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A maximal forcing axiom compatible with weak club guessing

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Abstract

We show there is no maximal forcing axiom compatible with tail club guessing. On the other hand, we may formulate a maximal forcing axiom compatible with a weak club guessing.

Introduction

We formulate a forcing axiom compatible with a tail club guessing in [M]. This note is a continuation to [M]. We show a maximal forcing axiom compatible with tail club guessing does not hold. On the other hand, we may force a maximal forcing axiom compatible with a weak club guessing. Namely, if a ladder system $\langle C_\delta \mid \delta \in A\rangle$ is weak club guessing and a supercompact cardinal exists, then there is a model of set theory where

(1) $\langle C_\delta \mid \delta \in A\rangle$ remains weak club guessing. Namely, for any club D of ω_1, there exists $\delta \in A$ such that $D \cap C_\delta$ is infinite.

(2) Let P be any preorder such that P preserves every stationary subset of ω_1 and that for any $B \subseteq A$ such that the ladder system $\langle C_\delta \mid \delta \in B\rangle$ is weak club guessing, P also preserves the ladder system $\langle C_\delta \mid \delta \in B\rangle$ to be weak club guessing. Then every system $\langle D_i \mid i < \omega_1\rangle$ of dense subsets of P has a filter which hits every D_i.

§1. No maximal forcing axioms are compatible with TCG

We set our notation.

1.1 Definition A sequence $\langle C_\delta \mid \delta \in A\rangle$ is a *ladder system*, if

- $A \subseteq \{\delta \mid \delta < \omega_1, \delta \text{ is a limit ordinal}\}$.
- For every $\delta \in A$, C_δ is a cofinal subset of δ and is of order type ω.

A ladder system $\langle C_\delta \mid \delta \in A\rangle$ is *tail club guessing*, if for any club $D \subseteq \omega_1$, there exists $\delta \in A$ such that $C_\delta \setminus D$ is finite. A ladder system $\langle C_\delta \mid \delta \in A\rangle$ is *weak club guessing*, if for any club $D \subseteq \omega_1$, there exists $\delta \in A$ such that $C_\delta \cap D$ is infinite. Hence if $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing, then it is weak club guessing.

Fix a ladder system $\langle C_\delta \mid \delta \in A\rangle$. We write for small sets and positive sets as follows;

- $(TCG) = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X\rangle \text{ fails to be tail club guessing}\}$.
- $(TCG)^+ = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X\rangle \text{ is tail club guessing}\}$.

Similarly,

- $(WCG) = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X\rangle \text{ fails to be weak club guessing}\}$.
- $(WCG)^+ = \{X \subseteq \omega_1 \mid \langle C_\delta \mid \delta \in A \cap X\rangle \text{ is weak club guessing}\}$.

We know of a forcing axiom which is compatible with tail club guessing.

Theorem. ([M]) Let $\langle C_\delta \mid \delta \in A\rangle$ be tail club guessing. Then we may force the following, assuming that a supercompact cardinal exists.

(1) $\langle C_\delta \mid \delta \in A\rangle$ remains tail club guessing.

(2) Forcing axiom$^+$ holds for the class of partially ordered sets P which are semiproper and $\langle C_\delta \mid \delta \in A\rangle$-$\omega$-semiproper,

□

On the other hand,

1.2 Proposition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system. Let $n_* < \omega$. Then there exists a partially ordered set P and a P-name $\dot{D}$ such that

(1) P is proper and $(TCG)^+$-preserving.
(2) $\Vdash_P$"$\dot{D}$ is a club in ω_1 such that for all $\delta \in A$, $|C_\delta \setminus \dot{D}| \geq n_*$".

□

We make use of this P in two ways. First, we observe TCG-sequences may get killed at limit stages of iterated forcing.

1.3 Corollary. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system. Then there exists an iterated forcing $\langle P_n \mid n < \omega\rangle$ such that

- If $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing, then for all $n < \omega$, $\Vdash_{P_n}$"$\langle C_\delta \mid \delta \in A\rangle$ remains tail club guessing".
- If P_ω is any limit of the P_n's, then $\Vdash_{P_\omega}$"$\langle C_\delta \mid \delta \in A\rangle$ must fail to be tail club guessing".

Second, we put above in terms of forcing axiom. Suppose $\langle C_\delta \mid \delta \in A\rangle$ is tail club guessing. Then no maximal forcing axioms hold for the class of partially ordered sets P which preserve all stationary subsets of ω_1 and all elements of $(TCG)^+$ (i.e. for any $X \subseteq \omega_1$, if $\langle C_\delta \mid \delta \in A \cap X\rangle$ is tail club guessing, then it remains so in the generic extensions of P).

1.4 Corollary. Let $\langle C_\delta \mid \delta \in A\rangle$ be tail club guessing. Let forcing axiom hold for the class of partially ordered sets P such that P are proper and that for any $B \subseteq A$ such that $\langle C_\delta \mid \delta \in B\rangle$ is tail club guessing, then $\Vdash_P$"$\langle C_\delta \mid \delta \in B\rangle$ remains to be tail club guessing". Then we have a contradiction.

□

Proof of proposition 1.2. Let $p \in P$, if $p = (\alpha^p, D^p)$ such that

(1) $\alpha^p < \omega_1$.
(2) $D^p \subseteq \alpha^p + 1$, $\alpha^p \in D^p$ and D^p is closed.
(3) For all $\delta \in A$ with $\delta \leq \alpha^p$, $|C_\delta \setminus D^p| \geq n_*$.

For $p, q \in P$, let $q \leq p$, if

(4) $\alpha^p \leq \alpha^q$.
(5) $D^p = D^q \cap (\alpha^p + 1)$.

Claim. (Dense) For any $p \in P$ and any η with $\alpha^p < \eta < \omega_1$, there exists $q \leq p$ such that $\alpha^q = \eta$.

Proof. Let $\alpha^q = \eta$ and $D^q = D^p \cup \{\eta\}$. Then this $q = (\alpha^q, D^q)$ works.

□

Claim. P is proper and σ-Baire.

Proof. Let θ be a sufficiently large regular cardinal. Let N be a countable elementary substructure of H_θ with $P \in N$. Let $p \in N \cap P$. We want $q \leq p$ such that q is (P, N)-generic. Let $\delta = N \cap \omega_1$. We construct a (P, N)-generic sequence $\langle p_n \mid n < \omega\rangle$ such that $p_0 \leq p$ and that $|(\alpha^{p_0} \cap C_\delta) \setminus D^{p_0}| \geq n_*$. Let $\alpha^q = \delta$ and $D^q = \bigcup\{D^{p_n} \mid n < \omega\} \cup \{\delta\}$. Then this $q = (\alpha^q, D^q)$ works.

□

Claim. If $X \subseteq \omega_1$ such that $\langle C_\delta \mid \delta \in A \cap X\rangle$ is tail club guessing, then $\Vdash_P$"$\langle C_\delta \mid \delta \in A \cap X\rangle$ remains to be tail club guessing".

Proof. Suppose $p \Vdash_P$ "$\dot{C}$ is a club in ω_1". Want $q \leq p$ and $\delta \in A \cap X$ such that $q \Vdash_P$ "$C_\delta \setminus \dot{C}$ is finite". To this end, take an $\in$-chain $\langle N_i \mid i < \omega_1 \rangle$ in H_θ, where θ is sufficiently large. Since $\langle C_\delta \mid \delta \in A \cap X \rangle$ is tail club guessing, there exists $\delta \in A \cap X$ such that $C_\delta \setminus \{N_i \cap \omega_1 \mid i < \omega_1\}$ is finite. By renaming, we have an $\in$-chain $\langle N_n \mid n < \omega \rangle$ in H_θ such that $\{N_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of C_δ. We may assme that $P, p, \dot{C} \in N_0$ and $| C_\delta \cap (\alpha^p, N_0 \cap \omega_1) | \geq n_*$. We construct a descending sequence $\langle q_n \mid n < \omega \rangle$ of conditions such that

- $q_0 \leq p$.
- $| (\alpha^{q_0} \cap C_\delta) \setminus D^{q_0} | \geq n_*$.
- $q_n \in N_{n+1} \cap P$ is (P, N_n)-generic.

Let $\alpha^q = \delta$ and $D^q = \bigcup \{D^{q_n} \mid n < \omega\} \cup \{\delta\}$. Then $q \Vdash_P$ "$N_n \cap \omega_1 \in \dot{C}$" for all $n < \omega$. Hence this q works.

Claim. Let G be P-generic over V. Let $\dot{D} = \bigcup \{D^p \mid p \in G\}$. Then $\dot{D}$ is a club in ω_1 and for all $\delta \in A$, $| C_\delta \setminus \dot{D} | \geq n_*$.

□

Proof of corollary 1.3. Iteratively force clubs D_n in ω_1 so that for all $\delta \in A$ and all $n < \omega$, $| C_\delta \setminus D_n | \geq n$. Then let $D = \bigcap \{D_n \mid n < \omega\}$. If ω_1 gets preserved, then D is a club in ω_1 such that for all $\delta \in A$, $C_\delta \setminus D$ is infinite. Hence $\langle C_\delta \mid \delta \in A \rangle$ fails to be tail club guessing. If ω_1 gets collapsed, then this entails the same conclusion.

Proof of corollary 1.4 is the same. Argue in V and get a sequence $\langle D_n \mid n < \omega \rangle$ of clubs in ω_1.

□

§2. A maximal forcing axiom is compatible with WCG

We have seen that there is no maximal forcing axiom compatible with tail club guessing (TCG). But a weak club guessing (WCG) admits maximal one.

2.1 Definition. Let $\langle C_\delta \mid \delta \in A \rangle$ be a ladder system. Let $\mathcal{F}$ denote the set of all cofinal subsequences of C_δ (viewed as sequences of length ω) for all $\delta \in A$. Let $Seq^\omega(\omega_1)$ denote the set of all sequences $\langle a_n \mid n < \omega \rangle$ such that each a_n is a countable subset of ω_1. Hence we have $\mathcal{F} \subseteq Seq^\omega(\omega_1)$. Let P be a preorder, we say P is *$\mathcal{F}$-limsup-semiproper*, if for all sufficiently large regular cardinals θ and all $\in$-chains $\langle N_n \mid n < \omega \rangle$ in H_θ with $P, \langle C_\delta \mid \delta \in A \rangle \in N_0$, if $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$, then for any $p \in N_0 \cap P$, there exists $q \leq p$ such that for infinitely many $n < \omega$, q is (P, N_n)-semi-generic.
Similary, we say P is *$\mathcal{F}$-liminf-semiproper*, if q is (P, N_n)-semi-generic for all but finitey many $n < \omega$.
Lastly, we say P is *$\mathcal{F}$-generic-limsup-semiproper*, if $q \Vdash_p$ "$N_n[G] \cap \omega_1^V = N_n \cap \omega_1^V$ for infinitely many $n < \omega$".

Hence we are looking at the set $\{n < \omega \mid N[G] \cap \omega_1^V = N \cap \omega_1^V\}$ in $V[G]$ which might be infinite and may or may not be in V.

For the notion of ω-stationary sets $S \subseteq Seq^\omega(K) = \{\langle a_n \mid n < \omega \rangle \mid a_n \in [K]^\omega \text{ for all } n < \omega\}$, may see [M]. They are analogously formulated as the stationary sets in $[K]^\omega$.

2.2 Proposition. Let P be a preorder.

- If P is ω_1-closed, then P is ω-semiproper.
- If P is ω-semiproper, then P is $\mathcal{F}$-liminf-semiproper.
- If P is $\mathcal{F}$-liminf-semiproper, then P is $\mathcal{F}$-limsup-semiproper.
- If P is $\mathcal{F}$-limsup-semiproper, then P is $\mathcal{F}$-generic-limsup-semiproper.
- If $\mathcal{F}$ is ω-stationary and P is $\mathcal{F}$-generic-limsup-semiproper, then P preserves ω_1.

2.3 Definition. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system. We formulate *(proper, $\langle C_\delta \mid \delta \in A\rangle$-limsup-semiproper, full) -Reflection Principle* (abusively, $\mathcal{F}$-*RP*) as follows; Given any (K, S, θ, a) such that

- K is a set with $K \supseteq \omega_1$.
- $S \subseteq Seq^\omega(K)$.
- θ is a regular cardinal with $K \in H_{|\mathrm{TC}(K)|^+} \in H_{(2^{|\mathrm{TC}(K)|})^+} \in H_\theta$.
- $a \in H_\theta$.

There exists $(D, \langle N_i \mid i < \omega_1\rangle)$ such that

- D is a club in ω_1.
- $\langle N_i \mid i < \omega_1\rangle$ is an $\in$-chain in H_θ with $a \in N_0$.
- For any $\delta \in Y^*(D) = \{\delta \in A \mid |C_\delta \cap D| = \omega\}$, let $\langle C_\delta(k_\delta(m)) \mid m < \omega\rangle$ enumerate $\{C_\delta(k) \mid C_\delta(k) \in D\}$. Then there exists $n_\delta < \omega$ such that we have either (1) or (2).

(1) $\langle N_{C_\delta(k_\delta(m))} \cap K \mid n_\delta \leq m < \omega\rangle \in S$.

(2) For any strictly increasing sequence $\langle m_l \mid l < \omega\rangle$ of natural numbers with $n_\delta \leq m_0$ and for any $\in$-chain $\langle M_l \mid l < \omega\rangle$ with $\langle M_l \mid l < \omega\rangle \sqsupseteq_{\omega_1} \langle N_{C_\delta(k_\delta(m_l))} \mid l < \omega\rangle$, we have $\langle M_l \cap K \mid l < \omega\rangle \notin S$.

We might call $i_\delta = C_\delta(k_\delta(n_\delta)))$ a *critical point* of C_δ with respect to D for each $\delta \in Y^*(D)$. Hence we are looking at $\langle N_i \mid i_\delta \leq i \in C_\delta \cap D\rangle$.

2.4 Theorem. Let $\mathcal{F}^V \subseteq Seq^\omega(\omega_1)$ be defined by $\langle C_\delta \mid \delta \in A\rangle$. Then $\mathcal{F}^{V[G_\alpha]}$-generic-limsup-semiproper combined with semiproper iterates under the simple iteration. (The $\mathcal{F}^{V[G_\alpha]}$ are uniformly defined from the ladder system $\langle C_\delta \mid \delta \in A\rangle$ in each intermediate universe $V[G_\alpha]$. The exact value of $\mathcal{F}^{V[G_\alpha]}$ increasingly changes as α gets bigger.)

□

2.5 Corollary. Let $\langle C_\delta \mid \delta \in A\rangle$ be weak club guessing so that $\mathcal{F}$ defined from $\langle C_\delta \mid \delta \in A\rangle$ is ω-stationary. Let us recall $\mathcal{F} = \{\langle C_\delta(k(m)) \mid m < \omega\rangle \mid \delta \in A, \langle k(m) \mid m < \omega\rangle$ is a sequence of strictly increasing natural numbers$\}$. We may force the following, if there exists a supercompact cardinal.

(1) If $\langle C_\delta \mid \delta \in A\rangle$ is weak club guessing in the ground model, then it remains to be so in the extensions.

(2) Forcing axiom$^+$ holds for the class of partially ordered sets P such that P is semiproper and that P is $\mathcal{F}$-generic-limsup-semiproper.

2.6 Lemma. Let $\langle C_\delta \mid \delta \in A\rangle$ be a ladder system. Let $\mathcal{F} \subseteq Seq^\omega(\omega_1)$ be defined from the system. Let $\mathcal{F}$-Reflection Principle hold. Let us consider $(WCG)^+$ with repect to the system. Let P be a preorder. Then the following are equivalent on P.

(1) P is $(WCG)^+$-preserving.

(2) P is $\mathcal{F}$-generic-limsup-semiproper.

□

The $\mathcal{F}$-RP gets forced by a little better notion of forcing than semiproper + $\mathcal{F}$-generic-limsup-semiproper partially ordered set.

2.7 Lemma. Let $\mathcal{F} \subseteq Seq^\omega(\omega_1)$ be as above. Let forcing axiom hold for the class of preordered sets P such that P is proper and that P is $\mathcal{F}$-limsup-semiproper. Then $\mathcal{F}$-Reflection Principle holds.

□

2.8 Corollary. The following is consistent, if there exists a supercompact cardinal.

(1) $\langle C_\delta \mid \delta \in A\rangle$ is weak club guessing.

(2) The forcing axiom$^+$ holds for the class of preordered sets P such that P preserves every stationary subset of ω_1 and that P preserves every member of $(WCG)^+$ with respect to $\langle C_\delta \mid \delta \in A\rangle$.

Proof of lemma 2.6. (2) implies (1): No use of $\mathcal{F}$-RP made in this direction. Let $X \subseteq \omega_1$ such that $\langle C_\delta \mid \delta \in X \cap A\rangle$ is weak club guessing. Suppose $p \Vdash_P$ "$\dot{C} \subseteq \omega_1$ is a club". Want $q \leq p$ and $\delta \in X \cap A$ such that $q \Vdash_P$ "$|C_\delta \cap \dot{C}| = \omega$". To this end, let θ be a sufficiently large regular cardinal. Since $\langle C_\delta \mid \delta \in X \cap A\rangle$ is weak club guessing, we may take an $\in$-chain $\langle N_n \mid n < \omega\rangle$ in H_θ such that $\{N_n \cap \omega_1 \mid n < \omega\}$ is a cofinal subset of C_δ, where $\delta = N_\omega \cap \omega_1$ and $N_\omega = \cup\{N_n \mid n < \omega\}$. We may assume $P, \langle C_\delta \mid \delta \in A\rangle, p, \dot{C} \in N_0$. Since P is $\mathcal{F}$-generic-limsup-semiproper and $\langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$, we have q such that $q \leq p$ and $q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$". Hence $q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1 \in \dot{C} \cap C_\delta$ for infinitely many $n < \omega$" and so $q \Vdash_P$ "$|C_\delta \cap \dot{C}| = \omega$".

(1) implies (2): Suppose P is $(WCG)^+$-preserving. Want to show P is $\mathcal{F}$-generic-limsup-semiproper. It suffices to show the following. There exists a club $C \subseteq [H_{\theta_0}]^\omega$, where $\theta_0 = |\mathrm{TC}(P)|^+$ $(\geq \omega_2)$, such that for any $\in$-chain $\langle N_n \mid n < \omega\rangle$ in H_{θ_0} through C (i.e. for all $n < \omega$, $N_n \in C$), if $\langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$, then for any $p \in N_0 \cap P$, there exists q such that $q \leq p$ and $q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$". We show this by contradiction. Suppose not. Let $S = \{\langle N_n \mid n < \omega\rangle \mid \langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$ and there is $p \in N_0 \cap P$ such that for all q with $q \leq p$, $\neg(q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$")$\}$. Then this S is ω-stationary. Namely, for any club C in $[H_{\theta_0}]^\omega$, there exists $\langle N_n \mid n < \omega\rangle \in S$ through C. By Fodor's Lemma for ω-stationary sets (may see [M]), we may assume that there exists $p_0 \in P$ such that $S \subseteq \{\langle N_n \mid n < \omega\rangle \mid \langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$, $p_0 \in N_0 \cap P$ and for all q with $q \leq p_0$, $\neg(q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$")$\}$.

Let S^* denote the set of the end-segments of the elements of S. Apply $\mathcal{F}$-RP to $(H_{\theta_0}, S^*, \lambda, (P, p_0))$, where λ is sufficiently large. Get a club D in ω_1 and an $\in$-chain $\langle M_i \mid i < \omega_1\rangle$ in H_λ as in $\mathcal{F}$-RF.

Claim. $B = \{\delta \in Y^*(D) \mid \delta \models$ "(1) in $\mathcal{F}$-RP"$\} \in (WCG)^+$.

Proof. Let E be a club in ω_1. Want $\delta \in B$ with $|C_\delta \cap E| = \omega$. Since S is ω-stationary in $Seq^\omega(H_{\theta_0})$, we may take an $\in$-chain $\langle N_n^* \mid n < \omega\rangle$ in H_λ such that $D, \langle M_i \mid i < \omega_1\rangle, E \in N_0^*$ and that $\langle N_n^* \cap H_{\theta_0} \mid n < \omega\rangle \in S$. Let $N_\omega^* = \bigcup\{N_n^* \mid n < \omega\}$ and $\delta = N_\omega^* \cap \omega_1$. Then $\langle N_n^* \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$ and is through $D \cap E$. In particular, $\delta \in Y^*(D)$ and $|C_\delta \cap E| = \omega$. Let $\{N_n^* \cap \omega_1 \mid n < \omega\} \subseteq C_\delta \cap D = \{C_\delta(k_\delta(m)) \mid m < \omega\}$. By considering an end-segment, we may assume $\langle N_n^* \cap H_{\theta_0} \mid n < \omega\rangle \in S^*$ with $C_\delta(k_\delta(n_\delta)) < N_0^* \cap \omega_1$.

Want (1) holds at this δ so that $\delta \in B$. Since $N_n^* \supseteq_{\omega_1} M_{N_n^* \cap \omega_1}$ for all $n < \omega$ and $\langle N_n^* \cap H_{\theta_0} \mid n < \omega\rangle \in S^*$, (2) in $\mathcal{F}$-RP fails. Hence (1) must hold in $\mathcal{F}$-RP.

□

Since $p_0 \Vdash_P$ "$\langle C_\delta \mid \delta \in B\rangle$ remains to be weak club guessing", $p_0 \Vdash_P$ "there exists $\delta \in B$ such that $|C_\delta \cap \dot{D}| = \omega$, where $\dot{D} = \{i < \omega_1 \mid M_i[G] \cap \omega_1 = M_i \cap \omega_1 = i \in D\}$ a club in ω_1". Take q and $\delta \in B$ such that $q \leq p_0$ and $q \Vdash_P$ "$|C_\delta \cap \dot{D}| = \omega$". Since $\delta \in B$, we have (1) in $\mathcal{F}$-RP. Hence $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega\rangle \in S^*$, where $\langle C_\delta(k) \mid k < \omega\rangle$ enumerates C_δ and $\langle C_\delta(k_\delta(m)) \mid m < \omega\rangle$ enumerates $C_\delta \cap D$.

Since $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega\rangle \in S^*$, there exists $\langle N_n \mid n < \omega\rangle \in S$ such that $\langle M_{C_\delta(k_\delta(m))} \cap H_{\theta_0} \mid n_\delta \leq m < \omega\rangle$ is an end-segment of $\langle N_n \mid n < \omega\rangle$. Since $\langle N_n \mid n < \omega\rangle \in S$, we have $\neg(q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$"). However, $q \Vdash_P$ "if $C_\delta(k_\delta(m)) \in C_\delta \cap \dot{D}$ and $N_n = M_{C_\delta(k_\delta(m))} \cap H_{\theta_0}$, then $N_n[G] \cap \omega_1 = (M_{C_\delta(k_\delta(m))} \cap H_{\theta_0})[G] \cap \omega_1 = M_{C_\delta(k_\delta(m))}[G] \cap \omega_1 = M_{C_\delta(k_\delta(m))} \cap \omega_1 = N_n \cap \omega_1 = C_\delta(k_\delta(m))$". Hence $q \Vdash_P$ "$N_n[G] \cap \omega_1 = N_n \cap \omega_1$ for infinitely many $n < \omega$". This is a contradiction.

□

Proof of lemma 2.7. Let (K, S, θ, a) be as in the hypothesis of $\mathcal{F}$-RF. We force $\dot{D}$, $\langle \dot{N}_i \mid i < \omega_1\rangle$ and $\langle \dot{n}_\delta \mid \delta \in A\rangle$ by initial segments. Let $p \in P$, if $p = (\alpha^p, D^p, \langle N_i^p \mid i \leq \alpha^p\rangle, \langle n_\delta^p \mid \delta \in A \cap (\alpha^p + 1)\rangle)$ satisfies the following:

(1) $\alpha^p < \omega_1$.

(2) $D^p \subseteq \alpha^p + 1$, $\alpha^p \in D^p$ and D^p is closed.

(3) $\langle N_i^p \mid i \leq \alpha^p\rangle$ is an $\in$-chain in H_θ with $a \in N_0^p$.

(4) For each $\delta \in A \cap (\alpha^p + 1)$, $n_\delta^p < \omega$. If $|C_\delta \cap D^p| = \omega$, then let $\langle C_\delta(k) \mid k < \omega\rangle$ enumerate C_δ and let $\langle C_\delta(k_\delta^p(m)) \mid m < \omega\rangle$ enumerate $C_\delta \cap D^p$. And we demand either (i) or (ii);

(i) $\langle N^p_{C_\delta(k^p_\delta(m))} \cap K \mid n^p_\delta \leq m < \omega\rangle \in S$.

(ii) For any strictly increasing sequence $\langle m_l \mid l < \omega\rangle$ of natural numbers with $n^p_\delta \leq m_0$ and any $\in$-chain $\langle M_l \mid l < \omega\rangle$ in H_θ such that $\langle M_l \mid l < \omega\rangle \sqsupseteq_{\omega_1} \langle N^p_{C_\delta(k^p_\delta(m_l))} \mid l < \omega\rangle$, we have $\langle M_l \cap K \mid l < \omega\rangle \notin S$.

For $p, q \in P$, we set $q \leq p$, if

(5) $\alpha^p \leq \alpha^q$.

(6) $D^p = D^q \cap (\alpha^p + 1)$.

(7) For all $i \leq \alpha^p$, $N^p_i = N^q_i$.

(8) For all $\delta \in A \cap (\alpha^p + 1)$, $n^p_\delta = n^q_\delta$.

Claim. (Dense, Extension by Escape) For any (p, η, x) such that $p \in P$, $\alpha^p < \eta < \omega_1$ and $x \in H_\theta$, there exists $q \in P$ such that $q \leq p$, $\alpha^q = \eta$ and $x \in N^q_\eta$.

Proof. Let $\alpha^q = \eta$, $D^q = D^p \cup \{\eta\}$ and $\langle N^q_i \mid i \leq \eta\rangle$ be any $\in$-chain which end-extends $\in$-chain $\langle N^p_i \mid i \leq \alpha^p\rangle$ and $x \in N^q_\eta$. Let $\langle n^q_\delta \mid \delta \in A \cap (\eta + 1)\rangle$ be any sequence of natural numbers which end-extends $\langle n_\delta \mid \delta \in A \cap (\alpha^p + 1)\rangle$. Since for any $\delta \in A \cap (\eta + 1)$, $|C_\delta \cap D^q| = \omega$ iff ($\delta \leq \alpha^p$ and $|C_\delta \cap D^p| = \omega$), this q works.

Claim. (Targeted-Extension) Let λ be a sufficiently large regular cardinal and M be a countable elementary substructure of H_λ with $P \in M$. Let $\delta_M = M \cap \omega_1$. Then for any (p, ξ, D) such that $p \in M \cap P$, $\xi < \delta_M$ and $D \in M$ is a dense subset of P, there exists $r \in M \cap D$ such that $r \leq p$, $\xi < \alpha^r$ and $C_{\delta_M} \cap D^r = C_{\delta_M} \cap D^p$.

Proof. We consider a family of maps indexed by $p \in P$. Let $p \in P$. Let η be such that $\alpha^p < \eta < \omega_1$. Let $r = f_p(\eta)$, where $r \in D, r \leq p, \eta < \alpha^r$ and $D^r \cap (\alpha^p, \eta] = \{\eta\}$. We may assume $\langle f_p \mid p \in P\rangle \in M$. Since $p \in M$, $f_p \in M$. Let $C(f_p) = \{\beta < \omega_1 \mid$ for all η with $\alpha^p < \eta < \beta$, $\alpha^{f_p(\eta)} < \beta\}$. Then $C(f_p)$ is a club in ω_1 and $C(f_p) \in M$. Take $\beta \in C(f_p) \cap M$ such that $\alpha^p, \xi < \beta$. Let η be such that $\alpha^p, \xi, max(C_{\delta_M} \cap \beta) < \eta < \beta$. Then $f_p(\eta) \in M$ and so there exists $r \in M \cap D$ such that $r \leq p$, $\xi < \alpha^r$ and $C_{\delta_M} \cap D^r = C_{\delta_M} \cap D^p$.

Claim. (Proper) P is proper and σ-Baire.

Proof. Let λ be a sufficiently large regular cardinal and M be a countable elementary substructure of H_λ with $P \in M$. Let $p \in M \cap P$. Let $\delta_M = M \cap \omega_1$. Then by repeating above claim, we may construct a (P, M)-generic sequence $\langle p_n \mid n < \omega\rangle$ such that $C_{\delta_M} \cap D^{p_n} = C_{\delta_M} \cap D^p$ for all $n < \omega$. Let $\alpha^q = \delta_M$, $D^q = \bigcup\{D^{p_n} \mid n < \omega\} \cup \{\delta_M\}$, $\langle N^q_i \mid i \leq \delta_M\rangle = \bigcup\{\langle N^{p_n}_i \mid i \leq \alpha^{p_n}\rangle \mid n < \omega\} \cup \{(\delta_M, M \cap H_\theta)\}$ and $\langle n^q_\delta \mid \delta \in A \cap (\delta_M + 1)\rangle$ be any end-extension of all of $\langle n^{p_n}_\delta \mid \delta \in A \cap (\alpha^{p_n} + 1)\rangle$. Notice that $C_{\delta_M} \cap D^q = C_{\delta_M} \cap D^p$ which is finite. Hence this q is a lower bound of the p_n's.

Claim. P is $\mathcal{F}$-limsup-semiproper.

Proof. Let λ be a sufficiently large regular cardinal. Let $\langle N_n \mid n < \omega\rangle$ be an $\in$-chain in H_λ such that $K, \theta, P, \langle C_\delta \mid \delta \in A\rangle \in N_0$ and $\langle N_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}$. Let $N_\omega = \bigcup\{N_n \mid n < \omega\}$ and let $\delta^* = N_\omega \cap \omega_1$. Let $\langle C_{\delta^*}(k) \mid k < \omega\rangle$ enumerate C_{δ^*} and $\langle C_{\delta^*}(k_{\delta^*}(n)) \mid n < \omega\rangle$ enumerate $\{N_n \cap \omega_1 \mid n < \omega\}$.

Let $p \in N_0 \cap P$. We want q such that $q \leq p$ and q is (P, N_n)-semi-generic for infinitely many $n < \omega$. To this end, we argue in two cases.

Case 1. There exists a sequence $\langle n_l \mid l < \omega\rangle$ of strictly increasing natural numbers and an $\in$-chain $\langle M_l \mid l < \omega\rangle$ in H_θ such that $\langle M_l \mid l < \omega\rangle \sqsupseteq_{\omega_1} \langle N_{n_l} \cap H_\theta \mid l < \omega\rangle$ and $\langle M_l \cap K \mid l < \omega\rangle \in S$:

Apply the Sequencial 3 H Lemma (see [M]) to get an $\in$-chain $\langle M^*_l \mid l < \omega\rangle$ in H_λ such that

- $\langle M^*_l \mid l < \omega\rangle \sqsupseteq_{\omega_1} \langle N_{n_l} \mid l < \omega\rangle$.

- For all $l < \omega$, $M_l^* \cap H_{|\mathrm{TC}(K)|^+} = M_l \cap H_{|\mathrm{TC}(K)|^+}$.

  And so

- $\langle M_l^* \cap K \mid l < \omega \rangle = \langle M_l \cap K \mid l < \omega \rangle \in S$.

Notice $p \in N_0 \subseteq N_{n_0} \subseteq M_0^*$. It suffices to get q such that $q \leq p$ and for all $l < \omega$, q is (P, M_l^*)-generic and so (P, N_{n_l})-semi-generic. Construct a descending sequence $\langle q_l \mid l < \omega \rangle$ of conditions such that

- $p \geq q_l \in M_{l+1}^*$ and q_l is (P, M_l^*)-generic.
- $D^{q_l} \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{M_0^* \cap \omega_1, \cdots, M_l^* \cap \omega_1\}$.
- $\alpha^{q_l} = M_l^* \cap \omega_1$ and $N^{q_l}_{\alpha^{q_l}} = M_l^* \cap H_\theta$.

Let q be defined by

- $\alpha^q = \delta^* = M_\omega^* \cap \omega_1$, where $M_\omega^* = \bigcup\{M_l^* \mid l < \omega\}$.
- $D^q = \bigcup\{D^{q_l} \mid l < \omega\} \cup \{\delta^*\}$.
- $\langle N_i^q \mid i \leq \alpha^q \rangle = \bigcup\{\langle N_i^{q_l} \mid i \leq \alpha^{q_l} \rangle \mid l < \omega\} \cup \{(\delta^*, M_\omega^* \cap H_\theta)\}$.
- $\langle n_\delta^q \mid \delta \in A \cap (\alpha^q + 1) \rangle = \bigcup\{\langle n_\delta^{q_l} \mid \delta \in A \cap (\alpha^{q_l} + 1) \rangle \mid l < \omega\} \cup \{(\delta^*, n_{\delta^*}^q)\}$,

  where $n_{\delta^*}^q < \omega$ gets specified as follows;

Let $C_{\delta^*} \cap D^q = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid m < \omega\}$. Since $D^q \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{M_l^* \cap \omega_1 \mid l < \omega\}$, there exists $n_{\delta^*}^q < \omega$ such that

$$\{M_l^* \cap \omega_1 \mid l < \omega\} = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid n_{\delta^*}^q \leq m < \omega\}.$$

And so

$$\langle N^q_{C_{\delta^*}(k^q_{\delta^*}(m))} \cap K \mid n^q_{\delta^*} \leq m < \omega \rangle = \langle (M_l^* \cap H_\theta) \cap K \mid l < \omega \rangle = \langle M_l \cap K \mid l < \omega \rangle \in S.$$

Hence this q works.

□

Case 2. For all sequences $\langle n_l \mid l < \omega \rangle$ of strictly increasing natural numbers and all $\in$-chains $\langle M_l \mid l < \omega \rangle$ in H_θ with $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N_{n_l} \cap H_\theta \mid l < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$:

It suffices to get $q \in P$ such that $q \leq p$ and for all $n < \omega$, q is (P, N_n)-generic. To this end, we may construct a descending sequence $\langle q_n \mid n < \omega \rangle$ of conditions such that

- $p \geq q_n \in N_{n+1}$ and q_n is (P, N_n)-generic.
- $D^{q_n} \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{N_0 \cap \omega_1, \cdots, N_n \cap \omega_1\}$.
- $\alpha^{q_n} = N_n \cap \omega_1$ and $N^{q_n}_{\alpha^{q_n}} = N_n \cap H_\theta$.

Let q be defined by

- $\alpha^q = \delta^* = N_\omega \cap \omega_1$.
- $D^q = \bigcup\{D^{q_n} \mid n < \omega\} \cup \{\delta^*\}$.
- $\langle N_i^q \mid i \leq \alpha^q \rangle = \bigcup\{\langle N_i^{q_n} \mid i \leq \alpha^{q_n} \rangle \mid n < \omega\} \cup \{(\delta^*, N_\omega \cap H_\theta)\}$.
- $\langle n_\delta^q \mid \delta \in A \cap (\alpha^q + 1) \rangle = \bigcup\{\langle n_\delta^{q_n} \mid \delta \in A \cap (\alpha^{q_n} + 1) \rangle \mid n < \omega\} \cup \{(\delta^*, n_{\delta^*}^q)\}$,

  where $n_{\delta^*}^q < \omega$ gets specified as follows;

Let $C_{\delta^*} \cap D^q = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid m < \omega\}$. Since $D^q \cap C_{\delta^*} = (D^p \cap C_{\delta^*}) \cup \{N_n \cap \omega_1 \mid n < \omega\}$, there exists $n_{\delta^*}^q < \omega$ such that

$$\{N_n \cap \omega_1 \mid n < \omega\} = \{C_{\delta^*}(k_{\delta^*}^q(m)) \mid n_{\delta^*}^q \leq m < \omega\}.$$

And so

$$\langle N^q_{C_{\delta^*}(k^q_{\delta^*}(m))} \mid n^q_{\delta^*} \leq m < \omega \rangle = \langle N^q_{N_n \cap \omega_1} \mid n < \omega \rangle = \langle N_n \cap H_\theta \mid n < \omega \rangle.$$

For all sequences $\langle m_l \mid l < \omega \rangle$ of strictly increasing natural numbers with $n^q_{\delta^*} \leq m_0$ and all $\in$-chains $\langle M_l \mid l < \omega \rangle$ in H_θ with $\langle M_l \mid l < \omega \rangle \supseteq_{\omega_1} \langle N^q_{C_{\delta^*}(k^q_{\delta^*}(m_l))} \mid l < \omega \rangle$ which is a subsequence of $\langle N_n \cap H_\theta \mid n < \omega \rangle$, we have $\langle M_l \cap K \mid l < \omega \rangle \notin S$.

Hence this q works.

Now apply forcing axiom to this P. We have a club D in ω_1 and an $\in$-chain $\langle N_i \mid i < \omega_1 \rangle$ which works for (K, S, θ, a) in $\mathcal{F}$-RP.

§3. Iteration theorem

We show $\mathcal{F}^{V[G_\alpha]}$-generic-limsup-semiproper combined with semiproper iterates under the simple iterations. For an account on the simple iterations, see [M].

3.1 Theorem. Let $I = \big(\langle P_\alpha \mid \alpha \leq \nu \rangle, \langle \dot{Q}_\alpha \mid \alpha < \nu \rangle\big)$ be a simple iteration such that

- For all $\alpha < \nu$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is semiproper".
- For all $\alpha < \nu$, $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is $\mathcal{F}^{V[G_\alpha]}$-generic-limsup-semiproper",

where $\mathcal{F}^{V[G_\alpha]}$ is formed in $V[G_\alpha]$ as the set of cofinal subsequences of the $\langle C_\delta(n) \mid n < \omega \rangle$'s for all $\delta \in A$. Hence $\mathcal{F}^{V[G_\alpha]}$ may contain new cofinal subsequences than the original $\mathcal{F} = \mathcal{F}^V$.

Then for all $\alpha \leq \nu$, we have $\Vdash_{P_\alpha}$ "the tails $P_{\alpha\nu}$ are semiproper and $\mathcal{F}^{V[G_\alpha]}$-generic-limsup-semiproper".

In particular, P_ν is semiproper and $\mathcal{F}^V$-generic-limsup-semiproper.

3.2 Iteration Lemma. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $I, \langle C_\delta \mid \delta \in A \rangle \in N$.

Let $(\alpha, \alpha^*, a, p, \langle \dot{M}_n \mid n < \omega \rangle)$ be such that

- $\alpha < \alpha^* \leq \nu$.
- $a \in P_\alpha$, $p \in P_{\alpha^*}$ and $a \leq p\lceil\alpha$.
- $a \Vdash_{P_\alpha}$ "$\langle \dot{M}_n \mid n < \omega \rangle$ is an $\in$-chain in $H_\theta^{V[G_\alpha]}$ such that $N \cup \{G_\alpha, p\} \subseteq \dot{M}_0$ and that $\langle \dot{M}_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}^{V[G_\alpha]}$".

Then there exists $q \in P_{\alpha^*}$ such that

- $q\lceil\alpha = a$ and $q \leq p$.
- $a \Vdash_{P_\alpha}$ " $q\lceil[\alpha, \alpha^*) \Vdash^{V[G_\alpha]}_{P_{\alpha\alpha^*}}$ "$\dot{M}_n[G_{\alpha\alpha^*}] \cap \omega_1 = \dot{M}_n \cap \omega_1$ for infnitely many $n < \omega$"".

□

We extract sort of typical constructions involved as (Technical construction 1–3).

Lemma. (Technical construction 1) Let $(\alpha, \alpha^*, a, p, \langle \dot{\delta}^p_k \mid k < \omega \rangle, \dot{x})$ be such that

- $\alpha < \alpha^* \leq \nu$ and α^* is limit.
- $a \in P_\alpha$, $p \in P_{\alpha^*}$ and $a \leq p\lceil\alpha$.
- $\langle \dot{\delta}^p_k \mid k < \omega \rangle$ are stages for p.

- $a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot\delta^p_0 = \alpha$".
- $a \Vdash_{P_\alpha}$ "$\dot x \le p$ and $\dot x\lceil\alpha \in G_\alpha$".

  Then there exists $(\beta, b, x, \langle\dot\delta^x_k \mid k < \omega\rangle)$ such that
- $\alpha < \beta < \alpha^*$.
- $b \in P_\beta$, $b\lceil\alpha \le a$, $x \in P_{\alpha^*}$ and $b \le x\lceil\beta$.
- $b\lceil\alpha \Vdash_{P_\alpha}$ "$\dot x = x$".
- $\langle\dot\delta^x_k \mid k < \omega\rangle$ are stages for x and for all $k < \omega$, $\Vdash_{P_{\alpha^*}}$ "$\dot\delta^p_{k+1} \le \dot\delta^x_k$" (a step ahead).
- $b^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot\delta^x_0 = \beta$".

□

Lemma. (Technical construction 2) Let $(\alpha, \alpha^*, a, p, \langle\dot\delta^p_k \mid k < \omega\rangle, \dot x)$ be as in technical construction 1. Then there exists $\langle(\beta_i, b_i, x_i, \langle\dot\delta^{x_i}_k \mid k < \omega\rangle) \mid i < \mu\rangle$ such that

- $\beta = \beta_i, b = b_i, x = x_i, \langle\dot\delta^x_k \mid k < \omega\rangle = \langle\dot\delta^{x_i}_k \mid k < \omega\rangle$ serve exactly as in technical construction 1.
- For $i, j < \mu$ such that $i \ne j$, we have that $b_i\lceil\alpha$ and $b_j\lceil\alpha$ are incompatible in P_α.
- $\{b_i\lceil\alpha \mid i < \mu\}$ forms a maximal antichain below a in P_α.

□

Lemma. (Technical construction 3) Let $(\alpha, \alpha^*, a, p, \langle\dot\delta^p_k \mid k < \omega\rangle, \langle\dot M_n \mid n < \omega\rangle)$ be such that

- $\alpha < \alpha^* \le \nu$ and α^* is limit.
- $a \in P_\alpha$, $p \in P_{\alpha^*}$ and $a \le p\lceil\alpha$.
- $\langle\dot\delta^p_k \mid k < \omega\rangle$ are stages for p in P_{α^*}.
- $a^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot\delta^p_0 = \alpha$".
- $a \Vdash_{P_\alpha}$ "$\langle\dot M_n \mid n < \omega\rangle$ is an $\in$-chain in $H^{V[G_\alpha]}_\theta$ with $N \cup \{G_\alpha, p, \langle\dot\delta^p_k \mid k < \omega\rangle\} \subseteq \dot M_0$".

  Let T be a tree such that $T \subset {}^{<\omega}ON$ with $\{\emptyset\} = T_0$. For $\sigma = \emptyset$, let
- $\alpha^\emptyset = \alpha$, $a^\emptyset = a$, $p^\emptyset = p$, $\langle\dot\delta^\emptyset_k \mid k < \omega\rangle = \langle\dot\delta^p_k \mid k < \omega\rangle$ and $\langle\dot M^\emptyset_n \mid n < \omega\rangle = \langle\dot M_n \mid n < \omega\rangle$.

  For all $\sigma \in T$, we have $(\alpha^\sigma, a^\sigma, p^\sigma, \langle\dot\delta^\sigma_k \mid k < \omega\rangle, \langle\dot M^\sigma_n \mid n < \omega\rangle)$ such that
- $\alpha \le \alpha^\sigma < \alpha^*$.
- $a^\sigma \in P_{\alpha^\sigma}$, $p^\sigma \in P_{\alpha^*}$, $a^\sigma\lceil\alpha \le a$, $p^\sigma \le p$ and $a^\sigma \le p^\sigma\lceil\alpha^\sigma$.
- $\langle\dot\delta^\sigma_k \mid k < \omega\rangle$ are stages for p^σ in P_{α^*}.
- $a^{\sigma\frown} 1 \Vdash_{P_{\alpha^*}}$ "$\dot\delta^{p^\sigma}_0 = \alpha^\sigma$".
- $a^\sigma \Vdash_{P_{\alpha^\sigma}}$ "$\langle\dot M^\sigma_n \mid n < \omega\rangle$ is an $\in$-chain in $H^{V[G_{\alpha^\sigma}]}_\theta$ with $N \cup \{G_{\alpha^\sigma}, p^\sigma, \langle\dot\delta^\sigma_k \mid k < \omega\rangle\} \subseteq \dot M^\sigma_0$".

  For all $\tau \in \mathrm{suc}_T(\sigma)$, we have $\langle\dot m(\tau, n) \mid n < \omega\rangle$ such that
- $\alpha^\sigma \le \alpha^\tau$.
- $a^\tau\lceil\alpha^\sigma \le a^\sigma$.
- $p^\tau \le p^\sigma$ in P_{α^*}.
- $\langle\dot\delta^\tau_k \mid k < \omega\rangle$ are a step ahead of $\langle\dot\delta^\sigma_k \mid k < \omega\rangle$.
- $a^\tau\lceil\alpha^\sigma \Vdash_{P_{\alpha^\sigma}}$ " $p^\tau\lceil[\alpha^\sigma, \alpha^*) \Vdash^{V[\dot G_{\alpha^\sigma}]}_{P_{\alpha^\sigma\alpha^*}}$ "$\dot M^\sigma_0[G_{\alpha^\sigma\alpha^*}] \cap \omega_1 = \dot M^\sigma_0 \cap \omega_1$"".
- $a^\tau \Vdash_{P_{\alpha^\tau}}$ "$\langle\dot m(\tau, n) \mid n < \omega\rangle$ is a sequence of strictly increasing natural numbers".
- $a^\tau \Vdash_{P_{\alpha^\tau}}$ "$\dot M^\tau_n = \dot M^\sigma_{\dot m(\tau,n)}[G_{\alpha^\sigma\alpha^\tau}]$".

- $a^{\tau} \Vdash_{P_{\alpha^\tau}}$ "for all $n<\omega$, $\dot{M}^{\tau}_n \cap \omega_1 = \dot{M}^{\sigma}_{m(\tau,n)} \cap \omega_1$".

 Let $q \le p$ be a fusion of a^{σ}'s in P_{α^*}.

 Then there exists a sequence $\langle \dot{n}_k \mid k<\omega\rangle$ of P_{α^*}-names such that

- $q \Vdash_{P_{\alpha^*}}$ "$\langle \dot{n}_k \mid k<\omega\rangle$ is a sequence of strictly increasing natural numbers".
- For all $k<\omega$, we have $q \Vdash_{P_{\alpha^*}}$ "$\dot{M}_{\dot{n}_k} \cap \omega_1 = \dot{M}_{\dot{n}_k}[G_{\alpha\alpha^*}] \cap \omega_1$".

More specifically, we may calculate $\langle i_k \mid k<\omega\rangle$, $\langle a_k \mid k<\omega\rangle$, $\langle \alpha_k \mid k<\omega\rangle$, $\langle p_k \mid k<\omega\rangle$ and $\langle m(k,n) \mid k,n<\omega\rangle$ in $V[G_{\alpha^*}]$ such that, where $M =_{\omega_1} N$ abbreviates $M\cap\omega_1 = N\cap\omega_1$,

- $M_0[G_{\alpha_0\alpha^*}] \subseteq M_0^{\emptyset}[G_{\alpha_0\alpha^*}] =_{\omega_1} M_0^{\emptyset} = M_0$.
- $M_{m(0,0)}[G_{\alpha_0\alpha^*}] \subseteq M^{\emptyset}_{m(0,0)}[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha^*}] = M_0^{\langle i_0\rangle}[G_{\alpha_1\alpha^*}] =_{\omega_1} M_0^{\langle i_0\rangle} =_{\omega_1} M^{\emptyset}_{m(0,0)} = M_{m(0,0)}$.
- $M_{m(0,m(1,0))}[G_{\alpha\alpha^*}] \subseteq M^{\emptyset}_{m(0,m(1,0))}[G_{\alpha_0\alpha_1}][G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^*}] = M^{\langle i_0\rangle}_{m(1,0)}[G_{\alpha_1\alpha_2}][G_{\alpha_2\alpha^*}] = M_0^{\langle i_0,i_1\rangle}[G_{\alpha_2\alpha^*}]$ $=_{\omega_1} M_0^{\langle i_0,i_1\rangle} =_{\omega_1} M^{\langle i_0\rangle}_{m(1,0)} =_{\omega_1} M^{\emptyset}_{m(0,m(1,0))} = M_{m(0,m(1,0))}$.

In general,

$$a^{\emptyset} = a = a_0.$$

$$a^{\langle i_0,\cdots,i_k\rangle} = a_{k+1}.$$

$$\alpha_k = l(a_k).$$

$$M_n^{\emptyset} = M_n.$$

$$M_n^{\langle i_0\rangle} = M_{m(0,n)}[G_{\alpha_0\alpha_1}].$$

$$M_n^{\langle i_0\rangle} =_{\omega_1} M_{m(0,n)}.$$

$$M_n^{\langle i_0,\cdots,i_k,i_{k+1}\rangle} = M^{\langle i_0,\cdots,i_k\rangle}_{m(k+1,n)}[G_{\alpha_{k+1}\alpha_{k+2}}].$$

$$M_n^{\langle i_0,\cdots,i_k,i_{k+1}\rangle} =_{\omega_1} M^{\langle i_0,\cdots,i_k\rangle}_{m(k+1,n)}.$$

$$M_0^{\emptyset}[G_{\alpha_0\alpha^*}] =_{\omega_1} M_0^{\emptyset}.$$

$$M_0^{\langle i_0,\cdots,i_k\rangle}[G_{\alpha_{k+1}\alpha^*}] =_{\omega_1} M_0^{\langle i_0,\cdots,i_k\rangle}.$$

And so

$$M_0[G_{\alpha_0\alpha^*}] =_{\omega_1} M_0.$$

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha_0\alpha^*}] =_{\omega_1} M_{m(0,m(1,\cdots,m(k,0)\cdots))}.$$

Where $m(k,n)$ abbreviates $m(\langle i_0,\cdots,i_k\rangle,n)$.

We give a proof of iteration lemma 3.2.

3.2 Lemma. (Iteration Lemma) Let θ be a sufficiently large regular cardinal. Let N be a countable elementary substructure of H_θ with $I, \langle C_\delta \mid \delta\in A\rangle \in N$.

Let $(\alpha,\alpha^*,a,p,\langle \dot{M}_n \mid n<\omega\rangle)$ satisfy

- $\alpha<\alpha^*\le\nu$.
- $a\in P_\alpha$, $p\in P_{\alpha^*}$ and $a \le p\lceil\alpha$.
- $a \Vdash_{P_\alpha}$ "$N\cup\{\dot{G}_\alpha,p\} \subseteq \dot{M}_0$, $\langle \dot{M}_n \mid n<\omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\alpha]}$ and $\langle \dot{M}_n\cap\omega_1 \mid n<\omega\rangle \in \mathcal{F}^{V[\dot{G}_\alpha]}$".

 Then there exists $(q,\langle \dot{m}(n) \mid n<\omega\rangle)$ such that

- $q \in P_{\alpha^*}$ and $q \leq p$.
- $q\lceil\alpha = a$.
- $q \Vdash_{P_{\alpha^*}}$ "$\dot{m}(n)$ are strictly increasing natural numbers and $\dot{M}_{\dot{m}(n)}[\dot{G}_{\alpha^*}\lceil[\alpha,\alpha^*)] \cap \omega_1 = \dot{M}_{\dot{m}(n)} \cap \omega_1$".

  Notational Remark. Let $\alpha < \beta$ and G_β be P_β-generic over V. Then

- G_α denotes $G_\beta\lceil\alpha = \{r\lceil\alpha \mid r \in G_\beta\}$ which is P_α-generic over V.
- H_α denotes $\{r(\alpha)[G_\alpha] \mid r \in G_\beta\}$ which is $Q_\alpha = \dot{Q}_\alpha[G_\alpha]$-generic over $V[G_\alpha]$.

If $\dot{x}$ is a P_α-name, then we may view $\dot{x}[G_\alpha]$ as a term $\check{\dot{x}}[\dot{G}_\beta\lceil\check{\alpha}]$ being interpreted by G_β in $V[G_\beta]$. We simply denote this by x for easier notation. For sequences $s = \langle \dot{x}_n \mid n < \omega\rangle$ of P_α-names, we abbreviate as follows;

- $x_n = \dot{x}_n[G_\alpha]$ (the n-th value of the interpretation of a term $\langle \check{s}(n)[\dot{G}_\alpha] \mid n < \check{\omega}\rangle$ by G_α in $V[G_\alpha]$) $= \dot{x}_n[G_\beta\lceil\alpha]$ (the n-th value of the interpretation of a term $\langle \check{s}(n)[\dot{G}_\beta\lceil\check{\alpha}] \mid n < \check{\omega}\rangle$ by G_β in $V[G_\beta]$) for each $n < \omega$.

Proof. By induction on α^* for all $(\alpha, a, p, \langle \dot{M}_n \mid n < \omega\rangle)$.

Case 1. (Successor Steps Essential) Let $(\alpha, \alpha+1, a, p, \langle \dot{M}_n \mid n < \omega\rangle)$ be as in the hypothesis. Since $\Vdash_{P_\alpha}$ "$\dot{Q}_\alpha$ is $\mathcal{F}^{V[\dot{G}_\alpha]}$-generic-limsup-semiproper" and $a \Vdash_{P_\alpha}$ "$p(\alpha) \in \dot{Q}_\alpha \cap \dot{M}_0$, $\langle \dot{M}_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\alpha]}$ and $\langle \dot{M}_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}^{V[\dot{G}_\alpha]}$", we have $a \Vdash_{P_\alpha}$ "there exists $\pi \leq p(\alpha)$ in $\dot{Q}_\alpha$ such that $\pi \Vdash^{V[\dot{G}_\alpha]}_{\dot{Q}_\alpha}$ "$\{n < \omega \mid M_n[H_\alpha] \cap \omega_1 = M_n \cap \omega_1\}$ is infinite"". Let $q \in P_{\alpha+1}$ such that $q\lceil\alpha = a$ and $a \Vdash_{P_\alpha}$ "$q(\alpha) = \pi$". Then $q \leq p$ in $P_{\alpha+1}$ and $q \Vdash_{P_{\alpha+1}}$ "$\dot{M}_n[G_{\alpha\alpha+1}] \cap \omega_1 = \dot{M}_n[H_\alpha] \cap \omega_1 = \dot{M}_n \cap \omega_1$ for infinitely many $n < \omega$".

Case 2. (Successor Steps General) Let $(\alpha, \beta+1, a, p, \langle \dot{M}_n \mid n < \omega\rangle)$ be as in the hypothesis. We may assume $\alpha < \beta$. Apply the hypothesis of induction to $(\alpha, \beta, a, p\lceil\beta, \langle \dot{M}_n \mid n < \omega\rangle)$. We have $(q', \langle \dot{m}(n) \mid n < \omega\rangle)$ such that

- $q' \in P_\beta$, $a = q'\lceil\alpha$ and $q' \leq p\lceil\beta$.
- $q' \Vdash_{P_\beta}$ "$\dot{m}(n)$ are strictly increasing natural numbers and $M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1 = M_{\dot{m}(n)} \cap \omega_1$ for all $n < \omega$".
  Hence $q' \Vdash_{P_\beta}$ "$\langle M_{\dot{m}(n)}[G_{\alpha\beta}] \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\beta]}$ such that $N \cup \{\dot{G}_\beta, p\} \subseteq M_{\dot{m}(0)}[G_{\alpha\beta}]$ and $\langle M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}^{V[\dot{G}_\beta]}$". Now we are in case 1 with $(\beta, \beta+1, q', p, \langle M_{\dot{m}(n)}[G_{\alpha\beta}] \mid n < \omega\rangle)$. Hence we have $q \in P_{\beta+1}$ such that $q\lceil\beta = q'$, $q \leq p$ and $q \Vdash_{P_{\beta+1}}$ "$M_{\dot{m}(n)}[G_{\alpha\beta}][G_{\beta\beta+1}] \cap \omega_1 = M_{\dot{m}(n)}[G_{\alpha\beta}] \cap \omega_1$ for infinitely many $n < \omega$". And so $q\lceil\alpha = a$ and $q \Vdash_{P_{\beta+1}}$ "$M_m[G_{\alpha\beta+1}] \cap \omega_1 = M_m \cap \omega_1$ for infinitely many $m < \omega$".

Case 3. (Limit) Let $(\alpha, \alpha^*, a, p, \langle \dot{M}_n \mid n < \omega\rangle)$ be as in the hypothesis. We assume α^* is limit. Construct a tree representation T and a map $\langle \sigma \mapsto (\alpha^\sigma, a^\sigma, p^\sigma, \langle \dot{\delta}^\sigma_n \mid n < \omega\rangle, \langle \dot{M}^\sigma_n \mid n < \omega\rangle) \mid \sigma \in T\rangle$ such that

For $\langle\rangle = \emptyset \in T_0$, we set

(0) $\alpha^{\langle\rangle} = \alpha, a^{\langle\rangle} = a, p^{\langle\rangle} = p$, $\langle \dot{\delta}^{\langle\rangle}_k \mid k < \omega\rangle$ be any stages for p such that $\dot{\delta}^{\langle\rangle}_0 = \check{\alpha}$ and that $a \Vdash_{P_\alpha}$ "$\langle \dot{\delta}^{\langle\rangle}_k \mid k < \omega\rangle \in \dot{M}_0$" and $\langle \dot{M}^{\langle\rangle}_n \mid n < \omega\rangle = \langle \dot{M}_n \mid n < \omega\rangle$.

In general, for $\sigma = \langle i_0, \cdots, i_{k-1}\rangle \in T_k$, we demand

(1) $\alpha \leq \alpha^\sigma < \alpha^*$.
(2) $a^\sigma \in P_{\alpha^\sigma}$ and $a^\sigma\lceil\alpha \leq a$.
(3) $p^\sigma \in P_{\alpha^*}$ and $p^\sigma \leq p$.
(4) $a^\sigma \leq p^\sigma\lceil\alpha^\sigma$.
(5) $\langle \dot{\delta}^\sigma_n \mid n < \omega\rangle$ are stages for p^σ.
(6) $p^\sigma\lceil{\alpha^\sigma}^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}^\sigma_0 = \alpha^\sigma$" (0th-stage self-decisive condition).

(7) $\langle \dot{M}^\sigma_n \mid n < \omega\rangle$ is a sequence of P_{α^σ}-names such that $a^\sigma \Vdash_{P_{\alpha^\sigma}}$ "$N \cup \{\dot{G}_{\alpha^\sigma}, p^\sigma, \langle \dot{\delta}^\sigma_k \mid k < \omega\rangle\} \subsetneq \dot{M}^\sigma_0$, $\langle \dot{M}^\sigma_n \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_{\alpha^\sigma}]}$ and $\langle \dot{M}^\sigma_n \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}^{V[\dot{G}_{\alpha^\sigma}]}$".

For $\tau = \sigma^\frown\langle i\rangle = \langle i_0, \cdots, i_{k-1}, i_k\rangle \in T_{k+1}$, there exists a sequence $\langle \dot{m}(\tau, n) \mid n < \omega\rangle$ of P_{α^τ}-names and we demand

(8) $\alpha^\sigma \leq \alpha^\tau$.

(9) $a^\tau \lceil \alpha^\sigma \leq a^\sigma$.

(10) $p^\tau \leq p^\sigma$.

(11) $a^\tau \lceil \alpha^\sigma \Vdash_{P_{\alpha^\sigma}}$ "$p^\tau, \langle \dot{\delta}^\tau_n \mid n < \omega\rangle \in \dot{M}^\sigma_1$ and $p^\tau \lceil [\alpha^\sigma, \alpha^*)$ is $(P_{\alpha^\sigma\alpha^*}, \dot{M}^\sigma_0)$-semi-generic".

(12) For all $n < \omega$, $\Vdash_{P_{\alpha^*}}$ "$\dot{\delta}^\sigma_{n+1} \leq \dot{\delta}^\tau_n$" (a step ahead).

(13) $p^\tau \lceil \alpha^{\tau\frown}1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}^\tau_0 = \alpha^\tau$" (0th-stage self-decisive condition).

(14) $a^\tau \Vdash_{P_{\alpha^\tau}}$ "$\langle \dot{m}(\tau, n) \mid n < \omega\rangle$ is a sequence of strictly increasing natural numbers and for all $n < \omega$, $\dot{m}(\tau, n) \geq 1$, $\dot{M}^\sigma_{\dot{m}(\tau,n)}[G_{\alpha^\sigma\alpha^\tau}] = \dot{M}^\tau_n$ and $\dot{M}^\sigma_{\dot{m}(\tau,n)} \cap \omega_1 = \dot{M}^\tau_n \cap \omega_1$".

The contruction is by recursion on $k < \omega$. For $k = 0$, we set $T_0 = \{\emptyset\}$ and set $\alpha^\emptyset, a^\emptyset, p^\emptyset, \langle \dot{\delta}^\emptyset_n \mid n < \omega\rangle$ and $\langle \dot{M}^\emptyset_n \mid n < \omega\rangle$ as specified. This is possible as $a \Vdash_{P_\alpha}$ "$N \cup \{\dot{G}_\alpha, p\} \subseteq \dot{M}_0$". Then it is easy to see that all the assumptions (1) through (7) for $\sigma = \emptyset$ are satisfied.

Suppose we have constructed T_k and $\alpha^\sigma, a^\sigma, p^\sigma, \langle \dot{\delta}^\sigma_n \mid n < \omega\rangle$ and $\langle \dot{M}^\sigma_n \mid n < \omega\rangle$ for each $\sigma \in T_k$ such that (1) through (7) are satisfied. Let $\gamma = \alpha^\sigma$, $w = a^\sigma$, $x = p^\sigma$, $\langle \dot{\delta}_n \mid n < \omega\rangle = \langle \dot{\delta}^\sigma_n \mid n < \omega\rangle$ and $\langle \dot{N}_n \mid n < \omega\rangle = \langle \dot{M}^\sigma_n \mid n < \omega\rangle$ for shorter notation. Then $w \in P_\gamma$ forces that

- $N \cup \{\dot{G}_\gamma, x, \langle \dot{\delta}_n \mid n < \omega\rangle\} \subseteq \dot{N}_0$ and $x\lceil\gamma \in \dot{G}_\gamma$.

Hence by the iteration lemma for semiproperness and lemmas on stages (please see [M] for an account), there exists $(\beta, y, \langle \dot{\delta}^y_n \mid n < \omega\rangle)$ in $V[\dot{G}_\gamma]$ such that

- $\gamma < \beta < \alpha^*$.
- $y \leq x$ in P_{α^*}.
- $\langle \dot{\delta}^y_n \mid n < \omega\rangle$ are stages for y.
- $y\lceil\beta^\frown 1 \Vdash_{P_{\alpha^*}}$ "$\dot{\delta}^y_0 = \beta$".
- For all $n < \omega$, $\Vdash_{P_{\alpha^*}}$ "$\dot{\delta}_{n+1} \leq \dot{\delta}^y_n$" (a step ahead).
- $y\lceil\gamma \in \dot{G}_\gamma$.
- $y\lceil[\gamma, \alpha^*)$ is $(P_{\gamma\alpha^*}, \dot{N}_0)$-semi-generic
- $(\beta, y, \langle \dot{\delta}^y_n \mid n < \omega\rangle) \in \dot{N}_1$.

Then for any $d \leq w$ in P_γ such that d decides the values of β, y and $\langle \dot{\delta}^y_n \mid n < \omega\rangle$, we may consider $(\gamma, \beta, d, y\lceil\beta, \langle \dot{N}_n \mid 1 \leq n < \omega\rangle)$ satisfying

- $\gamma < \beta < \alpha^*$.
- $d \in P_\gamma$, $y\lceil\beta \in P_\beta$ and $d \leq (y\lceil\beta)\lceil\gamma$.
- $d \Vdash_{P_\gamma}$ "$N \cup \{\dot{G}_\gamma, y\lceil\beta\} \subseteq \dot{N}_1$ and $\langle \dot{N}_n \mid 1 \leq n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\gamma]}$ and $\langle \dot{N}_n \cap \omega_1 \mid 1 \leq n < \omega\rangle \in \mathcal{F}^{V[\dot{G}_\gamma]}$".

Now we apply the hypothesis of induction at β. Hence there exists $(b, \langle \dot{m}(n) \mid n < \omega\rangle)$ such that

- $b \in P_\beta$, $b\lceil\gamma = d$ and $b \leq y\lceil\beta$.
- $b \Vdash_{P_\beta}$ "$\langle \dot{m}(n) \mid n < \omega\rangle$ is a sequence of strictly increasing natural numbers such that $1 \leq \dot{m}(n)$ and $\dot{N}_{\dot{m}(n)}[G_{\gamma\beta}] \cap \omega_1 = \dot{N}_{\dot{m}(n)} \cap \omega_1$".

And so

- $b \Vdash_{P_\beta}$ "$N \cup \{\dot{G}_\beta, y, \langle \dot{\delta}^y_n \mid n < \omega\rangle\} \subsetneq \dot{N}_{\dot{m}(0)}[G_{\gamma\beta}]$, $\langle \dot{N}_{\dot{m}(n)}[G_{\gamma\beta}] \mid n < \omega\rangle$ is an $\in$-chain in $H_\theta^{V[\dot{G}_\beta]}$ and $\langle \dot{N}_{\dot{m}(n)}[G_{\gamma\beta}] \cap \omega_1 \mid n < \omega\rangle \in \mathcal{F}^{V[\dot{G}_\beta]}$".

Since there exists d as above predense many below w, we may construct T_{k+1} and

$$\langle \tau \mapsto (\alpha^\tau, a^\tau, p^\tau, \langle \dot{\delta}_n^\tau \mid n < \omega\rangle, \langle \dot{M}_n^\tau \mid n < \omega\rangle, \langle \dot{m}(\tau, n) \mid n < \omega\rangle) \mid \tau \in T_{k+1}\rangle,$$

where the correspondences are $\alpha^\tau = \beta, a^\tau = b, p^\tau = y, \langle \dot{\delta}_n^\tau \mid n < \omega\rangle = \langle \dot{\delta}_n^y \mid n < \omega\rangle, \langle \dot{m}(\tau, n) \mid n < \omega\rangle = \langle \dot{m}(n) \mid n < \omega\rangle$ and $\langle \dot{M}_n^\tau \mid n < \omega\rangle = \langle \dot{N}_{\dot{m}(n)}[G_{\gamma\beta}] \mid n < \omega\rangle$. This completes the construction.

Let q be a fusion of the tree representation T. Let G_α^* be P_{α^*}-generic over V with $q \in G_{\alpha^*}$. Let us calculate $\langle i_n \mid n < \omega\rangle$ from the generic cofinal path through T so that for all $k < \omega$, $\langle i_n \mid n < k\rangle \in T_k$ and $a^{\langle i_n \mid n<k\rangle} \in G_{\alpha^{\langle i_n \mid n<k\rangle}}$.

Let

$$M_n = \dot{M}_n[G_\alpha],$$

$$\alpha^k = \alpha^{\langle i_n \mid n<k\rangle},\ a^k = a^{\langle i_n \mid n<k\rangle},\ p^k = p^{\langle i_n \mid n<k\rangle},$$

$$\dot{\delta}_m^k = \dot{\delta}_m^{\langle i_n \mid n<k\rangle},\ \dot{M}_m^k = \dot{M}_m^{\langle i_n \mid n<k\rangle},\ M_m^k = \dot{M}_m^k[G_{\alpha^k}],\ m(k,n) = \dot{m}(\langle i_0, \cdots, i_k\rangle, n)[G_{\alpha^{k+1}}].$$

Then

$$a^k \in G_{\alpha^k},\ p^k \in G_{\alpha^*}.$$

$$M_0 =_{\omega_1} M_0[G_{\alpha\alpha^*}],$$

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^1}]\cdots[G_{\alpha^k\alpha^{k+1}}] = M_0^{k+1},$$

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^1}]\cdots[G_{\alpha^k\alpha^{k+1}}][G_{\alpha^{k+1}\alpha^*}] = M_0^{k+1}[G_{\alpha^{k+1}\alpha^*}],$$

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^*}] \subseteq M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^1}]\cdots[G_{\alpha^k\alpha^{k+1}}][G_{\alpha^{k+1}\alpha^*}].$$

Hence

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^*}] \subseteq M_0^{k+1}[G_{\alpha^{k+1}\alpha^*}] \supseteq_{\omega_1} M_0^{k+1} \supseteq_{\omega_1} M_{m(0,m(1,\cdots,m(k,0)\cdots))}.$$

So

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^*}] \cap \omega_1 \leq M_0^{k+1}[G_{\alpha^{k+1}\alpha^*}] \cap \omega_1 = M_0^{k+1} \cap \omega_1 = M_{m(0,m(1,\cdots,m(k,0)\cdots))} \cap \omega_1.$$

So

$$M_{m(0,m(1,\cdots,m(k,0)\cdots))} \subseteq_{\omega_1} M_{m(0,m(1,\cdots,m(k,0)\cdots))}[G_{\alpha^0\alpha^*}].$$

Note that $m(0, m(1, \cdots, m(k, 0)\cdots))$ strictly increase.

□

References

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