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1 Introduction

The present paper contains new results on the modified Laplace transform

\[ \mathcal{L}f(p) = p Lf(p) = \int_0^\infty p e^{-pt} f(t) \, dt. \]

The results are intended to publish elsewhere.

2 Preliminaries

**Theorem 2.1.** The following estimates hold.

\[ \sup_{p > 0} |\mathcal{L}f(p)| \leq \sup_{t > 0} |f(t)| \]
\[ \int_0^\infty |\mathcal{L}f(p)| \, dp \leq \int_0^\infty \frac{|f(t)|}{t^2} \, dt. \]

If we interpolate the results above, we obtain the following inequality.

**Theorem 2.2.**

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp \leq 4 \int_0^\infty |f(t)|^2 \, \frac{dt}{t^2}. \]

**Proof.** By using the distribution function, we have

\[ \int_0^\infty |\mathcal{L}f(p)|^2 \, dp = 4 \int_0^\infty \lambda |\{p > 0 : |\mathcal{L}f(p)| > 2\lambda\}| \, d\lambda. \]
By the $L^\infty$-estimate, we obtain
\[
\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \lambda |\{p > 0 : |\mathcal{L}[\chi_{\{|f| \leq \lambda\}}f](p)| > \lambda\}| d\lambda.
\]

Next, we invoke the $L^1$-estimate and the Chebychev inequality. The result is
\[
\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \left( \int_0^\infty |\mathcal{L}[\chi_{\{|f| \leq \lambda\}}f](p)| dp \right) d\lambda.
\]

Having used up our estimates which were already proved, we have only to calculate the integral elaborately.
\[
\int_0^\infty |\mathcal{L}f(p)|^2 dp \leq 4 \int_0^\infty \left( \int_0^\infty \chi_{\{|f(t)| \leq \lambda\}}|f(t)| \frac{dt}{t^2} \right) d\lambda \leq 4 \int_0^\infty |f(t)|^2 \frac{dt}{t^2}.
\]

The power 2 is best possible in the following sense.

**Example 2.3.** Let us establish that
\[
\int_\mathbb{R} \mathcal{L}f(p)^2 dp \leq \int_\mathbb{R} |f(t)|^2 \frac{dt}{t^{1+\beta}}
\]
fails for $0 < \beta < 1$. Take $\alpha \in \mathbb{R}$ so that $\frac{\beta}{2} < \alpha < \frac{1}{2}$. Then
\[
f_\alpha(x) = (\chi_{[0,1]}(x)x)^\alpha
\]
satisfies
\[
\mathcal{L}f_\alpha(p) = p \int_0^1 t^\alpha e^{-tp} dt
\]
\[
= p \int_0^p (p^{-1}s)^{\alpha} e^{-s} d(p^{-1}s)
\]
\[
\simeq p^{-\alpha}
\]
as $p \to \infty$. As a result, we have $f_\alpha \in L^2 \left( (0, \infty), \frac{dt}{t^{1+\beta}} \right)$, $0 < \beta < \alpha$, while $\mathcal{L}f \notin L^2(0, \infty)$.

In the rest of this paper, we consider
\[\mathcal{H}_K = \{ f : [0, \infty) \to [0, \infty) : f(0) = 0, f \text{ is absolutely continuous and } \|f\|_{\mathcal{H}_K} < \infty \},\]
where the norm is given by
\[
\|f\|_{\mathcal{H}_K} = \left( \int_0^\infty |f'(t)|^2 \frac{e^t dt}{t} \right)^{\frac{1}{2}}.
\]

To prove that $\mathcal{L}$ is compact, we have only to establish the following.

**Theorem 2.4.** $\mathcal{H}_K \subset L^2 \left( (0, \infty), \frac{dt}{t^2} \right)$ in the sense of compact embedding.
Proof. This is because

\[ H_K \subset L^\infty((0, \infty), \max(|t|^{-1}, 1)) \]

is a continuous embedding and

\[ L^\infty((0, \infty), \max(|t|^{-1}, 1)) \subset L^2((0, \infty), \frac{dt}{t^2}) \]

is a compact embedding. \qed