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The Gap Equation in the BCS-Bogoliubov Theory of Superconductivity from the Viewpoint of an Integral Transform

超伝導におけるギャップ方程式と積分変換

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1 Introduction

Since the surprising discovery by Onnes that the electrical resistivity of mercury drops to zero below the temperature 4.2 K in 1911, the zero electrical resistivity is observed in many metals and alloys. Such a phenomenon is called superconductivity. In 1957 Bardeen, Cooper and Schrieffer [2] proposed the highly successful quantum theory of superconductivity, called the BCS theory. In 1958 Bogoliubov [4] obtained the results similar to those in the BCS theory using the canonical transformation called the Bogoliubov transformation. The theory by Bardeen, Cooper, Schrieffer and Bogoliubov is called the BCS-Bogoliubov theory.

The BCS-Bogoliubov theory also explains the experimental fact that it takes a finite energy to excite a quasi particle from the superconducting ground state to an upper energy state. This experimental fact implies the existence of an energy gap in the spectrum of the Hamiltonian. In the BCS-Bogoliubov theory, this energy gap results from the existence of the electron pairs called the Cooper pairs and is described in terms of the gap function. The gap function, denoted by $\Delta_k(T)$ (≥ 0), is a function both of the temperature $T \geq 0$ and of wave vector $k \in \mathbb{R}^3$ of an electron and satisfies, in the BCS-Bogoliubov theory, the following nonlinear equation called the gap equation (1.1) below. Let $k_B > 0$ and $\omega_D > 0$ stand for the Boltzmann constant and the Debye frequency, respectively. We denote Planck's constant by $h > 0$ and set $\hbar = h/(2\pi)$. Let $m > 0$ and $\mu > 0$ stand for the electron mass and the chemical potential, respectively. Set $\xi_k = \hbar^2|k|^2/(2m) - \mu$, which corresponds to the kinetic energy of an electron with wave vector k. The gap equation reads as follows :

$$(1.1) \qquad \Delta_k(T) = -\frac{1}{2}\sum_{k'} \frac{U_{k,k'}\,\Delta_{k'}(T)}{\sqrt{\xi_{k'}^2 + \Delta_{k'}(T)^2}} \tanh \frac{\sqrt{\xi_{k'}^2 + \Delta_{k'}(T)^2}}{2k_BT},$$

where $k' \in \mathbb{R}^3$ denotes wave vector and the potential $U_{k,k'}$ is a function of k and k' satisfying $U_{k,k'} \le 0$. In this connection, see [9] for a new gap equation of superconductivity.

The BCS-Bogoliubov theory makes the assumption that there is a unique solution with some nice properties such as continuity and smoothness to the gap equation (1.1). The sum in (1.1) is often replaced by an integral, and accordingly the gap equation is often regarded as a nonlinear integral equation. In such a situation, Odeh [7] and Billard and Fano [3] established the existence and uniqueness of the positive solution (the gap function) to the gap equation in the case $T = 0$. In the case $T \ge 0$, Vansevenant [8] and Yang [10] determined the transition temperature and showed that there is a unique positive solution to the gap equation. Recently, Hainzl, Hamza, Seiringer and Solovej [5], and Hainzl and Seiringer [6] proved that the existence of a positive solution to the gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature.

In the results just above, spaces of functions of wave vector only are dealt with. But, in this paper, we deal with a certain Banach space of continuous functions both of the temperature and of wave vector, and regard the gap function as an element of the Banach space and consider the gap equation as a nonlinear integral equation on the Banach space.

The BCS-Bogoliubov theory also makes the assumption that the solution: $T \mapsto \Delta_k(T)$ with k fixed is of class C^2 with respect to T. But a mathematical proof of this statement has not been given yet as far as we know. In this paper we first show that there is a unique solution of class C^2 (with respect to T) to the simplified gap equation (2.3) below and point out some more properties of the solution. We then give another proof that there is a unique solution to the gap equation on the basis of the Schauder theorem, and show that the solution is continuous with respect to both T and k.

2 The solution to the simplified gap equation

Suppose that $U_{k,k'}$ is given by (see [2])

$$U_{k,k'} = \begin{cases} -U_0 & (|\xi_k| \le \hbar\omega_D \text{ and } |\xi_{k'}| \le \hbar\omega_D), \\ 0 & (\text{otherwise}), \end{cases} \tag{2.1}$$

where $U_0 > 0$ is a constant. Then $\Delta_k(T)$ depends only on the temperatur T when $|\xi_k| \le \hbar\omega_D$, whereas $\Delta_k(T) = 0$ when $|\xi_k| > \hbar\omega_D$. Let $|\xi_k| \le \hbar\omega_D$. Then (1.1) leads to

$$1 = \frac{U_0}{2} \sum_{k' \ (|\xi_{k'}| \le \hbar\omega_D)} \frac{1}{\sqrt{\xi_{k'}^2 + \Delta(T)^2}} \tanh \frac{\sqrt{\xi_{k'}^2 + \Delta(T)^2}}{2k_B T}. \tag{2.2}$$

Here the symbol k' $(|\xi_{k'}| \le \hbar\omega_D)$ stands for k' satisfying $|\xi_{k'}| \le \hbar\omega_D$, and the gap function $\Delta_k(T)$ is denoted by $\Delta(T)$ simply because it does not depend on k when k satisfies

$|\xi_k| \le \hbar\omega_D$. Accordinly, in this case, the gap function $\Delta(T)$ becomes a function of the temperature T only.

We now replace the sum in (2.2) by the following integral (see [2]):

$$1 = \frac{U_0 N_0}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2 + \Delta(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta(T)^2}}{2k_B T}\, d\xi, \tag{2.3}$$

where $N_0 > 0$ stands for the density of states per unit energy at the Fermi surface. The simplified gap equation (2.3) as well as the hypothesis (2.1) is accepted widely in condensed matter physics (see e.g. [2] and [11, (11.45), p.392]).

It is well known that superconductivity occurs at temperatures below the temperature called the transition temperature (the critical temperature). Let us now define it.

Definition 2.1 ([2]). The transition temperature is the temperature $T_c^{smpl} > 0$ satisfying

$$\frac{1}{U_0 N_0} = \int_0^{\hbar\omega_D/(2k_B T_c^{smpl})} \frac{\tanh \eta}{\eta}\, d\eta \,.$$

Remark 2.2. The temperature T_c^{smpl} is determined uniquely. Its definition originates from the simplified gap equation (2.3) since the equality in Definition 2.1 is rewritten as

$$1 = \frac{U_0 N_0}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{1}{\sqrt{\xi^2}} \tanh \frac{\sqrt{\xi^2}}{2k_B T_c^{smpl}}\, d\xi \,,$$

which is obtained by setting $\Delta(T) = 0$ and $T = T_c^{smpl}$ in the simplified gap equation (2.3).

As mentioned above, in the BCS-Bogoliubov theory, it is assumed that there is a unique solution : $T \mapsto \Delta(T)$ to the simplified gap equation (2.3) and that it is of class C^2 on the interval $[\,0,\, T_c^{smpl}\,)$. One of our main results is the following, which gives a mathematical proof of this statement and points out some more properties of the solution. Let

$$\Delta_0 = \frac{\hbar\omega_D}{\sinh \frac{1}{U_0 N_0}} \,. \tag{2.4}$$

Proposition 2.3. *Let Δ_0 be as in (2.4). Then there is a unique nonnegative solution : $T \mapsto \Delta(T)$ to the simplified gap equation (2.3) such that the solution, i.e., the gap function is continuous and monotonically decreasing on the closed interval $[\,0,\, T_c^{smpl}\,]$:*

$$\Delta(0) = \Delta_0 > \Delta(T_1) > \Delta(T_2) > \Delta(T_c^{smpl}) = 0, \qquad 0 < T_1 < T_2 < T_c^{smpl}.$$

Moreover, it is of class C^2 on the interval $[\,0,\, T_c\,)$ and satisfies

$$\Delta'(0) = \Delta''(0) = 0 \quad \textit{and} \quad \lim_{T \uparrow T_c^{smpl}} \Delta'(T) = -\infty.$$

Remark 2.4. We set $\Delta(T) = 0$ for $T \ge T_c^{smpl}$.

3 Proof of Proposition 2.3

Let

$$h(T,\,Y,\,\xi) = \begin{cases} \dfrac{1}{\sqrt{\xi^2+Y}}\tanh\dfrac{\sqrt{\xi^2+Y}}{2k_BT} & (0<T\le T_c^{smpl}\,, \quad Y\ge 0), \\ \dfrac{1}{\sqrt{\xi^2+Y}} & (T=0, \quad Y>0) \end{cases}$$

and set

$$F(T,\,Y) = \int_0^{\hbar\omega_D} h(T,\,Y,\,\xi)\,d\xi - \frac{1}{U_0N_0}\,. \tag{3.1}$$

We consider the function F on the following domain $W \subset \mathbb{R}^2$:

$$W = W_1 \cup W_2 \cup W_3 \cup W_4\,,$$

where

$$\begin{aligned} W_1 &= \left\{(T,\,Y)\in\mathbb{R}^2 :\ 0<T<T_c^{smpl}\,,\ 0<Y<2\,\Delta_0^2\right\}, \\ W_2 &= \left\{(0,\,Y)\in\mathbb{R}^2 :\ 0<Y<2\,\Delta_0^2\right\}, \\ W_3 &= \left\{(T,\,0)\in\mathbb{R}^2 :\ 0<T\le T_c^{smpl}\right\}, \\ W_4 &= \left\{(T_c^{smpl}\,,\,Y)\in\mathbb{R}^2 :\ 0<Y<2\,\Delta_0^2\right\}. \end{aligned}$$

Here, Δ_0 is that in (2.4).

Remark 3.1. The simplified gap equation (2.3) is rewritten as $F(T,\,Y)=0$, where Y corresponds to $\Delta(T)^2$.

A straightforward calculation gives the following.

Lemma 3.2. *The function F is of class C^1 on W, and at each $(T,\,Y)\in W\setminus W_2$,*

$$\frac{\partial F}{\partial T}(T,\,Y)<0, \qquad \frac{\partial F}{\partial Y}(T,\,Y)<0.$$

Lemma 3.3. *The function F is of class C^2 on W_1 .*

Remark 3.4. One may prove Proposition 2.3 on the basis of the implicit function theorem in its well-known form. In this case, *an interior point* $(T_0,\,Y_0)$ of the domain W satisfying $F(T_0,\,Y_0)=0$ need to exist. But there are the two points $(0,\,\Delta_0^2)$ and $(T_c^{smpl}\,,\,0)$ in *the boundary* of W satisfying

$$F(0,\,\Delta_0^2) = F(T_c^{smpl}\,,\,0) = 0. \tag{3.2}$$

So one can not apply the implicit function theorem in its well-known form.

Lemma 3.5. *There is a unique nonnegative solution: $T \mapsto Y = f(T)$ to the gap equation $F(T,\, Y) = 0$ such that the function f is continuous on the closed interval $[0,\, T_c^{smpl}]$ and satisfies $f(0) = \Delta_0^2$ and $f(T_c^{smpl}) = 0$.*

Proof. By Lemma 3.2 and (3.2), the function: $Y \mapsto F(T_c^{smpl}\,,\, Y)$ is monotonically decreasing and there is a $Y_1 \quad (0 < Y_1 < 2\Delta_0^2)$ satisfying $F(T_c^{smpl}\,,\, Y_1) < 0$. Note that Y_1 is arbitrary as long as $0 < Y_1 < 2\Delta_0^2$. Hence, by Lemma 3.2, there is a $T_1 \quad (0 < T_1 < T_c^{smpl})$ satisfying $F(T_1\,,\, Y_1) < 0$. Hence, $F(T,\, Y_1) < 0$ for $T_1 \le T \le T_c^{smpl}$. On the other hand, the function: $T \mapsto F(T,\, 0)$ is monotonically decreasing and there is a $T_2 \quad (0 < T_2 < T_c^{smpl})$ satisfying $F(T_2\,,\, 0) > 0$. Note that T_2 is arbitrary as long as $0 < T_2 < T_c^{smpl}$. Hence, $F(T,\, 0) > 0$ for $T_2 \le T < T_c^{smpl}$.

Let $\max(T_1\,,\, T_2) \le T < T_c^{smpl}$ and fix T. It then follows from Lemma 3.2 that the function: $Y \mapsto F(T,\, Y)$ with T fixed is monotonically decreasing on $[0,\, Y_1]$. Since $F(T,\, 0) > 0$ and $F(T,\, Y_1) < 0$, there is a unique Y $(0 < Y < Y_1)$ satisfying $F(T,\, Y) = 0$. When $T = T_c^{smpl}$, there is a unique value $Y = 0$ satisfying $F(T_c^{smpl}\,,\, Y) = 0$ (see (3.2)).

Since F is continuous on W, there is a unique solution: $T \mapsto Y = f(T)$ to the gap equation $F(T,\, Y) = 0$ such that the function f is continuous on $[\max(T_1\,,\, T_2),\, T_c^{smpl}]$ and $f(T_c^{smpl}) = 0$.

Since $(\partial F/\partial Y)(0,\, Y) < 0$ $(0 < Y < 2\Delta_0^2)$, there is a unique value $Y = \Delta_0^2$ satisfying $F(0,\, Y) = 0$. Lemma 3.2 therefore implies that the function f is continuous on $[0,\, T_c^{smpl}]$ and satisfies $f(0) = \Delta_0^2$ and $f(T_c^{smpl}) = 0$. □

Proposition 2.3 thus follows from Lemmas 3.2, 3.3 and 3.5.

4 The gap equation from the viewpoint of a nonlinear integral transform

Set $x,\, \xi = \hbar^2|k|^2/(2m) - \mu$. Here we denote by x (or ξ) the kinetic energy of an electron with wave vector k. Suppose that the gap function is a function both of the temperature T and of $x \in \mathbb{R}$ and that the gap function is an even function with respect to x. Suppose further that the potential in (1.1) is a function of x and ξ. We then denote the gap function by $u(T,\, x)$. Set $a = \hbar\omega_D$ for simplicity. The gap equation then reads as follows :

$$u(T,\, x) = \int_\varepsilon^a \frac{U(x,\, \xi)\, u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}}\, \tanh \frac{\sqrt{\xi^2 + u(T,\, \xi)^2}}{2k_B T}\, d\xi, \tag{4.1}$$

where, $U(x,\, \xi)$ stands for the product of the potential in (1.1) and $-N_0$. Here we denote by N_0 the density of states per unit energy at the Fermi surface, and we let $\varepsilon > 0$ be small enough and fix it $(0 < \varepsilon < a)$.

We denote the right side of (4.1) by $Au(T,\, x)$. Then the map A is regarded as a nonlinear integral transform, and the gap function becomes a fixed point of A.

In the next section we deal with a certain Banach space and show that there is a unique fixed point of A on the basis of the Schauder theorem.

5 The solution to the gap equation

Let T_1, $T_2 > 0$ be small enough and let $(0 <)T_1 < T_2$. Set $K = [T_1, T_2] \times [\varepsilon, a] \subset \mathbb{R}^2$. Let $0 < U_1 < U_2$. We assume the following :

$$(5.1) \qquad U_1 \le U(x, \xi) \le U_2 \quad \text{for} \quad (x, \xi) \in [\varepsilon, a]^2, \qquad \text{and} \quad U(\cdot, \cdot) \in C^1([\varepsilon, a]^2).$$

Remark 5.1. In previous results [5, 6, 8, 10], spaces of functions of wave vector only are dealt with. But, in this paper, we deal with the Banach space $C(K)$ of continuous functions both of the temperature and of wave vector, and regard A as a nonlinear map from a certain subset of $C(K)$ into itself.

On one hand, let $U(x, \xi) = U_1$ on $[\varepsilon, a]^2$. Then an argument similar to that in sections 2 and 3 gives that there is a unique transition temperature $T_c^{smpl}(1) > 0$ and that there is a unique nonnegative solution : $T \mapsto \Delta_1(T)$ to the simplified gap equation

$$(5.2) \qquad 1 = U_1 \int_\varepsilon^a \frac{1}{\sqrt{\xi^2 + \Delta_1(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_1(T)^2}}{2k_B T} \, d\xi, \qquad 0 \le T \le T_c^{smpl}(1).$$

We let $\Delta_1(T) = 0$ for $T \ge T_c^{smpl}(1)$.

On the other hand, let $U(x, \xi) = U_2$ on $[\varepsilon, a]^2$. Then a similar argument gives that there is a unique transition temperature $T_c^{smpl}(2) > 0$ and that there is a unique nonnegative solution : $T \mapsto \Delta_2(T)$ to the simplified gap equation

$$(5.3) \qquad 1 = U_2 \int_\varepsilon^a \frac{1}{\sqrt{\xi^2 + \Delta_2(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_2(T)^2}}{2k_B T} \, d\xi, \qquad 0 \le T \le T_c^{smpl}(2).$$

We let $\Delta_2(T) = 0$ for $T \ge T_c^{smpl}(2)$.

Remark 5.2. The solutions $T \mapsto \Delta_1(T)$ and $T \mapsto \Delta_2(T)$ both satisfy properties similar to those in Proposition 2.3.

A straightforward calculation gives the following.

Lemma 5.3. *The inequality $T_c^{smpl}(1) < T_c^{smpl}(2)$ holds. Moreover, $\Delta_1(T) < \Delta_2(T)$ $(0 \le T < T_c^{smpl}(2))$, and $\Delta_1(T) = \Delta_2(T) = 0 \quad (T \ge T_c^{smpl}(2))$.*

Let

$$V = \{u \in C(K) : \ \Delta_1(T) \le u(T, x) \le \Delta_2(T) \quad \text{for} \quad (T, x) \in K\}$$

and let V be equicontinuous. The definition of V immediately implies that V is uniformly bounded since

$$\sup_{(T,\,x)\in K} |\, u(T,\, x)\,| \le \Delta_2(T_1).$$

By the Ascoli–Arzelà theorem, V is relatively compact.

Lemma 5.4. *The closure $\overline{V} \subset C(K)$ is compact and convex. Moreover,* $0 \notin \overline{V}$.

Define the map A mentioned above by

$$Au(T,\, x) = \int_\varepsilon^a \frac{U(x,\, \xi)\, u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T,\, \xi)^2}}{2k_B T}\, d\xi, \qquad u \in V. \tag{5.4}$$

A straightforward calculation gives $Au \in C(K)$ for $u \in V$.

Lemma 5.5. *Let $u \in V$. Then*

$$\Delta_1(T) \le Au(T,\, x) \le \Delta_2(T) \quad \textit{for} \quad (T,\, x) \in K.$$

Proof. We show $Au(T,\, x) \le \Delta_2(T)$. Since

$$\frac{u(T,\, \xi)}{\sqrt{\xi^2 + u(T,\, \xi)^2}} \le \frac{\Delta_2(T)}{\sqrt{\xi^2 + \Delta_2(T)^2}}\,,$$

it follows from (5.3) that

$$Au(T,\, x) \le \int_\varepsilon^a \frac{U_2\, \Delta_2(T)}{\sqrt{\xi^2 + \Delta_2(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_2(T)^2}}{2k_B T}\, d\xi = \Delta_2(T).$$

The rest can be shown similarly by (5.2). □

Lemma 5.6. *The set $\{Au :\ u \in V\}$ is equicontinuous.*

Proof. Let $v \in V$ and let $\varepsilon_1 > 0$. Since V be equicontinuous,

$$|\, v(T,\, x) - v(T_0,\, x_0)\,| < \varepsilon_1, \qquad \sqrt{(T - T_0)^2 + (x - x_0)^2} < \delta_1(\varepsilon_1). \tag{5.5}$$

Note that $\delta_1(\varepsilon_1)$ depends on ε_1 only. Then, for $u \in V$,

$$|\, Au(T,\, x) - Au(T_0,\, x_0)\,| \le |\, Au(T,\, x) - Au(T,\, x_0)\,| + |\, Au(T,\, x_0) - Au(T_0,\, x_0)\,|\,.$$

On one hand,

$$|Au(T,\, x) - Au(T,\, x_0)| \le |\, x - x_0\,| \cdot a \sup_{(x,\,\xi)\in[\varepsilon,\, a]^2} \left| \frac{\partial U}{\partial x}(x,\, \xi) \right|.$$

On the other hand, by (5.5),

$$
\begin{aligned}
& |\, Au(T,\, x_0) - Au(T_0,\, x_0)\, | \\
\leq\ & |\, T - T_0\, |\, \frac{a\, U_2 \sqrt{a^2 + \Delta_2(T_1)^2}}{2k_B T_1^2} + U_2 \int_\varepsilon^a \frac{|\, u(T,\, \xi) - u(T_0,\, \xi)\, |}{\xi}\, d\xi \\
\leq\ & |\, T - T_0\, |\, \frac{a\, U_2 \sqrt{a^2 + \Delta_2(T_1)^2}}{2k_B T_1^2} + \varepsilon_1\, U_2 \ln \frac{a}{\varepsilon}\, .
\end{aligned}
$$

Set $b = a \sup\limits_{(x,\,\xi) \in [\varepsilon,\, a]^2} \left| \dfrac{\partial U}{\partial x}(x,\, \xi) \right| + \dfrac{a\, U_2 \sqrt{a^2 + \Delta_2(T_1)^2}}{2k_B T_1^2}$. Then

$$
\begin{aligned}
|\, Au(T,\, x) - Au(T_0,\, x_0)\, | \ &\leq\ b \sqrt{(T - T_0)^2 + (x - x_0)^2} + \varepsilon_1\, U_2 \ln \frac{a}{\varepsilon} \\
&<\ \varepsilon_1 \left(1 + U_2 \ln \frac{a}{\varepsilon} \right),
\end{aligned}
$$

where

$$
\sqrt{(T - T_0)^2 + (x - x_0)^2} < \min\left(\delta_1(\varepsilon_1),\, \varepsilon_1 / b\right).
$$

Replacing $\varepsilon_1 \left(1 + U_2 \ln \dfrac{a}{\varepsilon} \right)$ by an arbitrary positive number ε_2 completes the proof. □

Lemma 5.7. *The map $A : V \longrightarrow V$ is continuous.*

Proof. We have only to show the continuity of A. To this end, let $u,\, v \in V$. Then

$$
\|\, Au - Av\, \| \leq 3\, U_2 \ln \frac{a}{\varepsilon} \cdot \|\, u - v\, \|\, . \tag{5.6}
$$

□

We thus have the following.

Lemma 5.8. *The map $A : \overline{V} \longrightarrow \overline{V}$ is continuous.*

The Schauder theorem then implies the following.

Lemma 5.9. *There is a $u_0 \in \overline{V}$ satisfying $u_0 = Au_0$.*

The element $u_0 \in \overline{V}$ may be a limit point of the set V, and so it is not obvious that Au_0 is of the form (5.4). The following shows that this is the case.

Lemma 5.10. *Let u_0 be as in Lemma 5.9. Then*

$$
Au_0(T,\, x) = \int_\varepsilon^a \frac{U(x,\, \xi)\, u_0(T,\, \xi)}{\sqrt{\xi^2 + u_0(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T,\, \xi)^2}}{2k_B T}\, d\xi.
$$

Proof. For $u_0 \in \overline{V}$, there is a sequence $\{u_n\} \subset V$ satisfying $u_n \to u_0$ in $C(K)$. Hence it follows from (5.6) that $Au_n \to Au_0$ in $C(K)$. Therefore,

$$\begin{aligned} & \left| Au_0(T,\, x) - \int_\varepsilon^a \frac{U(x,\, \xi)\, u_0(T,\, \xi)}{\sqrt{\xi^2 + u_0(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T,\, \xi)^2}}{2k_B T}\, d\xi \right| \\ \leq\ & \|\, Au_0 - Au_n \,\| + U_2 \ln \frac{a}{\varepsilon} \cdot \|\, u_n - u_0 \,\| \,. \end{aligned}$$

□

Remark 5.11. Lemma 5.10 holds not only for $u_0 \in \overline{V}$ but also for every $u \in \overline{V}$.

We now prove the uniqueness of $u_0 \in \overline{V}$.

Lemma 5.12. *There is a unique $u_0 \in \overline{V}$ satisfying $u_0 = Au_0$.*

Proof. We give a proof similar to that of Thereom 24.2 given by Amann [1]. Let $v_0 \in \overline{V}$ satisfy $v_0 = Av_0$. We fix $T = T_0$ $(T_1 \leq T_0 \leq T_2)$. We deal with the case where $u_0(T_0,\, x) < v_0(T_0,\, x)$ for every $\varepsilon \leq x \leq a$. Then there are a number t $(0 < t < 1)$ and a ponit $(T_0,\, x_0) \in K$ such that

$$u_0(T_0,\, x) \geq t\, v_0(T_0,\, x) \quad (\varepsilon \leq x \leq a) \quad \text{and} \quad u_0(T_0,\, x_0) = t\, v_0(T_0,\, x_0). \tag{5.7}$$

Then

$$\begin{aligned} u_0(T_0,\, x_0) &= \int_\varepsilon^a \frac{U(x_0,\, \xi)\, u_0(T_0,\, \xi)}{\sqrt{\xi^2 + u_0(T_0,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T_0,\, \xi)^2}}{2k_B T_0}\, d\xi \\ &\geq \int_\varepsilon^a \frac{U(x_0,\, \xi)\, t\, v_0(T_0,\, \xi)}{\sqrt{\xi^2 + t^2\, v_0(T_0,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + t^2\, v_0(T_0,\, \xi)^2}}{2k_B T_0}\, d\xi \\ &> t \int_\varepsilon^a \frac{U(x_0,\, \xi)\, v_0(T_0,\, \xi)}{\sqrt{\xi^2 + v_0(T_0,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + v_0(T_0,\, \xi)^2}}{2k_B T_0}\, d\xi \\ &= t\, v_0(T_0,\, x_0), \end{aligned}$$

which contradicts (5.7). We can deal with the other cases similarly. Thus $u_0 = v_0$. □

We now state our results as a theorem.

Theorem 5.13. *There is a unique nonnegative $u_0 \in \overline{V} \subset C(K)$ such that u_0 satisfies the gap equation, i.e.,*

$$u_0(T,\, x) = \int_\varepsilon^a \frac{U(x,\, \xi)\, u_0(T,\, \xi)}{\sqrt{\xi^2 + u_0(T,\, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T,\, \xi)^2}}{2k_B T}\, d\xi \,.$$

We are in a position to define the transition temperature T_c .

Definition 5.14. $T_c = \sup\{T > 0 :\ u_0(T,\, x) > 0 \quad (\varepsilon \leq x \leq a)\}$.

Remark 5.15. Definition 5.14 implies that $T_c^{smpl}(1) \le T_c \le T_c^{smpl}(2)$. Our Banach space was $C(K)$, where $K = [T_1, T_2] \times [\varepsilon, a] \subset \mathbb{R}^2$. We now let $T_2 = T_c$. Our Banach space then becomes $C([T_1, T_c] \times [\varepsilon, a])$.

Corollary 5.16. *If $U(x, \xi) = U_1$ for every $(x, \xi) \in [\varepsilon, a]^2$, then $u_0(T, x) = \Delta_1(T)$ and $T_c = T_c(1)$.*

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