Fourier expansion of Arakawa lifting

Atsushi Murase and Hiro-aki Narita

Abstract

This note is a report based on our talk at the conference on automorphic forms held at RIMS during January 21th-25th, 2008. We announce our recent results about Fourier coefficients of Arakawa lifting, i.e. a theta lifting to a cusp form on the quaternion unitary group $GSp(1,1)$ from a pair consisting of an elliptic cusp form $f$ and an automorphic form $f'$ on a definite quaternion algebra over $\mathbb{Q}$. We provide an explicit formula for the Fourier coefficients in terms of toral integrals of $f$ and $f'$. As an application, we show the existence of non-vanishing Arakawa lifts.

0 Introduction

To explain the background of our study we start with reviewing Böcherer's conjecture on Fourier coefficients of holomorphic Siegel modular forms of degree two. We let $F$ be a Hecke-eigen holomorphic Siegel cusp form of weight $k$ with respect to $Sp(2, \mathbb{Z})$. Its Fourier expansion is described as

$$F(Z) = \sum_{T \in \text{Sym}_2(\mathbb{Z})^*, T > 0} C(T) e^{2\pi \sqrt{-1} \text{Tr} TZ},$$

where $\text{Sym}_2(\mathbb{Z})^*$ denotes the set of half-integral symmetric matrices of degree 2, and $T > 0$ means that $T$ is positive definite. Now let $-D$ be a fundamental discriminant with $D > 0$. For such $D$ we consider the average $A(D)$ of the Fourier coefficients of $F$ as follows:

$$A(D) := \sum_{S \in \{T | \det T = D/4 \}/SL_2(\mathbb{Z})} \frac{C(S)}{\epsilon(S)}.$$

Here we put $\epsilon(S) = \# \{ \gamma \in SL_2(\mathbb{Z}) \mid ^t\gamma S \gamma = S \}$. We let $L_{\text{spin}}(F, (\frac{D}{*}), s)$ be the quadratic twist of the spinor $L$-function for $F$. Then Böcherer's conjecture [B] is formulated as

$$|A(D)|^2 = C_F D^{k-1} L_{\text{spin}}(F, (\frac{D}{*}), k - 1).$$
with a constant $C_F$ depending only on $F$. There are several evidences of this conjecture (cf. [B], [B-S], [K-K]). We note that, in the conjecture, the spinor $L$-function is evaluated at its central point $s = k - 1$. This conjecture can be regarded as a generalization of the formula by Waldspurger-Kohnen-Zagier (cf. [Wa-1], [K-Z]), which says that the twisted central $L$-value of an integral weight elliptic cusp form $f$ is proportional to the square of a Fourier coefficient of a half-integral weight elliptic cusp form associated with $f$ by Shimura correspondence. Furusawa and Shalika have made a further expectation that Böcherer conjecture would also hold for an inner form $G$ of $GSp(2)$:

$$\{g \in M_2(B) \mid {}^t \bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \nu(g) \in \mathbb{Q}^\times\},$$

where $B$ is a quaternion algebra over $\mathbb{Q}$. This expectation is based on their conjectural relative trace formula for $G$ (cf. [F-S]). In this note, we consider the case where $B$ is definite, i.e. $G = GSp(1,1)$.

Our target is the "Arakawa lifting", which is a theta lifting from a pair of elliptic cusp form $f$ and an automorphic form $f'$ on $B_{\mathbb{A}_Q}^\times$ to a vector-valued cusp form $L(f, f')$ on $GSp(1,1)_{\mathbb{A}_Q}$. Its representation type at the Archimedean place is a quaternionic discrete series representation (for the definition see [G-W]). Our result (Theorem 2.2) says that a certain average of the Fourier coefficients of $L(f, f')$ (an analogue of $A(D)$) is explicitly written in terms of a product of toral integrals of $(f, f')$.

Our formula leads us to two directions of further research. One is to show the existence of non-vanishing lifts, which is discussed in §3. In fact, if $(f, f')$ are Hecke eigenforms with non-vanishing toral integrals, we have $L(f, f') \neq 0$ in view of our formula. Another direction is to find an explicit formula for the constant of proportionality relating the square norms of the averages of the Fourier coefficients to central $L$-values. Indeed, Furusawa-Shalika [F-S] expects that such square norm is proportional to the central value of a "Rankin-Selberg $L$-function" of the Arakawa lift. Waldspurger [Wa-1, Proposition 7] and our theorem tell us that the square norm of the average is proportional to a product of central $L$-values for the quadratic base changes of the Jacquet-Langlands lifts of $f$ and $f'$ twisted by a Hecke character. Such a product is expected to be a central $L$-value of a (twisted) Rankin-Selberg $L$-function of the Arakawa lift. We leave the study in this direction to our further research.

1 Arakawa lifting

Let $B$ be a definite quaternion algebra over $\mathbb{Q}$ with the discriminant $d_B$. Let $x \mapsto \bar{x}$ be the main involution of $B$. We fix a maximal order $\mathcal{O}$ of $B$. We denote by $G = GSp(1,1)$ the $\mathbb{Q}$-algebraic group defined by

$$\{g \in M_2(B) \mid {}^t \bar{g} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \nu(g) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \nu(g) \in \mathbb{Q}^\times\}.$$
From now on we assume that every automorphic form dealt with in this note has the trivial central character. Let $D$ be a divisor of $d_B$ and $S_\kappa(\Gamma_0(D))$ the space of elliptic cusp forms of weight $\kappa$ and level $D$. We regard each element of $S_\kappa(\Gamma_0(D))$ as an automorphic form on $\text{GL}_2(\mathbb{A}_Q)$. We further denote by $A_\kappa = A_\kappa(\mathbb{B}_{\mathbb{A}_Q})$ the space of automorphic forms on $\mathbb{B}_{\mathbb{A}_Q}$ of weight $(\sigma_\kappa = \text{Sym}_\kappa, V_\kappa)$ and right $\prod_{v<\infty} \mathcal{O}_v^{x}$-invariant.

Let $r$ be the metaplectic representation of $G_{A_Q} \times (\text{GL}_2(\mathbb{A}_Q) \times \mathbb{B}_{\mathbb{A}_Q})$ introduced in [M-N-1, §3]. We then define a theta series on $G_{A_Q} \times (\text{GL}_2(\mathbb{A}_Q) \times \mathbb{B}_{\mathbb{A}_Q})$ by

$$\Theta_\kappa(g, h, h') := \sum_{(X,t) \in B^2 \times Q^x} (r(g, h, h') \Phi)(X, t).$$

Here we put $\Phi := \prod_{v \leq \infty} \Phi_v$ with

$$\Phi_v(X, t) := \begin{cases} \text{ch}(O_v^2 \times \mathbb{Z}_v^x)(X, t) & (v \nmid D^{-1}d_B), \\ \text{ch}((O_v \oplus \mathfrak{P}_v^{-1}) \times \mathbb{Z}_v^x)(X, t) & (v | D^{-1}d_B), \\ \text{ch}(t \in \mathbb{R}_+^x) t^{2|X|} \sigma_\kappa(X_1 + X_2) e^{-2\pi t^t \overline{X}X} & (v = \infty), \end{cases}$$

where $\mathfrak{P}_v$ is a maximal ideal at $v$ and $\text{ch}(S)$ denotes the characteristic function for a set $S$. Then the Arakawa lifting is defined as follows:

$$S_\kappa(\Gamma_0(D)) \times A_\kappa(\mathbb{B}_{\mathbb{A}_Q}) \ni (f, f') \mapsto L(f, f')(g) := \iiint (R_+^x)^2(GL_2 \times \mathbb{B}_{\mathbb{A}_Q})_{\mathbb{Q} \setminus \mathbb{A}_Q} \Theta_\kappa(g, h, h')f(h') dhdh'.$$

This is a cusp form on $G_{A_Q}$ belonging to the minimal $K_\infty$-type of a quaternionic discrete series representation at infinity (cf. [M-N-2, Theorem 3.3.2]), where $K_\infty$ denotes a maximal compact subgroup of the real points of $Sp(1, 1)$.

## 2 Main result

### 2.1

In general, a cuspidal automorphic form $F$ on $G_{A_Q}$ admits a Fourier expansion as follows:

$$F(g) = \sum_{\xi \in B^- \setminus \{0\}} F_\xi(g) = \sum_{\xi \in B^- \setminus \{0\}} \sum_{\chi \in X_\xi} F_\xi^\chi(g).$$

Here

$$F_\xi(g) := \int_{B^- \setminus B_{\mathbb{A}_Q}} F \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(- \text{tr}(\xi x)) dx, \quad F_\xi^\chi(g) := \int_{R_+^x Q(\xi)^x \setminus A_{Q(\xi)}^x} F_\xi(s12 \cdot g) \chi(s)^{-1} ds,$$

where $B^- = \{ x \in B \mid \bar{x} = -x \}$, $\psi$ is the standard additive character on $\mathbb{Q} \setminus A_{\mathbb{Q}}$ and $X_\xi$ denotes the set of Hecke characters of $A_{\mathbb{Q}Q(\xi)^x \setminus A_{Q(\xi)}^x}$. Our main result is an explicit formula for $F_\xi^\chi$ when $F = L(f, f')$. 


2.2
To state the main theorem, we let \((f, f') \in S_{\kappa}(D) \times \mathcal{A}_{\kappa}\) be Hecke eigenforms. We further assume that \(f\) and \(f'\) are eigenforms for the “Atkin-Lehler involution”:

\[
f \left( h \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix} \right) = \epsilon_{p} f(h), \quad f'(h' \Pi_{p}) = \epsilon'_{p} f'(h')
\]

with \(\epsilon_{p}, \epsilon'_{p} \in \{\pm 1\}\). Here \(\Pi_{p}\) is a prime element of \(B_{p}\). Note that \(\mathcal{L}(f, f') \equiv 0\) unless \(\epsilon_{p} = \epsilon'_{p}\) for any \(p|D\).

For \(p < \infty\), let \(a_{p} \doteq \begin{cases} \mathcal{O}_{p} & (p \not| DB \text{ or } p|D) \\ \mathfrak{P}_{p} & (p|D^{-1}DB) \end{cases}\). We say that \(\xi \in B^{-} \setminus \{0\}\) is primitive if \(\xi \in a_{p} \setminus p\mathfrak{a}_{p}\) for each finite prime \(p\). We note that a Fourier coefficient \(F_{\xi}\) of an automorphic form \(F\) on \(G_{\mathbb{A}_{Q}}\) satisfies

\[
F_{\xi}\left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g \right) = F_{t\xi}(g) \quad (t \in \mathbb{Q}^{x}).
\]

It follows that the calculation of the Fourier expansion of \(F\) is reduced to that of \(F_{\xi}\) for primitive \(\xi\).

2.3
We next introduce several notations. Let \(\xi \in B^{-} \setminus \{0\}\) and \(d_{\xi}\) denote the discriminant of an imaginary quadratic field \(E := \mathbb{Q}(\xi)\). We put

\[
a := \begin{cases} 2\sqrt{-n(\xi)}\sqrt{d_{\xi}} & (d_{\xi}\text{ is odd}) \\ \sqrt{-n(\xi)}\sqrt{d_{\xi}} & (d_{\xi}\text{ is even}) \end{cases}, \quad b := \xi^2 - \frac{a^2}{4}.
\]

With these \(a\) and \(b\) we define an embedding \(\iota_{\xi} : E^{x} \hookrightarrow GL_{2}(\mathbb{Q})\) by

\[
\iota_{\xi}(x + y\xi) = x \cdot 1_{2} + y \cdot \begin{pmatrix} a/2 & b \\ 1 & -a/2 \end{pmatrix} \quad (x, y \in \mathbb{Q}).
\]

The completion \(E_{\infty}\) of \(E\) at \(\infty\) is identified with \(\mathbb{C}\) by

\[
\delta_{\xi} : E_{\infty} \ni x + y\xi \mapsto x + y\sqrt{-n(\xi)} \in \mathbb{C} \quad (x, y \in \mathbb{R}).
\]

For a Hecke character \(\chi = \prod_{\nu \leq \infty} \chi_{\nu}\) of \(E\), we define \(w_{\infty}(\chi) \in \mathbb{Z}\) to be

\[
\chi_{\infty}(u) = (\delta_{\xi}(u)/|\delta_{\xi}(u)|)^{w_{\infty}(\chi)} \quad (u \in E_{\infty}^{x}).
\]

Furthermore, for each finite prime \(p < \infty\), \(i_{p}(\chi)\) denotes the exponent of the conductor of \(\chi\) at \(p\) and

\[
\mu_{p} := \frac{\text{ord}_{p}(2\xi)^{2} - \text{ord}_{p}(d_{\xi})}{2}.
\]

We then have the following (cf. [M-N-2, Theorem 5.1.1]):
Proposition 2.1. $\mathcal{L}(f, f')^\chi_\xi = 0$ unless
\[ i_p(\chi) = 0 \text{ for any } p|d_B \text{ and } w_\infty(\chi) = -\kappa. \] (1)

2.4 Statement of the main theorem.

In what follows, we assume that (1) holds. We need further notations to state the main theorem.

Define $\gamma_0 = (\gamma_0, p)_{p \leq \infty} \in GL_2(A_Q)$ and $\gamma_0' = (\gamma_0, p)_{p < \infty} \in B^\kappa_{A_Q}$ as follows:

\[
\gamma_0, p := \begin{cases}
\begin{pmatrix} 1 & 0 \\
0 & p^{-\mu_p+i_p(\chi)} \\
\end{pmatrix} & (p \not| D), \\
\begin{pmatrix} 1 & 0 \\
0 & p \\
\end{pmatrix} & (p|D \text{ and } p \text{ is inert in } E), \\
\begin{pmatrix} 1 & a/2 \\
0 & 1 \\
\end{pmatrix} & (p|D \text{ and } p \text{ ramifies in } E), \\
\begin{pmatrix} 1 & 0 \\
0 & 1 \\
\end{pmatrix} & (p = \infty),
\end{cases}
\]

\[
\gamma_0, p' := \begin{cases}
\begin{pmatrix} 1 & 0 \\
0 & p^{-\mu_p+i_p(\chi)} \\
\end{pmatrix} & (p \not| d_B), \\
\Pi_{B_1, p} & (p|d_B).
\end{cases}
\]

Here $\Pi_{B_1, p}$ is a prime element of $B_1$ for $p|d_B$. We furthermore define the following local constants:

\[
C_p(f, \xi, \chi) := \begin{cases}
p^{2\mu_p-i_p(\chi)}(1 - \delta(i_p(\chi) > 0)e_p(E)p^{-1}) & (p \not| d_B), \\
1 & (p|D^{-1}d_B), \\
2e_p & (p|D \text{ and } p \text{ is inert in } E), \\
(p + 1)^{-1} & (p|D \text{ and } p \text{ ramifies in } E),
\end{cases}
\]

where $\delta(P) = 1$ (resp. 0) if a condition $P$ holds (resp. does not hold), and

\[
e_p(E) = \begin{cases}
-1 & (p \text{ is inert in } E), \\
0 & (p \text{ ramifies in } E), \\
1 & (p \text{ splits in } E).
\end{cases}
\]

For $(f, f') \in S_\kappa(D) \times A_\kappa$ we introduce their toral integrals with respect to a Hecke character $\chi$ of $E$:

\[
P_\chi(f; h) := \int_{B^\kappa_{E^\kappa} \backslash A^\kappa_{E^\kappa}} f(t_\xi(s)h)\chi(s)^{-1}ds, \quad P_\chi(f'; h') := \int_{B^\kappa_{E^\kappa} \backslash A^\kappa_{E^\kappa}} f(sh')\chi(s)^{-1}ds,
\]
where \((h, h') \in GL_2(\mathbb{A}_Q) \times B_{A_Q}^X\). Here we normalize the measure \(ds\) of \(\mathbb{A}_E^X\) so that
\[
\text{vol}(\mathcal{O}_{E_p}) = \text{vol}(E_{\infty}^{(1)}) = 1
\]
for each finite prime \(p\), where \(\mathcal{O}_{E_p}\) is the \(p\)-adic completion of the integer ring of \(E\) and \(E_{\infty}^{(1)}\) denotes the group of elements in \(E_{\infty}\) with norm 1.

We denote by \(h(E)\) and \(w(E)\) the class number of \(E\) and the number of roots of unity in \(E\) respectively. We are now able to state our main result (cf. [M-N-2, Proposition 2.4.1, Theorem 5.2.1]).

**Theorem 2.2.** (1) When \(\xi = 0\), \(\mathcal{L}(f, f')_{\xi} \equiv 0\).
(2) Let \(\xi\) be a primitive element in \(B^- \setminus \{0\}\). Suppose that \(\chi\) satisfies (1) and that \(\epsilon_p = \epsilon'_p\) for any \(p|D\). We then have the following formula:
\[
\mathcal{L}(f, f')_{\xi} \left( g_0 \begin{pmatrix} \sqrt{\eta_{\infty}} & 0 \\ 0 & \sqrt{\eta_{\infty}}^{-1} \end{pmatrix} \right) = 2^{\kappa-1} N(\xi)^{\kappa/4} \frac{w(E)}{h(E)} \prod_{p<\infty} \mathcal{C}_p(f, \xi, \chi) \eta_{\infty}^{\kappa/2+1} \exp(-4\pi \sqrt{N(\xi)\eta_{\infty}}) \frac{P_{\chi}(f; \gamma_0)}{P_{\chi}(f'; \gamma_0)}.
\]

Here \(\eta_{\infty} \in \mathbb{R}_+^X\) and \(g_0 = (g_{0,p})_{p<\infty} \in G_{A_{Q,f}}\) is given by
\[
g_{0,p} := \begin{cases} 
\text{diag}(p^{i_p(\chi)-\mu_p}, p^{2(i_p(\chi)-\mu_p)}, 1, p^{i_p(\chi)-\mu_p}) & (p \parallel d_B), \\
1_2 & (p|d_B)
\end{cases}
\]

**Remark 2.3.** (1) \(\mathcal{L}(f, f')_{\xi}\) is determined by the value at \(g_0 \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta}^{-1} \end{pmatrix}\) due to Sugano’s result ([Su, Proposition 2-5]).
(2) Murase and Sugano have obtained a similar formula for “Kudla lifting”, i.e. a theta lift from \(U(1,1)\) to \(U(2,1)\) (cf. [M-S]).

**Remark 2.4.** Let \(\Pi\) (resp. \(\Pi'\)) be the base change to \(GL_2(\mathbb{A}_E)\) of the Jacquet-Langlands lift \(\pi_f\) (resp. \(\pi_{f'}\)) of the automorphic representation attached to \(f\) (resp. \(f'\)). Waldspurger [Wa-2, Proposition 7] proved the following formula:
\[
\frac{||P_{\chi}(f; \gamma_0)||^2}{\langle f, f \rangle} = C_{f, \chi} \cdot L(\Pi \otimes \chi^{-1}, \frac{1}{2}),
\]
\[
\frac{||P_{\chi}(f'; \gamma_0)||^2}{\langle f', f' \rangle} = C_{f', \chi} \cdot L(\Pi' \otimes \chi^{-1}, \frac{1}{2}),
\]
where
\[
C_{\varphi, \chi} = \frac{\sqrt{|d_{\xi}|}}{4\pi} \cdot \frac{\zeta(2)}{2L(\pi_{\varphi}, \text{Ad}, 1)} \prod_{v: \text{bad}} C_{\varphi, \chi, v}
\]
for $\varphi = f$ or $f'$, with the adjoint $L$-function $L(\pi_{\varphi}, \text{Ad}, s)$ of $\varphi = f$ or $f'$ and where $C_{\varphi, \chi, \nu}$ is a ratio of a local period and $L$-values. We now remark that there does not appear $\frac{\sqrt{|d|}}{4\pi}$ in Waldspurger's formula [Wa-2, Proposition 7]. This is due to the difference between normalizations of Waldspurger's measure and ours for $A_{E}^{x}$.

Our theorem and Waldspurger's formula then imply

$$\frac{||L(f, f')_{\xi}(g_{0, f})||^{2}}{(f, f)(f', f')} = C_{f, f', \chi}L(\Pi \otimes \chi^{-1}, \frac{1}{2})L(\Pi' \otimes \chi^{-1}, \frac{1}{2})$$

with

$$C_{f, f', \chi} := 2^{2(\kappa-1)}N(\xi)^{\kappa}\tau\frac{w(E)}{h(E)}|\prod_{p<\infty}C_{p}(f, \xi, \chi)|^{2}\exp(-8\pi\sqrt{N(\xi)})\cdot C_{f, \chi} \cdot C_{f_{2}', \chi}.$$

It would be interesting to find a more explicit form of the constant $C_{f, f', \chi}$.

3 Application (Non-vanishing lifts)

A general approach to verify the non-vanishing of theta lifts is to study their Petersson inner products. This technique is due to S. Rallis [R] and J. S. Li [L] etc. Via the Siegel-Weil formula (cf. [We]), it reduces the problem to the non-vanishing of a special value of the standard $L$-function for the preimages of the theta lifts. This method is useful when the Siegel-Weil formula is available, but this is not the case for our theta lifts.

Our approach to show the existence of the non-vanishing Arakawa lifts is to find examples of $(f, f')$ with non-vanishing toral integrals involved in our formula for Fourier coefficients of the lifts (Theorem 2.2).

3.1 Result

We now specialize the situation. Let $B = \mathbb{Q} + \mathbb{Q} \cdot i + \mathbb{Q} \cdot j + \mathbb{Q} \cdot ij$ with $i^{2} = j^{2} = -1$ and $ij = -ji$. It is known that $d_{B} = 2$ and the class number of $B$ is one. We note that $D = 1$ or 2. Let $O = \mathbb{Z}1 + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}(1 + i + j + k)/2$, and put $\xi = \frac{1}{2}i$, which is primitive. Suppose that the Hecke character $\chi$ of $E$ is unramified at all finite places. This assumption implies that $w_{\infty}(\chi)$ is divisible by 4.

Proposition 3.1. Let $B$, $\xi$ and $\chi$ be as above. Then there exist Hecke eigenforms $(f, f')$ such that

$$P_{\chi}(f; \gamma_{0})P_{\chi}(f'; \gamma'_{0}) \neq 0$$

for every $\kappa \geq \begin{cases} 12 & (D = 1) \\ 8 & (D = 2) \end{cases}$ with $4|\kappa$.

Theorem 3.2. Let $(f, f')$ be as above. Then $L(f, f') \neq 0$. 
3.2 Outline of the proof

Theorem 3.2 is a direct consequence of Proposition 3.1 and Theorem 2.2. This subsection is thus devoted to the outline of our proof of Proposition 3.1. If one finds a pair \((f, f')\) such that \(P_{\chi}(f; \gamma_{0})P_{\chi}(f'; \gamma_{0}') \neq 0\), there exists a pair of Hecke eigenforms with the same property. This follows from the fact that \(S_{\kappa}(\Gamma_{0}(D))\) and \(A_{\kappa}(B_{AQ}^{x})\) have basis consisting of Hecke eigenforms.

To begin with, we find \(f' \in A_{\kappa}(B_{AQ}^{x})\) such that \(P_{\chi}(f'; \gamma_{0}') \neq 0\). Eichler's trace formula of Brandt matrices (cf. [E, Theorem 5]) says that

\[
\dim_{\mathbb{C}} A_{\kappa}(B_{AQ}^{x}) = \begin{cases} 
\frac{\kappa+12}{12} & (\kappa \equiv 0 \mod 12), \\
\frac{\kappa-4}{12} & (\kappa \equiv 4 \mod 12), \\
\frac{\kappa+4}{12} & (\kappa \equiv 8 \mod 12)
\end{cases}
\]

and hence \(\dim_{\mathbb{C}} A_{\kappa}(B_{AQ}^{x}) \neq 0\) if \(\kappa \geq 8\). By a direct calculation we see that \(P_{\chi}(f'; \gamma_{0}) = \pm 1 \times (f'(1), v_{\kappa}^{*}) v_{\kappa}\), where \(v_{\kappa}\) is a highest weight vector of \(V_{\kappa}\). Since the class number of \(B\) is one, \(f' \mapsto f'(1)\) induces an isomorphism \(A_{\kappa}(B_{AQ}^{x}) \cong V_{\kappa}^{\mathcal{O}^{x}}\). Let \(f'\) be an element of \(A_{\kappa}(B_{AQ}^{x})\) corresponding to \(\sum_{u \in \mathcal{O}^{x}} \sigma_{\kappa}(u) v_{\kappa}\). We then have \(P_{\chi}(f'; \gamma_{0}') \neq 0\).

Next let us find \(f \in S_{\kappa}(\Gamma_{0}(D))\) such that \(P_{\chi}(f; \gamma_{0}) \neq 0\). We view \(f\) as a modular form on the complex upper half plane. A direct calculation shows that the non-vanishing of \(P_{\chi}(f; \gamma_{0})\) is reduced to that of

\[
\begin{cases} 
f(\sqrt{-1}) & (D = 1), \\
f(\frac{1+i\sqrt{-1}}{2}) & (D = 2).
\end{cases}
\]

When \(D = 1\), set

\[
f = \begin{cases} 
\Delta^{\kappa/12} & (\kappa \equiv 0 \mod 12), \\
\Delta^{(\kappa-4)/12} E_{4} & (\kappa \equiv 4 \mod 12), \\
\Delta^{(\kappa-8)/12} E_{4}^{2} & (\kappa \equiv 8 \mod 12),
\end{cases}
\]

where \(\Delta\) denotes the Ramanujan delta function and \(E_{4}\) the Eisenstein series of weight 4. We then have \(P_{\chi}(f; \gamma_{0}) \neq 0\). When \(D = 2\), set

\[f = \left(\frac{\eta^{16}(2z)}{\eta^{8}(z)}\right)^{\kappa/4}\]

with the Dedekind eta function \(\eta\). Since \(\eta^{16}(2z)/\eta^{8}(z) \in S_{4}(\Gamma_{0}(2))\) (cf. [C, §2.1]) and \(\eta(z)\) has no zero on the upper half plane, we have \(f \in S_{\kappa}(\Gamma_{0}(2))\) and \(P_{\chi}(f; \gamma_{0}) \neq 0\).

Remark 3.3. The level raising of the modular forms of level \(D = 1\) introduced above to forms of level 2 also yields modular forms with non-vanishing toral integrals.
References


Atsushi Murase:
Department of Mathematical Sciences, Faculty of Science,
Kyoto Sangyo University, Motoyama, Kamigamo, Kita-ku, Kyoto 603-8555, Japan.
E-mail: murase@cc.kyoto-su.ac.jp

Hiro-aki Narita:
Department of Mathematics, Kumamoto University,
Kurokami, Kumamoto 860-8555, Japan
E-mail: narita@sci.kumamoto-u.ac.jp