

1.2 Nondegenerate dual quadratic mappings

Let W be a real vector space with dimension m and W^* the vector space dual to W. Suppose that we are given quadratic mappings $Q : W \to V$ and $Q^* : W^* \to V^*$. The mappings $B_Q : W \times W \to V$ and $B_{Q^*} : W^* \times W^* \to V^*$ defined by

$$B_Q(w_1, w_2) := Q(w_1+w_2)-Q(w_1)-Q(w_2), \quad B_{Q^*}(w_1^*, w_2^*) := Q^*(w_1^*+w_2^*)-Q^*(w_1^*)-Q^*(w_2^*)$$

are bilinear. For given $v \in V$ and $v^* \in V^*$, the mappings $Q_{v^*} : W \to \mathbb{R}$ and $Q_v^* : W^* \to \mathbb{R}$ defined by

$$Q_{v^*}(w) = \langle Q(w), v^*\rangle, \quad Q_v^*(w^*) = \langle v, Q^*(w^*)\rangle$$

are quadratic forms on W and W^*, respectively. We assume that Q and Q^* are nondegenerate and dual to each other with respect to the biregular mapping ϕ in (A.1). This means that Q and Q^* satisfy the following:

(A.3) (i) (Nondegeneracy) The algebraic set $\tilde{\Omega} := Q^{-1}(\Omega)$ (resp. $\tilde{\Omega}^* = Q^{*-1}(\Omega^*)$) is open dense in W (resp. W^*) and the rank of the differential of Q (resp. Q^*) at $w \in \tilde{\Omega}$ (resp. $w^* \in \tilde{\Omega}^*$) is equal to n. (In particular, $m \geq n$.)
(ii) (Duality) For any $v \in \Omega$, the quadratic forms $Q_{\phi(v)}$ and Q_v^* are dual to each other. Namely, fix a basis of W and the basis of W^* dual to it, and denote by S_{v^*} and S_v^* the matrices of the quadratic forms Q_{v^*} and Q_v^* with respect to the bases. Then $S_{\phi(v)}$ and S_v^* $(v \in \Omega)$ are nondegenerate and $S_{\phi(v)} = (S_v^*)^{-1}$.

Now we collect some elementary consequences of the assumptions (A.1) and (A.3). First note that a rational function defined over $\mathbb{R}$ with no zeros and no poles on Ω (resp. Ω^*) is a monomial of $P_1, \ldots, P_r$ (resp. $P_1^*, \ldots, P_r^*$). Hence the assumptions (A.1) and (A.3) (ii) imply the following lemma.

Lemma 2 *If we replace P_i, P_j^*, ϕ by their suitable real constant multiples (if necessary),*
(1) *there exists a $B = (b_{ij}) \in GL_r(\mathbb{Z})$ such that*

$$P_i^*(\phi(v)) = \prod_{j=1}^{r} P_j(v)^{b_{ij}} \quad (i = 1, \ldots, r).$$

(2) *There exist $\kappa, \kappa^* \in \mathbb{Z}^r$ and a non-zero constant α such that*

$$\det S_v^* = \alpha^{-1} P^{\kappa}(v), \quad \det S_{v^*} = \alpha P^{*\kappa^*}(v^*).$$

(3) *The mapping ϕ is of degree -1 and there exists a $\mu \in \mathbb{Z}^r$ such that*

$$\det\left(\frac{\partial \phi(v)_i}{\partial v_j}\right) = \pm P^{\mu}(v).$$

If $P_1, \ldots, P_r$ and $P_1^*, \ldots, P_r^*$ are the fundamental relative invariants of a regular prehomogeneous vector space (G, ρ, V) and its dual (G, ρ^*, V^*), then we have $B = A^{-1}$. Indeed, by the regularity, there exists a relative invariant P for which $\phi(v) = \operatorname{grad}\log P$ is a G-equivariant morphism satisfying (A.1). From the G-equivariance of the mapping ϕ ([7, §4, Prop. 9]), we have $B = A^{-1}$ (see [5]). It is very likely that the identity $B = A^{-1}$ always holds under the assumption (A.1) and (A.2) and, for simplicity, we assume

(A.4) $B = A^{-1}$.

Since we assumed that Ω_i (resp. Ω_i^*) are connected components, the signature of the quadratic form $Q_v^*(w^*)$ (resp. $Q_{v^*}(w)$) on W^* (resp. W) do not change when v (resp. v^*) varies on Ω_i (resp. Ω_i^*). Let p_i and q_i be the numbers of positive and negative eigenvalues of Q_v^* for $v \in \Omega_i$ and put

$$\gamma_i = \exp\left(\frac{(p_i - q_i)\pi\sqrt{-1}}{4}\right) \quad (i = 1, \ldots, \nu). \tag{4}$$

For $\Psi \in \mathcal{S}(W)$, we denote by $\hat{\Psi}$ the Fourier transform of Ψ:

$$\hat{\Psi}(w^*) = \int_W \Psi(w) \exp(2\pi\sqrt{-1}\langle w, w^*\rangle)\, dw.$$

Then, by (A.3) (ii) and the celebrated identity by Weil ([10, n°14, Théorème 2]), we have

$$\int_{W^*} \exp(2\pi\sqrt{-1}Q_v^*(w^*))\,\hat{\Psi}(w^*)\, dw^*$$
$$= 2^{-m/2}\,|\alpha|^{1/2}\,\gamma_i\,|P(v)|_i^{-\kappa/2} \int_W \exp\left(-\frac{\pi\sqrt{-1}}{2}\cdot Q_{\phi(v)}(w)\right)\Psi(w)\, dw \quad (v \in \Omega_i, \Psi \in \mathcal{S}(W)),$$

where dw and dw^* are the Euclidean measures dual to each other. This identity is the key to the proof of our main theorem.

1.3 Main theorem

We put

$$\tilde{P}_i(w) = P_i(Q(w)), \quad \tilde{P}_i^*(w^*) = P_i^*(Q^*(w^*)) \quad (i = 1, \ldots, r)$$
$$\tilde{\Omega}_i = Q^{-1}(\Omega_i), \quad \tilde{\Omega}_i^* = Q^{*-1}(\Omega_i^*) \quad (i = 1, \ldots, \nu).$$

Some of $\tilde{\Omega}_i$'s and $\tilde{\Omega}_i^*$'s may be empty. We define $\left|\tilde{P}(w)\right|_i^s$ and $\left|\tilde{P}^*(w^*)\right|_i^s$ in the same manner as in §1.1. The zeta functions associated with these polynomials are defined by

$$\tilde{\zeta}_i(s, \Psi) = \int_W \left|\tilde{P}(w)\right|_i^s \Psi(w)\, dw, \quad \tilde{\zeta}_i^*(s, \Psi^*) = \int_{W^*} \left|\tilde{P}^*(w^*)\right|_i^s \Psi^*(w^*)\, dw^*.$$

Then our main result is that the functional equation (2) for P_i's and P_j^*'s implies a functional equation for $\tilde{P}_i$'s and $\tilde{P}_j^*$'s and the gamma factors in the new functional equation can be written explicitly. Namely, we have the following theorem.

Theorem 3 ([6], Theorem 4) *Under the assumptions* (A.1)–(A.4), *the zeta functions* $\tilde{\zeta}_i(s, \Psi)$ *and* $\tilde{\zeta}_i^*(s, \Psi^*)$ *satisfy the functional equation*

$$\tilde{\zeta}_i^*((s + 2\lambda + \kappa/2 + \mu)A, \hat{\Psi}) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s)\tilde{\zeta}_j(s, \Psi),$$

where the gamma factors $\tilde{\Gamma}_{ij}(s)$ *are given by*

$$\tilde{\Gamma}_{ij}(s) = 2^{-2d(s)-m/2}\,|\alpha|^{1/2} \sum_{k=1}^{\nu} \gamma_k \Gamma_{ik}(s + \lambda + \kappa/2 + \mu)\Gamma_{kj}(s).$$

Here we denote by $d(s)$ ($s \in \mathbb{C}^r$) *the homogeneous degree of* P^s, *namely,* $d(s) = \sum_{i=1}^r s_i \deg P_i$.

By Lemma 1, we have the following formula expressing the b-functions $\tilde{b}_{\mathbf{m}}(s)$ of $\{\tilde{P}_1, \ldots, \tilde{P}_r\}$ in terms of the b-functions $b_{\mathbf{m}}(s)$ of $\{P_1, \ldots, P_r\}$.

Corollary to Theorem 3 *For* $\mathbf{m} \in \mathbb{Z}^r$, *we have*

$$\tilde{b}_{\mathbf{m}}(s) = b_{\mathbf{m}}(s) b_{\mathbf{m}}(s + \lambda + \kappa/2 + \mu)$$

up to a constant multiple.

In the case of one variable zeta functions, namely, in the case of $r = 1$, writing $P = P_1$ and $P^* = P_1^*$, we have the following lemma.

Lemma 4 *Assume that* $r = 1$. *Then we have*

$$A = B = -1, \quad d := \deg P = \deg P^*, \quad \lambda = \frac{n}{d}, \quad \mu = -\frac{2n}{d}, \quad \kappa = \frac{m}{d}.$$

By Lemma 4, if $r = 1$, then the functional equation for local zeta functions takes the form

$$\tilde{\zeta}_i^* \left(-s - \frac{m}{2d}, \hat{\Psi} \right) = \sum_{j=1}^{\nu} \tilde{\Gamma}_{ij}(s) \tilde{\zeta}_j(s, \Psi),$$
$$\tilde{\Gamma}_{ij}(s) = 2^{-2ds - m/2} |\alpha|^{1/2} \sum_{k=1}^{\nu} \gamma_k \Gamma_{ik} \left(s + \frac{m - 2n}{2d} \right) \Gamma_{kj}(s) \qquad (6)$$

and the b-function is given by

$$\tilde{b}(s) = b(s) b \left(s + \frac{m - 2n}{2d} \right), \qquad (7)$$

where $b(s)$ and $\tilde{b}(s)$ are defined by $P^*(\partial_v) P^s(v) = b(s) P^{s-1}(v)$ and $\tilde{P}^*(\partial_w) \tilde{P}^s(w) = \tilde{b}(s) \tilde{P}^{s-1}(v)$.

1.4 Representations of Euclidean Jordan Algebras

In [4, Chap. 8], Faraut and Koranyi proved that, starting from a representation of a Euclidean Jordan algebra, one can construct polynomials satisfying local functional equations. Their result was later generalized by Clerc [3] to zeta functions of several variables. Here we explain how their results can be incorporated in our Theorem 3.

Let V be a real simple Euclidean Jordan algebra with unity e, of dimension n and rank r. Denote by $P(v) = \det v$ the generic norm of V. Then $\Omega := \{v \in V \mid \det v \neq 0\}$ coincides with the set $V^\times$ of invertible elements in V. Let Ω_1 be the connected component of Ω containing e, the symmetric cone associated with V. Let G be the identity component of the group of linear transformations that preserve Ω_1, which is a real reductive Lie group. Then it is known that (G, V) is (a real form of) a prehomogeneous vector space, and the norm $P(v) = \det v$ of V is its fundamental relative invariant. More generally, restricting the G-action on V to the action of a minimal parabolic subgroup of G, we still have a prehomogeneous vector space with r fundamental relative invariants of minor determinant type. The prehomogeneous vector space is regular and we obtain a local functional equation (A.2) for zeta functions of r variables. Moreover the mapping $\phi : \Omega \to \Omega$ defined by $\phi(v) = v^{-1}$ satisfies the condition (A.1).

Let W be a Euclidean space of dimension m, Φ a representation of V in the space $Sym(W)$ of self adjoint endomorphism of W such that

$$\Phi(vv') = \frac{1}{2}\big(\Phi(v)\Phi(v') + \Phi(v')\Phi(v)\big), \quad v, v' \in V$$

and $Q : W \to V$ the quadratic mapping associated to Φ defined by

$$(Q(w)|v)_V = (\Phi(v)w|w)_W, \quad v \in V, w \in W. \tag{8}$$

Assume that Φ is *regular*, namely, there exists a $w \in W$ such that $\det Q(w) \neq 0$. Then the quadratic mapping Q is nondegenerate in the sense of (A.3) (i) , and we have $Q(W) = \overline{\Omega_1}$. We also assume that $\Phi(e) = \mathrm{id}_W$.

For an invertible $v \in V$, there exists a polynomial $q(v)$ of degree r such that $v^{-1} = \frac{q(v)}{\det v}$ ([4, Prop. II.2.4]). Since Φ is a Jordan algebra representation, $\Phi(v)$ and $\Phi(v^{-1})$ commute. Hence

$$\mathrm{id}_W = \Phi(v \cdot v^{-1}) = \frac{1}{2}\big(\Phi(v)\Phi(v^{-1}) + \Phi(v^{-1})\Phi(v)\big) = \Phi(v)\Phi(v^{-1}).$$

This implies that Q is self-dual with respect to $\phi(v) = v^{-1}$.

Thus our Theorem 3 shows that the compositions of the fundamental relative invariants with Q satisfy a local functional equation. This recovers the results of Faraut-Koranyi and Clerc. Concrete examples are described in Clerc [3].

In [3], it is noted that, if the Jordan algebra V is of rank 2, then the generic norm det is a quadratic form of signature $(1, n-1)$ and the polynomials Q of degree 4 constructed as above are *not* relative invariants of prehomogeneous vector spaces (except for some low-dimensional cases). However, it seems that no simple criterion on prehomogeneity has been known yet. This problem will be discussed in the next senction in a more general setting.

Remark. In [3], Clerc proved local functional equations also for zeta functions with harmonic polynomials. This part is not covered by Theorem 3.

2 Quartic polynomials obtained from representations of Clifford algebras

Let p, q be non-negative integers and consider the quadratic form $P(x) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} x_{p+j}^2$ of signature (p, q). We identify $V = \mathbb{R}^{p+q}$ with its dual vector space via the standard inner product $(x, y) = x_1y_1 + \cdots + x_{p+q}y_{p+q}$. Put $\Omega = V \setminus \{P = 0\}$. We determine the quadratic mappings $Q : W \to V$ that is self-dual with respect to the biregular mapping $\phi : \Omega \longrightarrow \Omega$ defined by

$$\phi(v) := \frac{1}{2}\mathrm{grad}\log P(v) = \frac{1}{P(v)}(v_1, \ldots, v_p, -v_{p+1}, \ldots, -v_{p+q}).$$

By Theorem 3, for such a quadratic mapping Q, the complex powers of the quartic polynomials $\tilde{P}(w) := P(Q(w))$ satisfy a functional equation with explicit gamma factors.

For a quadratic mapping Q of $W = \mathbb{R}^m$ to $V = \mathbb{R}^{p+q}$, there exist symmetric matrices $S_1, \ldots, S_{p+q}$ of size m such that

$$Q(w) = ({}^t w S_1 w, \ldots, {}^t w S_{p+q} w).$$

For $v \in \mathbb{R}^{p+q}$, we put

$$S(v) = \sum_{i=1}^{p+q} x_i S_i.$$

Then the mapping Q is self-dual with respect to ϕ if and only if

$$S(v)S(\phi(v)) = I_m \quad (v \in \Omega).$$

If we define ϵ_i to be 1 or -1 according as $i \leq p$ or $i > p$, this condition is equivalent to the polynomial identity

$$\sum_{i=1}^{p} x_i^2 S_i^2 - \sum_{j=1}^{q} x_{p+j}^2 S_{p+j}^2 + \sum_{1 \leq i < j \leq p+q} x_i x_j \left(\epsilon_j S_i S_j + \epsilon_i S_j S_i\right) = P(x) I_m.$$

This identity holds if and only if

$$\begin{aligned} S_i^2 &= I_m \ (1 \leq i \leq p+q), \\ S_i S_j &= \begin{cases} S_j S_i & (1 \leq i \leq p < j \leq p+q \text{ or } 1 \leq j \leq p < i \leq p+q) \\ -S_j S_i & (1 \leq i, j \leq p \text{ or } p+1 \leq i, j \leq p+q). \end{cases} \end{aligned}$$

This means that the mapping $S : V \to Sym_m(\mathbb{R})$ can be extended to a representation of the tensor product of the Clifford algebra C_p of $x_1^2 + \cdots + x_p^2$ and the Clifford algebra C_q of $x_{p+1}^2 + \cdots + x_{p+q}^2$.

Conversely, if we are given a representation $S : C_p \otimes C_q \longrightarrow M_m(\mathbb{R})$, then the representation S is a direct sum of simple modules and a simple $C_p \otimes C_q$-module is a tensor product of simple modules of C_p and C_q. Since one can choose a basis of the represetation space so that $S(\mathbb{R}^{p+q})$ is contained in $Sym_m(\mathbb{R})$, we have proved that

Theorem 5 *Self-dual quadratic mappings Q of $W = \mathbb{R}^m$ to the quadratic space (V, P) correspond to representations S of $C_p \otimes C_q$ such that $S(V) \subset Sym_m(\mathbb{R})$.*

The construction above is a generalization of a result of Faraut-Koranyi [4] on the functional equation associated with representations of simple Euclidean Jordan algebra of rank 2. In this case $(p, q) = (1, q)$. Then the self-dual quadratic mappings over the quadratic space of signature $(1, q)$ correspond to repsentations of $C_1 \otimes C_q \cong C_q \oplus C_q$. Representations of $C_1 \otimes C_q$ can be identified with the direct sum of 2 C_q-modules M_+ and M_-. On M_+ (resp. M_-), e_1 acts as multiplication by $+1$ (resp. -1). The case obtained from the Faraut-Koranyi construction is the one for which $M_- = \{0\}$.

Prehomogeneous or Non-prehomogeneous?

Most of the quartic polynomials $\tilde{P}$ and $\tilde{P}^*$ are conjectured not to be relative invariants of prehomogeneous vector spaces except for low-dimensional cases.

Theorem 6 *If $p+q = \dim V \leq 4$, then the polynomials $\tilde{P}$ and $\tilde{P}^*$ are relative invariants of prehomogeneous vector spaces.*

The prehomogeneous vector spaces appearing in the case $p + q \leq 4$ are given in the following table:

(p,q)	prehomogeneous vector space
$(1,0)$	$(GL(1,\mathbb{R}) \times SO(k_1,k_2), \mathbb{R}^{k_1+k_2})$
$(2,0)$	$(GL(1,\mathbb{C}) \times SO(k,\mathbb{C}), \mathbb{C}^k)$
$(1,1)$	$(GL(1,\mathbb{R}) \times SO(k_1,k_2), \mathbb{R}^{k_1+k_2}) \oplus (GL(1,\mathbb{R}) \times SO(k_3,k_4), \mathbb{R}^{k_3+k_4})$
$(3,0)$	$(GL(1,\mathbb{R}) \times SU(2) \times SO^*(2k), \mathbb{C}^{2k})$
$(2,1)$	$(GL(2,\mathbb{R}) \times SO(k_1,k_2), M(2,k_1+k_2,\mathbb{R}))$
$(4,0)$	$(GL(1,\mathbb{H}) \times GL(1,\mathbb{H}) \times GL(k,\mathbb{H}), M(2,k,\mathbb{H}))$
$(3,1)$	$(GL_2(\mathbb{C}) \times SU(k_1,k_2), M(2,k_1+k_2;\mathbb{C}))$
$(2,2)$	$(GL_2(\mathbb{R}) \times GL_2(\mathbb{R}) \times SL(k,\mathbb{R}), M(2,k;\mathbb{R})^{\oplus 2})$

It seems that, if $p + q \geq 5$, then $\tilde{P}$ and $\tilde{P}^*$ are relative invariants of prehomogeneous vector spaces only for few exceptional cases.

Let $\mathfrak{g}$ be the Lie algebra of the group $G = \left\{g \in GL(W) \;\middle|\; \tilde{P}(gw) \equiv \tilde{P}(w)\right\}$ and $\mathfrak{h}$ ($= \mathfrak{h}_{p,q}$) the Lie algebra of the group $H = \{h \in GL(W) \mid Q(hw) \equiv Q(w)\}$. We can prove that $\mathfrak{g}$ contains a Lie subalgebra isomorphic to $\mathfrak{so}(p,q) \oplus \mathfrak{h}$.

Conjecture 1. We have

$$\mathfrak{g} \cong \mathfrak{so}(p,q) \oplus \mathfrak{h}.$$

The Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ depend on p, q and the choice of the representation of $C_p \otimes C_q$. By the periodicity of Clifford algebras $C_{p+8} = M(16, C_p)$, there exists a natural correspondence between representations of $C_{p+8} \otimes C_q$ and representations of $C_p \otimes C_q$ and it can be proved that the structure of $\mathfrak{h}$ is the same for corresponding representations. This implies the isomorhisms

$$\mathfrak{h}_{p,q} \cong \mathfrak{h}_{q,p} \cong \mathfrak{h}_{p+8,q} \cong \mathfrak{h}_{p,q+8} \cong \mathfrak{h}_{p+4,q\pm4}. \tag{9}$$

If $\dim V$ and $\dim W$ are relatively small, then we can calculate $\mathfrak{h}$ explicitly by using a symbolic calculation engine (such as Mathematica and Maple) and we have the following conjecture on the structure of the Lie algebra $\mathfrak{h}$.

Conjecture 2. The Lie algebra $\mathfrak{h}$ is isomorphic to the reductive lie algebra given in the following table:

$\bar{p}\backslash\bar{q}$	0	1	2	3
0	$\mathfrak{gl}(k,\mathbb{R})$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{so}(k,\mathbb{C})$	$\mathfrak{so}^*(2k)$
1	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{so}(k_1,k_2)\times\mathfrak{so}(k_3,k_4)$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{u}(k_1,k_2)$
2	$\mathfrak{so}(k,\mathbb{C})$	$\mathfrak{so}(k_1,k_2)$	$\mathfrak{gl}(k,\mathbb{R})$	$\mathfrak{sp}(k,\mathbb{R})$
3	$\mathfrak{so}^*(2k)$	$\mathfrak{u}(k_1,k_2)$	$\mathfrak{sp}(k,\mathbb{R})$	$\mathfrak{sp}(k_1\mathbb{R})\times\mathfrak{sp}(k_2,\mathbb{R})$
4	$\mathfrak{gl}(k,\mathbb{H})$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{sp}(k,\mathbb{C})$	$\mathfrak{sp}(k,\mathbb{R})$
5	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{sp}(k_1,k_2)\times\mathfrak{sp}(k_3,k_4)$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{u}(k_1,k_2)$
6	$\mathfrak{sp}(k,\mathbb{C})$	$\mathfrak{sp}(k_1,k_2)$	$\mathfrak{gl}(k,\mathbb{H})$	$\mathfrak{so}^*(2k)$
7	$\mathfrak{sp}(k,\mathbb{R})$	$\mathfrak{u}(k_1,k_2)$	$\mathfrak{so}^*(2k)$	$\mathfrak{so}^*(2k_1)\times\mathfrak{so}^*(2k_2)$

Here $\bar{p} = p \bmod 8$ and $\bar{q} = q \bmod 8$ and k_1, k_2, k_3, k_4, k are non-negative integers determined by the multiplicities of irreducible representations in the representation of $C_p \otimes C_q$ corresponding to the quadratic mapping Q.

Note that, by (9), it is sufficient to give the table only for $0 \le \bar{p} \le 7$ and $0 \le \bar{q} \le 3$

Using Conjectures 1 and 2, we can determine all the cases where $\tilde{P}$ is prehomegeneous. For exmple, if $p+q \ge 13$, then $\tilde{P}$ is non-prehomogeneous for any representation of $C_p \otimes C_q$; namely it does not come from any prehomogeneous vector space.

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