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<th>タイトル</th>
<th>ラプラス変換の実逆変換への再生核空間の応用 (II) (再生核の応用についての研究)</th>
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<tbody>
<tr>
<td>著者</td>
<td>Kajino, Naotaka; Sawano, Yoshihiro; Fujiwara, Hiroshi</td>
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<tr>
<td>引用文献</td>
<td>数理解析研究所講究録 (2008), 1618: 189-191</td>
</tr>
<tr>
<td>発行日</td>
<td>2008-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140182">http://hdl.handle.net/2433/140182</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
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Kyoto University
1 Introduction

The present paper contains new results on the modified Laplace transform
\[ \mathcal{L}f(p) = pL(f(p) = \int_0^\infty p e^{-pt} f(t) \, dt. \]

The results are intended to publish elsewhere.

2 Preliminaries

Theorem 2.1. The following estimates hold.

\[ \sup_{p>0} |\mathcal{L}f(p)| \leq \sup_{t>0} |f(t)| \]
\[ \int_0^\infty |\mathcal{L}f(p)| \, dp \leq \int_0^\infty \frac{|f(t)|}{t^2} \, dt. \]

If we interpolate the results above, we obtain the following inequality.

Theorem 2.2.

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}. \]

Proof. By using the distribution function, we have

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp = 4 \int_0^\infty \lambda \{p > 0 : |\mathcal{L}f(p)| > 2\lambda \} \, d\lambda. \]
By the $L^\infty$-estimate, we obtain

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \lambda \{ p > 0 : |\mathcal{L}[\chi_{\{|f|\leq \lambda\}} f](p)| > \lambda \} \, d\lambda. \]

Next, we invoke the $L^1$-estimate and the Chebychev inequality. The result is

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \left( \int_0^\infty |\mathcal{L}[\chi_{\{|f|\leq \lambda\}} f](p)| \, dp \right) \, d\lambda. \]

Having used up our estimates which were already proved, we have only to calculate the integral elaborately.

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \left( \int_0^\infty \chi_{\{|f(t)|\leq \lambda\}} |f(t)| \frac{dt}{t^2} \right) \, d\lambda \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}. \]

The power 2 is best possible in the following sense.

**Example 2.3.** Let us establish that

\[ \int_\mathbb{R} |\mathcal{L}f(p)|^2 \, dp \leq \int_\mathbb{R} |f(t)|^2 \frac{dt}{t^{1+\beta}} \]

fails for $0 < \beta < 1$. Take $\alpha \in \mathbb{R}$ so that $\frac{\beta}{2} < \alpha < \frac{1}{2}$. Then

\[ f_\alpha(x) = (\chi_{[0,1]}(x)x)^\alpha \]

satisfies

\[ \mathcal{L}f_\alpha(p) = p \int_0^1 t^\alpha e^{-t} \, dt = p \int_0^p (p^{-1}s)^\alpha e^{-s} \, d(p^{-1}s) \approx p^{-\alpha} \]

as $p \to \infty$. As a result, we have $f_\alpha \in L^2 \left( (0, \infty), \frac{dt}{t^{1+\beta}} \right), 0 < \beta < \alpha$, while $\mathcal{L}f \notin L^2(0, \infty)$.

In the rest of this paper, we consider

\[ H_K = \{ f : [0, \infty) \to [0, \infty) : f(0) = 0, f \text{ is absolutely continuous and } \|f\|_{H_K} < \infty \}, \]

where the norm is given by

\[ \|f\|_{H_K} = \left( \int_0^\infty |f'(t)|^2 \frac{e^t \, dt}{t} \right)^{\frac{1}{2}}. \]

To prove that $\mathcal{L}$ is compact, we have only to establish the following.

**Theorem 2.4.** $H_K \subset L^2 \left( (0, \infty), \frac{dt}{t^2} \right)$ in the sense of compact embedding.
Proof. This is because

\[ H_K \subset L^\infty((0, \infty), \max(|t|^{-1}, 1)) \]

is a continuous embedding and

\[ L^\infty((0, \infty), \max(|t|^{-1}, 1)) \subset L^2((0, \infty), \frac{dt}{t^2}) \]

is a compact embedding. \qed