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ラプラス変換の実逆変換への再生核空間の応用II

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1 Introduction

The present paper contains new results on the modified Laplace transform

$$\mathcal{L}f(p) = p\,Lf(p) = \int_0^\infty p\,e^{-p\,t} f(t)\,dt.$$

The results are intended to publish elsewhere.

2 Preliminaries

Theorem 2.1. *The following estimates hold.*

$$\sup_{p>0} |\mathcal{L}f(p)| \le \sup_{t>0} |f(t)|$$

$$\int_0^\infty |\mathcal{L}f(p)|\,dp \le \int_0^\infty \frac{|f(t)|}{t^2}\,dt.$$

If we interpolate the results above, we obtain the following inequality.

Theorem 2.2.

$$\int_0^\infty |\mathcal{L}f(p)|^2\,dp \le 4\int_0^\infty |f(t)|^2\,\frac{dt}{t^2}.$$

Proof. By using the distribution function, we have

$$\int_0^\infty |\mathcal{L}f(p)|^2\,dp = 4\int_0^\infty \lambda\,|\{p>0\,:\,|\mathcal{L}f(p)|>2\lambda\}|\,d\lambda.$$

By the L^∞-estimate, we obtain

$$\int_0^\infty |\mathcal{L}f(p)|^2\,dp \le 4\int_0^\infty \lambda\,|\{p>0\,:\,|\mathcal{L}[\chi_{\{|f|\le\lambda\}}f](p)|>\lambda\}|\,d\lambda.$$

Next, we invoke the L^1-estimate and the Chebychev inequality. The result is

$$\int_0^\infty |\mathcal{L}f(p)|^2\,dp \le 4\int_0^\infty \left(\int_0^\infty |\mathcal{L}[\chi_{\{|f|\le\lambda\}}f](p)|\,dp\right)\,d\lambda.$$

Having used up our estimates which were already proved, we have only to calculate the integral elaborately.

$$\int_0^\infty |\mathcal{L}f(p)|^2\,dp \le 4\int_0^\infty \left(\int_0^\infty \chi_{\{|f(t)|\le\lambda\}}|f(t)|\,\frac{dt}{t^2}\right)\,d\lambda \le 4\int_0^\infty |f(t)|^2\,\frac{dt}{t^2}.$$

□

The power 2 is best possible in the following sense.

Example 2.3. Let us establish that

$$\int_{\mathbf{R}} \mathcal{L}f(p)^2\,dp \lesssim \int_{\mathbf{R}} |f(t)|^2\frac{dt}{t^{1+\beta}}$$

fails for $0<\beta<1$. Take $\alpha\in\mathbf{R}$ so that $\frac{\beta}{2}<\alpha<\frac{1}{2}$. Then

$$f_\alpha(x)=(\chi_{[0,1]}(x)x)^\alpha$$

satisfies

$$\begin{aligned}\mathcal{L}f_\alpha(p) &= p\int_0^1 t^\alpha e^{-tp}\,dt\\ &= p\int_0^p (p^{-1}s)^\alpha e^{-s}\,d(p^{-1}s)\\ &\simeq p^{-\alpha}\end{aligned}$$

as $p\to\infty$. As a result, we have $f_\alpha\in L^2\left((0,\infty),\frac{dt}{t^{1+\beta}}\right)$, $0<\beta<\alpha$, while $\mathcal{L}f\notin L^2(0,\infty)$.

In the rest of this paper, we consider

$$H_K=\{f:[0,\infty)\to[0,\infty)\,:\,f(0)=0,\ f \text{ is absolutely continuous and } \|f\|_{H_K}<\infty\},$$

where the norm is given by

$$\|f\|_{H_K}=\left(\int_0^\infty |f'(t)|^2\,\frac{e^t\,dt}{t}\right)^{\frac{1}{2}}.$$

To prove that $\mathcal{L}$ is compact, we have only to establish the following.

Theorem 2.4. *$H_K\subset L^2\left((0,\infty),\frac{dt}{t^2}\right)$ in the sense of compact embedding.*

Proof. This is because

$$H_K \subset L^\infty((0,\infty), \max(|t|^{-1}, 1))$$

is a continuous embedding and

$$L^\infty((0,\infty), \max(|t|^{-1}, 1)) \subset L^2\left((0,\infty), \frac{dt}{t^2}\right)$$

is a compact embedding. □