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<tr>
<td>タイトル</td>
<td>ラプラス変換の実逆変換への再生核空間の応用（II）再生核の応用についての研究</td>
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<tr>
<td>作者</td>
<td>Kajino, Naotaka; Sawano, Yoshihiro; Fujiwara, Hiroshi</td>
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<tr>
<td>引用</td>
<td>数理解析研究所講究録（2008）1618: 189-191</td>
</tr>
<tr>
<td>発行日</td>
<td>2008-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/140182">http://hdl.handle.net/2433/140182</a></td>
</tr>
<tr>
<td>型</td>
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1 Introduction

The present paper contains new results on the modified Laplace transform

\[ \mathcal{L}f(p) = pLf(p) = \int_0^\infty p e^{-pt} f(t) \, dt. \]

The results are intended to publish elsewhere.

2 Preliminaries

**Theorem 2.1.** The following estimates hold.

\[
\sup_{p > 0} |\mathcal{L}f(p)| \leq \sup_{t > 0} |f(t)| \\
\int_0^\infty |\mathcal{L}f(p)| \, dp \leq \int_0^\infty \frac{|f(t)|}{t^2} \, dt.
\]

If we interpolate the results above, we obtain the following inequality.

**Theorem 2.2.**

\[
\int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \frac{|f(t)|^2}{t^2} \, dt.
\]

**Proof.** By using the distribution function, we have

\[
\int_0^\infty |\mathcal{L}f(p)|^2 \, dp = 4 \int_0^\infty \lambda \delta \{ p > 0 : |\mathcal{L}f(p)| > 2\lambda \} \, d\lambda.
\]
By the $L^\infty$-estimate, we obtain
\[
\int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \lambda |\{p > 0 : |\mathcal{L}\chi_{\{|f|\leq \lambda\}}f(p)| > \lambda\}| \, d\lambda.
\]
Next, we invoke the $L^1$-estimate and the Chebychev inequality. The result is
\[
\int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \left( \int_0^\infty |\mathcal{L}\chi_{\{|f|\leq \lambda\}}f(p)| \, dp \right) \, d\lambda.
\]
Having used up our estimates which were already proved, we have only to calculate the integral elaborately.
\[
\int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty \left( \int_0^\infty \chi_{\{|f(t)|\leq \lambda\}} |f(t)| \frac{dt}{t^2} \right) \, d\lambda \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}.
\]

The power 2 is best possible in the following sense.

Example 2.3. Let us establish that
\[
\int_\mathbb{R} |\mathcal{L}f(p)|^2 \, dp \leq \int_\mathbb{R} |f(t)|^2 \frac{dt}{t^{1+\beta}}
\]
fails for $0 < \beta < 1$. Take $\alpha \in \mathbb{R}$ so that $\frac{\beta}{2} < \alpha < \frac{1}{2}$. Then
\[
f_{\alpha}(x) = (\chi_{[0,1]}(x)x)^\alpha
\]
satisfies
\[
\mathcal{L}f_{\alpha}(p) = p \int_0^1 t^\alpha e^{-tp} \, dt
\]

\[
= p \int_0^1 (p^{-1}s)^\alpha e^{-s} \, d(p^{-1}s)
\]

\[
\simeq p^{-\alpha}
\]
as $p \to \infty$. As a result, we have $f_{\alpha} \in L^2 \left((0, \infty), \frac{dt}{t^{1+\beta}}\right), 0 < \beta < \alpha$, while $\mathcal{L}f \notin L^2(0, \infty)$.

In the rest of this paper, we consider
\[
H_K = \{f : [0, \infty) \to [0, \infty) : f(0) = 0, f \text{ is absolutely continuous and } \|f\|_{H_K} < \infty\},
\]
where the norm is given by
\[
\|f\|_{H_K} = \left( \int_0^\infty |f'(t)|^2 \frac{e^t \, dt}{t} \right)^{\frac{1}{2}}.
\]

To prove that $\mathcal{L}$ is compact, we have only to establish the following.

Theorem 2.4. $H_K \subset L^2 \left((0, \infty), \frac{dt}{t^2}\right)$ in the sense of compact embedding.
Proof. This is because
\[ H_K \subset L^\infty((0, \infty), \max(|t|^{-1}, 1)) \]
is a continuous embedding and
\[ L^\infty((0, \infty), \max(|t|^{-1}, 1)) \subset L^2 \left( (0, \infty), \frac{dt}{t^2} \right) \]
is a compact embedding. \qed