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A maximal forcing axiom compatible with weak club guessing
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Abstract

We show there is no maximal forcing axiom compatible with tail club guessing. On the other hand, we may formulate a maximal forcing axiom compatible with a weak club guessing.

Introduction

We formulate a forcing axiom compatible with a tail club guessing in [M]. This note is a continuation to [M]. We show a maximal forcing axiom compatible with tail club guessing does not hold. On the other hand, we may force a maximal forcing axiom compatible with a weak club guessing. Namely, if a ladder system \( (C_\delta \mid \delta \in A) \) is weak club guessing and a supercompact cardinal exists, then there is a model of set theory where

1. \( (C_\delta \mid \delta \in A) \) remains weak club guessing. Namely, for any club \( D \) of \( \omega_1 \), there exists \( \delta \in A \) such that \( D \cap C_\delta \) is infinite.
2. Let \( P \) be any preorder such that \( P \) preserves every stationary subset of \( \omega_1 \) and that for any \( B \subseteq A \) such that the ladder system \( (C_\delta \mid \delta \in B) \) is weak club guessing, \( P \) also preserves the ladder system \( (C_\delta \mid \delta \in B) \) to be weak club guessing. Then every system \( (D_i \mid i < \omega_1) \) of dense subsets of \( P \) has a filter which hits every \( D_i \).

§1. No maximal forcing axioms are compatible with TCG

We set our notation.

1.1 Definition A sequence \( (C_\delta \mid \delta \in A) \) is a ladder system, if

- \( A \subseteq \{ \delta \mid \delta < \omega_1, \delta \) is a limit ordinal\).
- For every \( \delta \in A \), \( C_\delta \) is a cofinal subset of \( \delta \) and is of order type \( \omega \).

A ladder system \( (C_\delta \mid \delta \in A) \) is tail club guessing, if for any club \( D \subseteq \omega_1 \), there exists \( \delta \in A \) such that \( C_\delta \setminus D \) is finite. A ladder system \( (C_\delta \mid \delta \in A) \) is weak club guessing, if for any club \( D \subseteq \omega_1 \), there exists \( \delta \in A \) such that \( C_\delta \cap D \) is infinite. Hence if \( (C_\delta \mid \delta \in A) \) is tail club guessing, then it is weak club guessing.

Fix a ladder system \( (C_\delta \mid \delta \in A) \). We write for small sets and positive sets as follows;

- \( (TCG) = \{ X \subseteq \omega_1 \mid (C_\delta \mid \delta \in A \cap X) \) fails to be tail club guessing\}.
- \( (TCG)^+ = \{ X \subseteq \omega_1 \mid (C_\delta \mid \delta \in A \cap X) \) is tail club guessing\}.

Similarly,

- \( (WCG) = \{ X \subseteq \omega_1 \mid (C_\delta \mid \delta \in A \cap X) \) fails to be weak club guessing\}.
- \( (WCG)^+ = \{ X \subseteq \omega_1 \mid (C_\delta \mid \delta \in A \cap X) \) is weak club guessing\}.

We know of a forcing axiom which is compatible with tail club guessing.

Theorem. \((M)\) Let \( (C_\delta \mid \delta \in A) \) be tail club guessing. Then we may force the following, assuming that a supercompact cardinal exists.

1. \( (C_\delta \mid \delta \in A) \) remains tail club guessing.
2. Forcing axiom \( ^+ \) holds for the class of partially ordered sets \( P \) which are semiproper and \( (C_\delta \mid \delta \in A)\)-\( \omega \)-semiproper,
On the other hand,

1.2 Proposition. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be a ladder system. Let \( n_* < \omega \). Then there exists a partially ordered set \( P \) and a \( P \)-name \( \check{D} \) such that

(1) \( P \) is proper and \( (TCG)^+ \)-preserving.
(2) \( \models \neg \vDash_{\check{D}} \text{is a club in } \omega_1 \text{ such that for all } \delta \in A, |C_{\delta} \setminus \check{D}| \geq n_* \).

\( \square \)

We make use of this \( P \) in two ways. First, we observe TCG-sequences may get killed at limit stages of iterated forcing.

1.3 Corollary. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be a ladder system. Then there exists an iterated forcing \( \langle P_n \mid n < \omega \rangle \) such that

- If \( \langle C_{\delta} \mid \delta \in A \rangle \) is tail club guessing, then for all \( n < \omega, \models \neg \vDash_{P_n} \langle C_{\delta} \mid \delta \in A \rangle \) remains tail club guessing".
- If \( P_n \) is any limit of the \( P_n \)'s, then \( \models \neg \vDash_{P_n} \langle C_{\delta} \mid \delta \in A \rangle \) must fail to be tail club guessing".

Second, we put above in terms of forcing axiom. Suppose \( \langle C_{\delta} \mid \delta \in A \rangle \) is tail club guessing. Then no maximal forcing axioms hold for the class of partially ordered sets \( P \) which preserve all stationary subsets of \( \omega_1 \) and all elements of \( (TCG)^+ \) (i.e. for any \( X \subseteq \omega_1, \text{ if } \langle C_{\delta} \mid \delta \in A \cap X \rangle \text{ is tail club guessing, then it remains so in the generic extensions of } P \)).

1.4 Corollary. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be tail club guessing. Let forcing axiom hold for the class of partially ordered sets \( P \) such that \( P \) are proper and that for any \( B \subseteq A \) such that \( \langle C_{\delta} \mid \delta \in B \rangle \) is tail club guessing, then \( \models \neg \vDash_{\check{D}} \langle C_{\delta} \mid \delta \in B \rangle \) remains to be tail club guessing". Then we have a contradiction.

\( \square \)

Proof of proposition 1.2. Let \( p \in P \), if \( p = (\alpha^p, D^p) \) such that

(1) \( \alpha^p < \omega_1 \).
(2) \( D^p \subseteq \alpha^p + 1, \alpha^p \in D^p \) and \( D^p \) is closed.
(3) For all \( \delta \in A \) with \( \delta \leq \alpha^p, |C_{\delta} \setminus D^p| \geq n_* \).

For \( p, q \in P \), let \( q \leq p \), if

(1) \( \alpha^p \leq \alpha^q \).
(5) \( D^p = D^q \cap (\alpha^p + 1) \).

Claim. (Dense) For any \( p \in P \) and any \( \eta \) with \( \alpha^p < \eta < \omega_1 \), there exists \( q \leq p \) such that \( \alpha^q = \eta \).

Proof. Let \( \alpha^q = \eta \) and \( D^q = D^p \cup \{ \eta \} \). Then this \( q = (\alpha^q, D^q) \) works.

\( \square \)

Claim. \( P \) is proper and \( \sigma \)-Baire.

Proof. Let \( \theta \) be a sufficiently large regular cardinal. Let \( N \) be a countable elementary substructure of \( H_\theta \) with \( P \in N \). Let \( p \in N \cap P \). We want \( q \leq p \) such that \( q \) is \( (P, N) \)-generic. Let \( \delta = N \cap \omega_1 \). We construct a \( (P, N) \)-generic sequence \( \langle \eta_n \mid n < \omega \rangle \) such that \( p_0 \leq p \) and that \( \{ \alpha^p \cap C_{\delta} \} \setminus D^{p_0} \geq n_* \). Let \( \alpha^q = \delta \) and \( D^q = \{ D^{p_n} \mid n < \omega \} \cup \{ \delta \} \). Then this \( q = (\alpha^q, D^q) \) works.

\( \square \)

Claim. If \( X \subseteq \omega_1 \) such that \( \langle C_{\delta} \mid \delta \in A \cap X \rangle \) is tail club guessing, then \( \models \neg \vDash_{\check{D}} \langle C_{\delta} \mid \delta \in A \cap X \rangle \) remains to be tail club guessing".
Proof. Suppose $p \Vdash \text{"C" is a club in } \omega_1^\omega$. Want $q \leq p$ and $\delta \in A \cap X$ such that $q \Vdash \text{"C}_\delta \setminus \text{C" is finite".}$ To this end, take an $\in$-chain $\langle N_i \mid i < \omega_1 \rangle$ in $H_\theta$, where $\theta$ is sufficiently large. Since $\langle \text{C}_\delta \mid \delta \in A \cap X \rangle$ is tail club guessing, there exists $\delta \in A \cap X$ such that $\text{C}_\delta \setminus \{N_i \cap \omega_1 \mid i < \omega_1 \}$ is finite. By renaming, we have an $\in$-chain $(N_n \mid n < \omega)$ in $H_\theta$ such that $\{N_n \cap \omega_1 \mid n < \omega\}$ is an end-segment of $\text{C}_\delta$. We may assume that $P, p, \text{C} \in N_0$ and $|\text{C}_\delta \setminus \{N_0 \cap \omega_1\}| \geq n_*$. We construct a descending sequence $(q_n \mid n < \omega)$ of conditions such that

- $q_0 \leq p$.
- $\|(\alpha^{q_0} \cap \text{C}_\delta) \setminus D^{q_0}\| \geq n_*$.
- $q_n \in N_{n+1} \cap P$ is $(P, N_n)$-generic.

Let $\alpha^{q_0} = \delta$ and $D^{q_0} = \bigcup\{D^{p} \mid n < \omega \} \cup \{\delta\}$. Then $q \Vdash P_n \cap \omega_1 \in \text{C}_\delta$ for all $n < \omega$. Hence this $q$ works.

Claim. Let $G$ be $P$-generic over $V$. Let $D = \bigcup\{D^{p} \mid p \in G\}$. Then $D$ is a club in $\omega_1$ and for all $\delta \in A$, $|\text{C}_\delta \setminus D| \geq n_*$.

Proof of corollary 1.3. Iteratively force clubs $D_n$ in $\omega_1$ so that for all $\delta \in A$ and all $n < \omega_1$, $|\text{C}_\delta \setminus D_n| \geq n$. Then let $D = \bigcap\{D_n \mid n < \omega\}$. If $\omega_1$ gets preserved, then $D$ is a club in $\omega_1$ such that for all $\delta \in A$, $\text{C}_\delta \setminus D$ is infinite. Hence $\langle \text{C}_\delta \mid \delta \in A \rangle$ fails to be tail club guessing. If $\omega_1$ gets collapsed, then this entails the same conclusion.

Proof of corollary 1.4 is the same. Argue in $V$ and get a sequence $(D_n \mid n < \omega)$ of clubs in $\omega_1$.

§2. A maximal forcing axiom is compatible with WCG

We have seen that there is no maximal forcing axiom compatible with tail club guessing (TCG). But a weak club guessing (WCG) admits maximal one.

2.1 Definition. Let $\langle \text{C}_\delta \mid \delta \in A \rangle$ be a ladder system. Let $\mathcal{F}$ denote the set of all cofinal subsequences of $\text{C}_\delta$ (viewed as sequences of length $\omega$) for all $\delta \in A$. Let $\text{Seq}^{\omega}(\omega_1)$ denote the set of all sequences $(a_n \mid n < \omega)$ such that each $a_n$ is a countable subset of $\omega_1$. Hence we have $\mathcal{F} \subseteq \text{Seq}^{\omega}(\omega_1)$. Let $P$ be a preorder, we say $P$ is $\mathcal{F}$-limsup-semiproper, if for all sufficiently large regular cardinals $\theta$ and all $\in$-chains $\langle N_n \mid n < \omega \rangle$ in $H_\theta$ with $P, \langle \text{C}_\delta \mid \delta \in A \rangle \in N_0$, if $\langle N_n \cap \omega_1 \mid n < \omega \rangle \in \mathcal{F}$, then for any $p \in N_0 \cap P$, there exists $q \leq p$ such that for infinitely many $n < \omega$, $q$ is $(P, N_n)$-semi-generic.

Similarly, we say $P$ is $\mathcal{F}$-liminf-semiproper, if $q$ is $(P, N_n)$-semi-generic for all but finitely many $n < \omega$.

Lastly, we say $P$ is $\mathcal{F}$-generic-limsup-semiproper, if $q \Vdash P_{\omega_1}[G] \cap \omega_1^\omega = N \cap \omega_1^\omega$ for infinitely many $n < \omega$.

Hence we are looking at the set $\{n < \omega \mid N[G] \cap \omega_1^\omega = N \cap \omega_1^\omega \}$ in $V[G]$ which might be infinite and may or may not be in $V$.

For the notion of $\omega$-stationary sets $S \subseteq \text{Seq}^{\omega}(K) = \{a_n \mid n < \omega \} \cap K$ for all $n < \omega$, may see [M]. They are analogously formulated as the stationary sets in $|K|^\omega$.

2.2 Proposition. Let $P$ be a preorder.

- If $P$ is $\omega_1$-closed, then $P$ is $\omega$-semiproper.
- If $P$ is $\omega$-semiproper, then $P$ is $\mathcal{F}$-liminf-semiproper.
- If $P$ is $\mathcal{F}$-liminf-semiproper, then $P$ is $\mathcal{F}$-limsup-semiproper.
- If $P$ is $\mathcal{F}$-limsup-semiproper, then $P$ is $\mathcal{F}$-generic-limsup-semiproper.
- If $\mathcal{F}$ is $\omega$-stationary and $P$ is $\mathcal{F}$-generic-limsup-semiproper, then $P$ preserves $\omega_1$. 
2.3 Definition. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be a ladder system. We formulate (proper, \( \langle C_{\delta} \mid \delta \in A \rangle \)-limsup-semiproper, full) -Reflection Principle (abusively, \( \mathcal{F}\)-RP) as follows: Given any \((K, S, \theta, a)\) such that

- \( K \) is a set with \( K \supseteq \omega_{1} \).
- \( S \subseteq \text{Seq}^{\omega}(K) \).
- \( \theta \) is a regular cardinal with \( K \in H_{|\text{TC}(K)|}^{+} \in H_{(2|\text{TC}(K)|)^{+}} \in H_{\theta} \).
- \( a \in H_{\theta} \).

There exists \( (D, \langle N_{i} \mid i < \omega_{1} \rangle) \) such that

- \( D \) is a club in \( \omega_{1} \).
- \( \langle N_{i} \mid i < \omega_{1} \rangle \) is an \( \varepsilon \)-chain in \( H_{\theta} \) with \( a \in N_{0} \).
- For any \( \delta \in \mathcal{Y}^{*}(D) = \{ \delta \in A \mid |C_{\delta} \cap D| = \omega \} \), let \( \langle C_{\delta}(k_{\delta}(m)) \mid m < \omega \rangle \) enumerate \( \{ C_{\delta}(k) \mid C_{\delta}(k) \in D \} \).

Then there exists \( n_{\delta} < \omega \) such that we have either (1) or (2).

1. \( \langle N_{i} \mid i < \omega_{1} \rangle \cap K \cap n_{\delta} \leq m < \omega \rangle \in S \).
2. For any strictly increasing sequence \( \langle m_{l} \mid l < \omega \rangle \) of natural numbers with \( n_{\delta} \leq m_{0} \) and for any \( \varepsilon \)-chain \( \langle M_{l} \mid l < \omega \rangle \) defined on \( \langle N_{i} \cap K \mid l < \omega \rangle \notin S \).

We might call \( i_{\delta} = C_{\delta}(k_{\delta}(n_{\delta})) \) a critical point of \( C_{\delta} \) with respect to \( D \) for each \( \delta \in \mathcal{Y}^{*}(D) \). Hence we are looking at \( \langle N_{i} \mid i_{\delta} \leq i < \omega \rangle \cap D \).

2.4 Theorem. Let \( \mathcal{F}^{\mathcal{Y}} \subseteq \text{Seq}^{\omega}(\omega_{1}) \) be defined by \( \langle C_{\delta} \mid \delta \in A \rangle \). Then \( \mathcal{F}^{\mathcal{Y}^{[G_{\alpha}]}} \)-generic-limsup-semiproper combined with semiproper iterates under the simple iteration. (The \( \mathcal{F}^{\mathcal{Y}^{[G_{\alpha}]}} \) are uniformly defined from the ladder system \( \langle C_{\delta} \mid \delta \in A \rangle \) in each intermediate universe \( V[G_{\alpha}] \). The exact value of \( \mathcal{F}^{V[G_{\alpha}]} \) increasingly changes as \( \alpha \) gets bigger.)

2.5 Corollary. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be weak club guessing so that \( \mathcal{F} \) defined from \( \langle C_{\delta} \mid \delta \in A \rangle \) is \( \omega \)-stationary. Let us recall \( \mathcal{F} = \{ \langle C_{\delta}(k(m)) \mid m < \omega \rangle \mid \delta \in A \} \). \( \langle k(m) \mid n < \omega \rangle \) is a sequence of strictly increasing natural numbers. We may force the following, if there exists a supercompact cardinal.

1. If \( \langle C_{\delta} \mid \delta \in A \rangle \) is weak club guessing in the ground model, then it remains to be so in the extensions.
2. Forcing axiom \( \square^{+} \) holds for the class of partially ordered sets \( P \) such that \( P \) is \( \mathcal{F} \)-generic-limsup-semiproper.

2.6 Lemma. Let \( \langle C_{\delta} \mid \delta \in A \rangle \) be a ladder system. Let \( \mathcal{F} \subseteq \text{Seq}^{\omega}(\omega_{1}) \) be defined from the system. Let \( \mathcal{F} \)-Reflection Principle hold. Let us consider \( (WCG)^{+} \) with respect to the system. Let \( P \) be a preorder. Then the following are equivalent on \( P \).

1. \( P \) is \( (WCG)^{+} \)-preserving.
2. \( P \) is \( \mathcal{F} \)-generic-limsup-semiproper.

The \( \mathcal{F}\)-RP gets forced by a little better notion of forcing than semiproper + \( \mathcal{F} \)-generic-limsup-semiproper partially ordered set.

2.7 Lemma. Let \( \mathcal{F} \subseteq \text{Seq}^{\omega}(\omega_{1}) \) be as above. Let forcing axiom hold for the class of preorder set \( P \) such that \( P \) is proper and that \( P \) is \( \mathcal{F} \)-limsup-semiproper. Then \( \mathcal{F} \)-Reflection Principle holds.

2.8 Corollary. The following is consistent, if there exists a supercompact cardinal.

1. \( \langle C_{\delta} \mid \delta \in A \rangle \) is weak club guessing.
2. The forcing axiom \( \square^{+} \) holds for the class of preorder set \( P \) such that \( P \) preserves every stationary subset of \( \omega_{1} \) and that \( P \) preserves every member of \( (WCG)^{+} \) with respect to \( \langle C_{\delta} \mid \delta \in A \rangle \).
Proof of lemma 2.6. (2) implies (1): No use of $\mathcal{F}$-RP made in this direction. Let $X \subseteq \omega_1$ such that $(C_\delta \mid \delta \in X \cap A)$ is a club guessing. Suppose $p \vdash \neg \langle C \mid \delta \in X \cap A \rangle$ is a club. Want $q \leq p$ and $\delta \in X \cap A$ such that $q \vdash \neg \langle C \rangle$. To this end, let $\theta$ be a sufficiently large regular cardinal. Since $(C_\delta \mid \delta \in X \cap A)$ is club guessing, we may take an $\epsilon$-chain $(N_\delta \mid n < \omega)$ in $H_\delta$ such that $(N_\delta \cap \omega_1 \mid n < \omega) \in H_\delta$ is a cofinal subset of $C_\delta$, where $\delta = N_\omega \cap \omega_1$ and $N_\lambda = \bigcup\{N_\delta \mid n < \omega\}$. We may assume $P$, $(C_\delta \mid \delta \in A), p, \dot{C} \in N_0$. Since $P$ is $\mathcal{F}$-generic-limsup-semi-proper and $(N_\delta \cap \omega_1 \mid n < \omega) \in \mathcal{F}$, we have a $q$ such that $q \vdash \neg p$ and $q \vdash \neg \langle N_\delta \mid n < \omega \rangle \in \mathcal{F}$ for infinitely many $n < \omega$. Hence $q \vdash \neg \langle N_\delta \mid n < \omega \rangle \in \mathcal{F}$ for infinitely many $n < \omega$ and so $q \vdash \neg \langle C \rangle$.

Claim. $B = \{\delta \in Y^*(D) \mid \delta \models \text{(1) in $\mathcal{F}$-RP}$\} \in (WCG)^+$.

Proof. Let $E$ be a club in $\omega_1$. Want $\delta \in B$ with $\langle C_\delta \cap E \rangle = \omega$. Since $S$ is $\omega$-stationary in Seq$^\omega(H_\theta)$, we may take an $\epsilon$-chain $(N_\delta \mid n < \omega)$ in $H_\delta$ such that $D, \langle M_i \mid i < \omega_1 \rangle, E \in N_\delta$ and that $(N_\delta \cap H_\theta \mid n < \omega) \in S$. Let $N^*_\delta = \bigcup\{N_\delta \mid n < \omega\}$ and $\dot{\delta} = N^*_\delta \cap \omega_1$. Then $(N_\delta \cap \omega_1 \mid n < \omega) \in S$ and is through $D \cap E$. In particular, $\delta \in Y^*(D)$ and $\langle C_\delta \cap E \rangle = \omega$. Let $\{N_\delta \cap \omega_1 \mid n < \omega\} \subseteq C_\delta \cap D = \langle C_\delta(k_m) \rangle \mid m < \omega \rangle$. By considering an end-segment, we may assume $(N_\delta \cap H_\theta \mid n < \omega) \in S^*$ with $C_\delta(k_m) < N^*_\delta \cap \omega_1$.

Want (1) holds at this $\delta$ so that $\delta \in B$. Since $N^*_\delta \supseteq \omega_1, M(N^*_\delta \cap \omega_1) = \omega$ for all $n < \omega$ and $(N_\delta \cap H_\theta \mid n < \omega) \in S^*$, (2) in $\mathcal{F}$-RP fails. Hence (1) must hold in $\mathcal{F}$-RP.

Proof of lemma 2.7. Let $(K, S, \theta, a)$ be as the hypothesis of $\mathcal{F}$-RF. We force $D, \langle N_i \mid i < \omega \rangle$ and $\langle n_\delta \mid \delta \in A \rangle$ by initial segments. Let $p \in P$, if $p = \langle \alpha^p, D^p, \langle N_i^p \mid i \leq \alpha^p \rangle, \langle n_\delta^p \mid \delta \in A \cap (\alpha^p + 1) \rangle \rangle$ satisfies the following:

1. $\alpha^p < \omega_1$.
2. $D^p \subseteq \alpha^p + 1, \alpha^p \in D^p$ and $D^p$ is closed.
3. $\langle N_i^p \mid i \leq \alpha^p \rangle$ is an $\epsilon$-chain in $H_\theta$ with $a \in N^p_0$.
4. For each $\delta \in A \cap (\alpha^p + 1), n_\delta^p < \omega$. If $C_\delta \cap D^p = \omega$, then let $\langle C_\delta(k) \mid k < \omega \rangle$ enumerate $C_\delta$ and let $\langle C_\delta(k_m^p) \mid m < \omega \rangle$ enumerate $C_\delta \cap D^p$. And we demand either (i) or (ii).
(i) \(\langle N_{C_{k}(k_{m})}^{p} \cap K \mid n_{m}^{p} \leq m < \omega \rangle \in S\).

(ii) For any strictly increasing sequence \(\langle n_{m} \mid l < \omega \rangle\) of natural numbers with \(n_{m}^{p} \leq n_{0}\) and any \(\varepsilon\)-chain \(\langle M_{l} \mid l < \omega \rangle\) in \(H_{\theta}\) such that \(\langle M_{l} \mid l < \omega \rangle \supseteq_{\omega_{1}} \langle N_{C_{k}(k_{m})}^{p} \mid l < \omega \rangle\), we have \(\langle M_{l} \cap K \mid l < \omega \rangle \notin S\).

For \(p, q \in P\), we set \(q \leq p\), if

(5) \(\alpha^{p} \leq \alpha^{q}\).

(6) \(D^{\alpha} = D^{\alpha} \cap (\alpha^{p} + 1)\).

(7) For all \(i \leq \alpha^{p}\), \(N_{i}^{p} = N_{i}^{q}\).

(8) For all \(\delta \in A \cap (\alpha^{p} + 1)\), \(n_{\delta}^{p} = n_{\delta}^{q}\).

Claim. (Dense, Extension by Escape) For any \((p, \eta, x)\) such that \(p \in P\), \(\alpha^{p} < \eta < \omega_{1}\) and \(x \in H_{\theta}\), there exists \(q \in P\) such that \(q \leq p\), \(\alpha^{q} = \eta\) and \(x \in N_{q}\).

Proof. Let \(\alpha^{q} = \eta\), \(D^{q} = D^{p} \cup \{\eta\}\) and \(\langle N_{i}^{q} \mid i \leq \alpha^{q}\rangle\) be any \(\varepsilon\)-chain which end-extends \(\varepsilon\)-chain \(\langle N_{i}^{p} \mid i \leq \alpha^{p}\rangle\) and \(x \in N_{q}\). Let \(\langle n_{\delta}^{q} \mid \delta \in A \cap (\alpha^{q} + 1)\rangle\) be any sequence of natural numbers which end-extends \(\langle n_{\delta} \mid \delta \in A \cap (\alpha^{p} + 1)\rangle\). Since for any \(\delta \in A \cap (\alpha^{q} + 1)\), \(|C_{\delta} \cap D^{q}| = \omega\) iff \((\delta \leq \alpha^{p}\) and \(|C_{\delta} \cap D^{p}| = \omega)\), this \(q\) works.

Claim. (Targeted-Extension) Let \(\lambda\) be a sufficiently large regular cardinal and \(M\) be a countable elementary substructure of \(H_{\lambda}\) with \(P \in M\). Let \(\delta_{M} = M \cap \omega_{1}\). Then for any \((p, \xi, D)\) such that \(p \in M \cap P\), \(\xi < \delta_{M}\) and \(D \in M\) is a dense subset of \(P\), there exists \(r \in M \cap D\) such that \(r \leq p\), \(\xi < \alpha^{r}\) and \(C_{\delta_{M}} \cap D^{r} = C_{\delta_{M}} \cap D^{p}\).

Proof. We consider a family of maps indexed by \(p \in P\). Let \(p \in P\). Let \(\eta\) be such that \(\alpha^{p} < \eta < \omega_{1}\). Let \(r = f_{p}(\eta)\), where \(r \in D, r \leq p, \eta < \alpha^{r}\) and \(D^{r} \cap (\alpha^{p}, \eta] = \{\eta\}\). We may assume \((f_{p} \mid p \in P) \subseteq M\). Since \(p \in M\), \(f_{p} \subseteq M\). Let \(C(f_{p}) = \{\beta < \omega_{1} \mid \forall \eta \in M, \alpha^{p} < \eta < \beta, \alpha^{f_{p}(\eta)} < \beta\}\). Then \(C(f_{p})\) is a club in \(\omega_{1}\) and \(C(f_{p}) \subseteq M\). Take \(\beta \in C(f_{p}) \subseteq M\) such that \(\alpha^{q}, \xi < \beta\). Let \(\eta\) be such that \(\alpha^{p}, \xi, max(C_{\delta_{M}} \cap \beta) < \eta < \beta\). Then \(f_{p}(\eta) \in M\) and so there exists \(r \in M \cap D\) such that \(r \leq p, \xi < \alpha^{r}\) and \(C_{\delta_{M}} \cap D^{r} = C_{\delta_{M}} \cap D^{p}\).

Claim. (Proper) \(P\) is proper and \(\sigma\)-Baire.

Proof. Let \(\lambda\) be a sufficiently large regular cardinal and \(M\) be a countable elementary substructure of \(H_{\lambda}\) with \(P \in M\). Let \(p \in M \cap P\). Let \(\delta_{M} = M \cap \omega_{1}\). Then by repeating above claim, we may construct a \((P, M)\)-generic sequence \(\langle p_{n} \mid n < \omega \rangle\) such that \(C_{\delta_{M}} \cap D^{p_{n}} = C_{\delta_{M}} \cap D^{p}\) for all \(n < \omega\). Let \(\alpha^{q} = \delta_{M}\), \(D^{q} = \bigcup\{D^{p_{n}} \mid n < \omega\} \cup \{\delta_{M}\}\). \(\langle N_{q}^{q} \mid i \leq \delta_{M}\rangle = \bigcup\{\langle N_{q}^{p_{n}} \mid i \leq \alpha^{p_{n}}\rangle \mid n < \omega\} \cup \{\langle \delta_{M}, M \cap H_{\theta}\rangle\}\). \(\langle n_{\delta}^{q} \mid \delta \in A \cap (\delta_{M} + 1)\rangle\) be any end-extension of all of \(\langle n_{\delta}^{p_{n}} \mid \delta \in A \cap (\alpha^{p_{n}} + 1)\rangle\). Notice that \(C_{\delta_{M}} \cap D^{q} = C_{\delta_{M}} \cap D^{p}\) which is finite. Hence this \(q\) is a lower bound of the \(p_{n}\)'s.

Claim. \(P\) is \(\mathcal{F}\)-limsup-semiproper.

Proof. Let \(\lambda\) be a sufficiently large regular cardinal. Let \(\langle N_{n} \mid n < \omega \rangle\) be an \(\varepsilon\)-chain in \(H_{\lambda}\) such that \(K, \theta, P, (C_{k} \mid \delta \in A) \in N_{0}\) and \(\langle N_{n} \cap \omega_{1} \mid n < \omega \rangle \subseteq F\). Let \(N_{\omega} = \bigcup\{N_{n} \mid n < \omega\}\) and let \(\delta^{*} = N_{\omega} \cap \omega_{1}\). Let \(\langle C_{k}^{*} \mid k < \omega \rangle\) enumerate \(C_{\delta^{*}}\) and \(\langle C_{(k)} \mid k < \omega \rangle\) enumerate \(\langle C_{(k^{*})} \mid n < \omega \rangle\).

Let \(p \in N_{0} \cap P\). We want \(q\) such that \(q \leq p\) and \(q = (P, N_{q})\)-semi-generic for infinitely many \(n < \omega\). To this end, we argue in two cases.

Case 1. There exists a sequence \(\langle n_{l} \mid l < \omega \rangle\) of strictly increasing natural numbers and an \(\varepsilon\)-chain \(\langle M_{l} \mid l < \omega \rangle\) in \(H_{\theta}\) such that \(\langle M_{l} \mid l < \omega \rangle \supseteq_{\omega_{1}} \langle N_{n_{l}} \cap H_{\theta} \mid l < \omega \rangle\) and \(\langle M_{l} \cap K \mid l < \omega \rangle \in S\).

Apply the Sequential 3 H Lemma (see [M]) to get an \(\varepsilon\)-chain \(\langle M_{l}^{*} \mid l < \omega \rangle\) in \(H_{\lambda}\) such that

- \(\langle M_{l}^{*} \mid l < \omega \rangle \supseteq_{\omega_{1}} \langle N_{n_{l}} \mid l < \omega \rangle\).


\[ M_\ast \cap H | l < \omega = M_\ast \cap H | l < \omega \in S. \]

Notice that \( \alpha^q = \delta^\ast = \cap \omega_{1} \) and \( \alpha^q = \delta^\ast = \cup \{ \delta^\ast, \cap \omega_{1} \} \). Let \( \alpha^q = \cup \{ \delta^\ast, \cap \omega_{1} \} \). exists

\[ \text{Case 2. For all sequences } (n_l | l < \omega) \text{ of strictly increasing natural numbers and all } \in \text{-chains } (M_l | l < \omega) \text{ in } H_{\delta} \text{ with } (M_l | l < \omega) \subseteq (N_{n_l} \cap H_{\delta} | l < \omega), \text{ we have } (M_l \cap K | l < \omega) \notin S: \]

It suffices to get \( q \in P \) such that \( q \leq p \) and for all \( n < \omega, q \) is \( (P, N_{n}) \)-generic. To this end, we may construct a descending sequence \( (q_n | n < \omega) \) of conditions such that

\[ p \geq q_n \in N_{n+1} \text{ and } q_n \text{ is } (P, N_{n}) \text{-generic}. \]

\[ D^\omega \cap C_{\delta} = (D^\omega \cap C_{\delta}) \cup \{ M_\ast \cap H | l < \omega \}, \]

\[ \alpha^q = \cup \{ M_\ast \cap H | l < \omega \} \]

\[ \text{Let } C_{\delta} \cap D^\omega = \{ C_{\delta}(k^\omega_{\ast} (m)) | m < \omega \}. \]

\[ \text{Since } D^\omega \cap C_{\delta} = (D^\omega \cap C_{\delta}) \cup \{ M_\ast \cap H | l < \omega \}, \text{ there exists } n^\omega_{\ast} \text{ such that} \]

\[ \{ M_\ast \cap H | l < \omega \} = \{ C_{\delta}(k^\omega_{\ast} (m)) | n^\omega_{\ast} \leq m < \omega \}. \]

And so

\[ \langle N^\omega_{C_{\delta}(k^\omega_{\ast} (m))} | l < \omega \rangle = \langle M_\ast \cap H | l < \omega \rangle = \langle M_l \cap H | l < \omega \rangle \in S. \]

Hence this \( q \) works.

\[ \square \]
For all sequences \( \langle m_l | l < \omega \rangle \) of strictly increasing natural numbers with \( n^{\omega}_l \leq m_0 \) and all \( \varepsilon \)-chains \( \langle M_l | l < \omega \rangle \) in \( H_\theta \) with \( \langle M_l | l < \omega \rangle \supseteq_{\varepsilon\omega_1} \langle N^\theta_{C_\delta, (\delta, (m_l))} | l < \omega \rangle \) which is a subsequence of \( \langle N_n \cap H_\theta | n < \omega \rangle \), we have \( \langle M_l \cap K | l < \omega \rangle \notin S \).

Hence this \( \eta \) works.

Now apply forcing axiom to this \( P \). We have a club \( D \) in \( \omega_1 \) and an \( \varepsilon \)-chain \( \langle N_i | i < \omega_1 \rangle \) which works for \( (K, S, \theta, a) \) in \( \mathcal{F}-RP \).

§3. Iteration theorem

We show \( \mathcal{F}^{V[G_\alpha]} \)-generic-limsup-semiproper combined with semiproper iterates under the simple iterations. For an account on the simple iterations, see [M].

3.1 Theorem. Let \( I = ((P_\alpha | \alpha \leq \nu), (\dot{Q}_\alpha | \alpha < \nu)) \) be a simple iteration such that

- For all \( \alpha < \nu \), \( \models_{P_\alpha} \dot{Q}_\alpha \) is semiproper".
- For all \( \alpha < \nu \), \( \models_{P_\alpha} \dot{Q}_\alpha \) is \( \mathcal{F}^{V[G_\alpha]} \)-generic-limsup-semiproper".

where \( \mathcal{F}^{V[G_\alpha]} \) is formed in \( V[G_\alpha] \) as the set of cofinal subsequences of the \( \langle C_\delta(n) | n < \omega \rangle \)'s for all \( \delta \in A \). Hence \( \mathcal{F}^{V[G_\alpha]} \) may contain new cofinal subsequences than the original \( \mathcal{F} = \mathcal{F}^{V} \).

Then for all \( \alpha \leq \nu \), we have \( \models_{P_\alpha} \" the tails \( P_\alpha \) are semiproper and \( \mathcal{F}^{V[G_\alpha]} \)-generic-limsup-semiproper".

In particular, \( P_\nu \) is semiproper and \( \mathcal{F}^{V} \)-generic-limsup-semiproper.

3.2 Iteration Lemma. Let \( \theta \) be a sufficiently large regular cardinal and \( N \) be a countable elementary substructure of \( H_\theta \) with \( I, \langle C_\delta | \delta \in A \rangle \in N \).

Let \( (\alpha, \alpha^*, a, p, \langle M_n | n < \omega \rangle) \) be such that

- \( \alpha < \alpha^* \leq \nu \).
- \( a \in P_\alpha \), \( p \in P_\alpha \) and \( a \leq p[\alpha] \).
- \( a \models_{P_\alpha} \langle M_n | n < \omega \rangle \) is an \( \varepsilon \)-chain in \( H_\theta^{V[G_\alpha]} \) such that \( N \cup \{G_\alpha, p\} \subseteq M_0 \) and that \( \langle M_n \cap \omega_1 | n < \omega \rangle \in \mathcal{F}^{V[G_\alpha]} \).

Then there exists \( q \in P_\alpha \) such that

- \( q[\alpha] = a \) and \( q \leq p \).
- \( a \models_{P_\alpha} q[\alpha, \alpha^*] \models_{P_{\alpha^*}} M_n[G_{\alpha^*}] \cap \omega_1 = M_n \cap \omega_1 \) for infinitely many \( n < \omega \)".

We extract sort of typical constructions involved as (Technical construction 1-3).

Lemma. (Technical construction 1) Let \( (\alpha, \alpha^*, a, p, \langle \delta^k | k < \omega \rangle, \check{\dot{X}}) \) be such that

- \( \alpha < \alpha^* \leq \nu \) and \( \alpha^* \) is limit.
- \( a \in P_\alpha \), \( p \in P_\alpha \) and \( a \leq p[\alpha] \).
- \( \langle \delta^k | k < \omega \rangle \) are stages for \( p \).
Then there exists \((\beta, \alpha, x, \langle \delta_k \mid k < \omega \rangle \rangle \) such that
- \(\alpha < \beta < \alpha^*\).
- \(b \in P_\beta, b \upharpoonright \alpha \leq \alpha, x \in P_\alpha^*\) and \(b \leq x \upharpoonright \beta\).
- \(b(\alpha) \upharpoonright P_\alpha \cdot x = x^\tau\).
- \(\langle \delta_k \mid k < \omega \rangle\) are stages for \(x\) and for all \(k < \omega, \vdash P_\alpha \cdot \delta_k^{x^\tau} \leq \delta_k^n\) (a step ahead).
- \(b^{-1} \vdash P_\alpha \cdot \delta_0^n = \beta^n\).

**Lemma.** (Technical construction 2) Let \((\alpha, \alpha^*, a, p, \langle \delta_k^n \mid k < \omega \rangle, \dot{x})\) be as in technical construction 1. Then there exists \((\langle (\beta_i, b_i, x_i, \langle \delta_k^{x_i} \mid k < \omega \rangle \rangle \rangle \rangle | i < \mu \rangle \rangle \) such that
- \(\beta = \beta_i, b = b_i, x = x_i, \langle \delta_k^{x_i} \mid k < \omega \rangle = \langle \delta_k^{x_i} \mid k < \omega \rangle\) serve exactly as in technical construction 1.
- For \(i, j < \mu\) such that \(i \neq j\), we have that \(b_i \upharpoonright \alpha\) and \(b_j \upharpoonright \alpha\) are incompatible in \(P_\alpha\).
- \(\{b_i \upharpoonright \alpha \mid i < \mu\}\) forms a maximal antichain below \(a\) in \(P_\alpha\).

**Lemma.** (Technical construction 3) Let \((\alpha, \alpha^*, a, p, \langle \delta_k^n \mid k < \omega \rangle, \langle \dot{M}_n \mid n < \omega \rangle\rangle\) be such that
- \(\alpha < \alpha^* \leq \nu\) and \(\alpha^*\) is limit.
- \(a \in P_\alpha, p \in P_\alpha^*\) and \(a \leq p \upharpoonright \alpha\).
- \(\langle \delta_k^n \mid k < \omega \rangle\) are stages for \(p\) in \(P_\alpha^*\).
- \(a^{-1} \vdash P_\alpha \cdot \delta_0^n = \alpha^n\).
- \(a \vdash \langle \dot{M}_n \mid n < \omega \rangle\rangle\) is an \(\mathcal{E}\)-chain in \(H_\theta[G_\alpha]\) with \(N \cup \{G_\alpha, a, \langle \delta_k^n \mid k < \omega \rangle\} \subseteq \dot{M}_0^n\).

Let \(T\) be a tree such that \(T \subseteq <\omega ON\) with \(\emptyset = T_0\). For \(\sigma = \emptyset\), let
- \(\mathcal{A}^0 = \alpha, a^0 = a, p^0 = p, \langle \delta_k^n \mid k < \omega \rangle = \langle \delta_k^n \mid k < \omega \rangle\) and \(\langle \dot{M}_n^0 \mid n < \omega \rangle = \langle \dot{M}_n \mid n < \omega \rangle\).

For all \(\sigma \in T\), we have \((\alpha^\sigma, a^\sigma, p^\sigma, \langle \delta_k^n \mid k < \omega \rangle, \langle \dot{M}_n^\sigma \mid n < \omega \rangle\rangle\) such that
- \(\alpha < \alpha^\sigma < \alpha^*\).
- \(a^\sigma \in P_{\alpha^\sigma}, p^\sigma \in P_{\alpha^\sigma}^*\). \(a^\sigma \upharpoonright \alpha \leq a, p^\sigma \leq p\) and \(a^\sigma \leq p^\sigma \upharpoonright a^\sigma\).
- \(\langle \delta_k^n \mid k < \omega \rangle\rangle\) are stages for \(p^\sigma\) in \(P_{\alpha^\sigma}^*\).
- \(a^\sigma^{-1} \vdash P_\alpha \cdot \delta_0^n = \alpha^\sigma^n\).
- \(a^\sigma \vdash \delta_0^n = \alpha^\sigma^n\).
- \(a^\sigma \vdash \langle \dot{M}_n^\sigma \mid n < \omega \rangle\rangle\) is an \(\mathcal{E}\)-chain in \(H_\theta[G_\alpha^\sigma]\) with \(N \cup \{G_\alpha^\sigma, a^\sigma, \langle \delta_k^n \mid k < \omega \rangle\} \subseteq \dot{M}_0^n\).

For all \(\tau \in \text{suc}_T(\sigma),\) we have \(\langle \dot{m}(\tau, n) \mid n < \omega \rangle\rangle\) such that
- \(\alpha^\sigma \leq \alpha^\tau\).
- \(a^\tau \upharpoonright a^\sigma \leq \alpha^\sigma\).
- \(p^\tau \leq p^\sigma\) in \(P_{\alpha^\sigma}^*\).
- \(\langle \dot{M}_n^\tau \mid n < \omega \rangle\rangle\) are a step ahead of \(\langle \delta_k^n \mid k < \omega \rangle\).
- \(a^\tau \vdash \alpha^\sigma \vdash P_\alpha \cdot \langle \alpha^\sigma, a^\tau \rangle = \dot{M}_0^n[G_\alpha^\sigma[G_\alpha^\sigma^\tau]] \cap \omega_1 = \dot{M}_0^n \cap \omega_1^\tau\).
- \(a^\tau \vdash \langle \dot{m}(\tau, n) \mid n < \omega \rangle\rangle\) is a sequence of strictly increasing natural numbers.
- \(a^\sigma \vdash \dot{M}_n^\tau = \dot{M}_n^\tau[G_\alpha^\sigma[G_\alpha^\sigma^\tau]]\).
\begin{itemize}
\item $a^\tau \models \neg \rho_{\alpha^*}$ "for all $n < \omega$, $M_n^\tau \cap \omega_1 = M_n^\sigma \cap \omega_1$".
\item Let $q \leq p$ be a fusion of $a^\sigma$'s in $P_{\alpha^*}$.
\item Then there exists a sequence $(\check{n}_k | k < \omega)$ of $P_{\alpha^*}$-names such that
\item $q \models \neg \rho_{\alpha^*}$, "$(\check{n}_k | k < \omega)$ is a sequence of strictly increasing natural numbers".
\item For all $k < \omega$, we have $q \models \neg \rho_{\alpha^*}$ "$n_k \cap \omega_1 = M_{n_k} \cap \omega_1$".
\end{itemize}

More specifically, we may calculate $(i_k | k < \omega)$, $(\alpha_k | k < \omega)$, $(\alpha_k | k < \omega)$, $(p_k | k < \omega)$ and $(m(k, n) | k, n < \omega)$ in $V[G_{\alpha^*}]$ such that, where $M = \omega_1 N$ abbreviates $M \cap \omega_1 = N \cap \omega_1$,

- $M_0[G_{\alpha_{00}}] \subseteq M_0^\emptyset[G_{\alpha_{00}}] = \omega_1 M_0^\emptyset = M_0$.
- $M_0(m(0,0))[G_{\alpha_{00}}] \subseteq M_0^\emptyset(m(0,0))[G_{\alpha_{00}}] = M_0^\emptyset(n) = \omega_1 M_0^\emptyset = M_0(m(0,0))$.
- $M_0(m(0,0))[G_{\alpha_{00}}] \subseteq M_0^\emptyset(m(0,0))[G_{\alpha_{00}}] = M_0^\emptyset(n) = M_0^\emptyset(m(0,0)) = M_0(m(0,0))$.

In general,

- $a^\emptyset = a = a_0$.
- $a^{(i_0, \ldots, i_k)} = a_{k+1}$.
- $\alpha_k = \lambda(\alpha_k)$.
- $M_n^\emptyset = M_n^\emptyset$.
- $M_n^\emptyset(m) = M_0(m, n)[G_{\alpha_{00}}]$.
- $M_n^\emptyset = \omega_1 M_n$.
- $M_n^\emptyset(m, \alpha_{k+1}) = M_n^\emptyset(m, \alpha_{k+1})[G_{\alpha_{k+1} \alpha_{k+2}}]$.
- $M_n^\emptyset(m, \alpha_{k+1}) = \omega_1 M_n^\emptyset(m, \alpha_{k+1})$.
- $M_0^\emptyset[G_{\alpha_{00}}] = \omega_1 M_0^\emptyset$.
- $M_0^\emptyset[G_{\alpha_{00}}] = \omega_1 M_0^\emptyset$.

And so

\begin{align*}
M_0[G_{\alpha_{00}}] &= \omega_1 M_0.
M_0(m, \cdots, m(k, n), \cdots)[G_{\alpha_{00}}] &= \omega_1 M_0(m, \cdots, m(k, n), \cdots).
\end{align*}

Where $m(k, n)$ abbreviates $\langle i_0, \cdots, i_k \rangle$. $n$).

We give a proof of iteration lemma 3.2.

**3.2 Lemma.** (Iteration Lemma) Let $\theta$ be a sufficiently large regular cardinal. Let $N$ be a countable elementary substructure of $H_\theta$ with $I, (C_\delta | \delta \in A) \in N$.

Let $(\alpha, \alpha^*, \alpha, p, (M_n | n < \omega))$ satisfy

- $\alpha < \alpha^* \leq \nu$.
- $a \in P_{\alpha^*}$, $p \in P_{\alpha^*}$ and $a \leq p|\alpha$.
- $a \models \neg \rho_{\alpha^*}$, $N \cup \{G_\alpha, p\} \subseteq M_0$, $(M_n | n < \omega)$ is an $e$-chain in $H_\theta \cap V[G_\alpha]$ and $(M_n \cap \omega_1 | n < \omega) \in \mathcal{F}^V[G_{\alpha^*}]$.

Then there exists $(q, (\check{m}(n) | n < \omega))$ such that
\[ q \in P_{\alpha^*}, \text{ and } q \leq p. \]
\[ q \vdash \alpha = a. \]
\[ q \vdash \pi(n) \text{ are strictly increasing natural numbers and } M_{m(0)}[G_{\alpha^*}] \cap \omega_1 = M_{m(0)} \cap \omega_1. \]

**Notational Remark.** Let \( \alpha < \beta \) and \( G_{\beta} \) be \( P_\beta \)-generic over \( V \). Then
\[ G_{\alpha} \text{ denotes } G_{\beta}[\alpha] = \{ r | \alpha \in G_{\beta} \} \text{ which is } P_{\alpha^*} \text{-generic over } V. \]
\[ H_{\alpha} \text{ denotes } \{ r(\alpha) | G_{\alpha} \} \text{ which is } Q_{\alpha} \text{-} generic \text{ over } V[G_{\alpha}]. \]

If \( \dot{x} \) is a \( P_{\alpha^*} \)-name, then we may view \( \dot{x}[G_{\alpha}] \) as a term \( \dot{x}[\dot{G}_{\beta}[\dot{\alpha}] \) being interpreted by \( G_{\beta} \) in \( V[G_{\beta}] \). We simply denote this by \( \dot{x} \) for easier notation. For sequences \( s = (\dot{x}_n | n < \omega) \) of \( P_{\alpha^*} \)-names, we abbreviate as follows:

\[ x_n = \dot{x}_n[G_{\alpha}] \text{ (the } n \text{-th value of the interpretation of a term } \langle s(n)[G_{\alpha}] | n < \omega \rangle \text{ by } G_{\alpha} \text{ in } V[G_{\alpha}] \]
\[ = \dot{x}_n[G_{\beta}[\alpha] \text{ (the } n \text{-th value of the interpretation of a term } \langle s(n)[G_{\beta}[\dot{\alpha}] | n < \omega \rangle \text{ by } G_{\beta} \text{ in } V[G_{\beta}] \text{ for each } n < \omega. \]

**Proof.** By induction on \( \alpha^* \) for all \( (\alpha, a, p, (M_n | n < \omega)). \)

**Case 1.** (Successor Steps Essential) Let \( (\alpha, \alpha + 1, a, p, (M_n | n < \omega)) \) be as in the hypothesis. Since \( \Pi_{\alpha}^n, \dot{Q}_{\alpha} \) is \( F^{\mathcal{H}V[G_{\alpha}]} \)-generic-\( \Pi^{\mathcal{H}V[G_{\alpha}]} \)-semiproper and \( a \vdash \pi(n) \alpha^* \) \( \in \dot{M}_0 \cap \dot{M}_0 \cap \omega_1 = M_{m(0)} \cap \omega_1 \). Hence \( \alpha^* \vdash \pi(n) \text{ all } n < \omega \).
\[ \text{Hence } \alpha^* \vdash \pi(n) \text{ all } n < \omega \text{ and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]
\[ \text{and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]
\[ \text{Hence } \alpha^* \vdash \pi(n) \text{ all } n < \omega \text{ and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]
\[ \text{Hence } \alpha^* \vdash \pi(n) \text{ all } n < \omega \text{ and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]

**Case 2.** (Successor Steps General) Let \( (\alpha, \beta + 1, a, p, (M_n | n < \omega)) \) be as in the hypothesis. We may assume \( \alpha < \beta \). Apply the hypothesis of induction to \( (\alpha, \beta, a, p, (M_n | n < \omega)). \) We have \( q', \langle \dot{m}(n) | n < \omega \rangle \) such that

\[ q' \in P_{\beta}, a = q' \text{ and } q' \leq p[\beta]. \]
\[ q' \vdash \dot{m}(n) \text{ are strictly increasing natural numbers and } M_{m(0)}[G_{\alpha}] \cap \omega_1 = M_{m(0)} \cap \omega_1 \text{ for all } n < \omega. \]
\[ \text{Hence } q' \vdash \pi(n) \text{ all } n < \omega \text{ and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]
\[ \text{and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]

**Case 3.** (Limit) Let \( (\alpha, \alpha^*, a, p, (M_n | n < \omega)) \) be as in the hypothesis. We assume \( \alpha^* \) is limit. Construct a tree representation \( T \) and a map \( \langle \sigma \mapsto (\alpha^*, a^*, p^*, \langle \delta_{\sigma}^n(n) | n < \omega \rangle, (M_{\sigma}^n | n < \omega)) | \sigma \in T \rangle \) such that

\[ \{ \} = \emptyset \in T_0, \text{ we set } \]
\[ (0) \alpha^0 = \alpha, a^0 = a, p^0 = p, \langle \delta_{\sigma}^0(n) | k < \omega \rangle \text{ are any stages for } p \text{ such that } \delta_{\sigma}^0 = \alpha^0 \text{ and } p \vdash \alpha^0 \text{ all } \langle \delta_{\sigma}^0(n) | n < \omega \rangle = (M_{\sigma}^0 | n < \omega). \]
\[ \text{In general, for } \sigma = \langle i_0, \ldots, i_{k-1} \rangle \in T_k, \text{ we demand } \]
\[ (1) \alpha \leq \alpha^* \leq \alpha^*. \]
\[ (2) a^* \in P_{\alpha}^* \text{ and } a^* \vdash \alpha \leq a. \]
\[ (3) p^* \in P_{\alpha^*} \text{ and } p^* \vdash p. \]
\[ (4) a^* \leq p^* \vdash \alpha^*. \]
\[ (5) \langle \delta_{\sigma}^n | n < \omega \rangle \text{ are stages for } p^*. \]
\[ (6) p^* \vdash \alpha^* \vdash \pi(n) \text{ all } n < \omega, \text{ and } \Pi_{\alpha}^n, \dot{Q}_{\alpha} \text{ is } \mathcal{H}V[G_{\alpha}] \text{-generic.} \]
(7) \( (M^\sigma_m | n < \omega) \) is a sequence of \( P_{\alpha^*} \)-names such that \( \alpha^* \vDash \mu_{\alpha^*} \text{“} N \cup \{ G_{\alpha^*}, p^\tau, (\delta^\sigma_n | k < \omega) \} \subseteq M^\sigma_0 \), \( (M^\sigma_m | n < \omega) \) is an \( \epsilon \)-chain in \( H^V_{\theta(G_{\alpha^*})} \) and \( (M^\sigma_m | \omega_1 | n < \omega) \in \mathcal{F}^V[G_{\alpha^*}] \).

For \( \tau = \sigma^-(i) = (i_0, \ldots, i_{k-1}, i_k) \in T_{k+1} \), there exists a sequence \( \langle \hat{m}(\tau, n) | n < \omega \rangle \) of \( P_{\alpha^*} \)-names and we demand

(8) \( \alpha^* \leq \alpha^\tau. \)

(9) \( \alpha^\tau \vDash \alpha^* \leq \alpha^\tau. \)

(10) \( p^\tau \leq p^\sigma. \)

(11) \( \alpha^\tau \vDash \alpha^* \leq \alpha^\tau \) is \( (P_{\alpha^*}, M^\sigma_0) \)-semi-generic, \( (P_{\alpha^*}, M^\sigma_0) \)-semi-generic.

(12) For all \( n < \omega, n \vDash \xi_{\alpha^*} \leq \xi_{\alpha^\tau} \) (a step ahead).

(13) \( p^\tau \vDash \alpha^\tau \leq \alpha^\tau \) (0th-stage self-decisive condition).

(14) \( \alpha^\tau \vDash \alpha^* \) is a sequence of strictly increasing natural numbers and for all \( n < \omega, \hat{m}(\tau, n) \geq 1, M^\sigma_{\hat{m}(\tau, n)} \cap \omega_1 = M^\sigma_n \cap \omega_1 = M^\sigma_n \cap \omega_1 \).

The contradiction is by recursion on \( k < \omega \). For \( k = 0 \), we set \( T_0 = \{ \emptyset \} \) and set \( a^0, a^\sigma, p^\sigma, (\delta^\sigma_n | n < \omega) \) and \( (M^\sigma_m | n < \omega) \) as specified. This is possible as a \( n \vDash \mu_{\alpha^*} \text{“} N \cup \{ G_{\alpha^*}, p \} \subseteq M^\sigma_0 \). Then it is easy to see that all the assumptions (1) through (7) for \( \sigma = \emptyset \) are satisfied.

Suppose we have constructed \( T_k \) and \( \alpha^\sigma, a^\sigma, p^\sigma, (\delta^\sigma_n | n < \omega) \) and \( (M^\sigma_m | n < \omega) \) for each \( \sigma \in T_k \) such that (1) through (7) are satisfied. Let \( \gamma = \alpha^\sigma, w = a^\sigma, x = p^\sigma, \hat{\delta}_n | n < \omega = \delta^\sigma_n \) and \( \langle \hat{N}_n | n < \omega \rangle = \langle M^\sigma_n | n < \omega \rangle \) for shorter notation. Then \( w \in P_{\alpha^*} \) forces that

- \( N \cup \{ G_{\gamma}, x, (\hat{\delta}_n | n < \omega) \} \subseteq \hat{N}_0 \) and \( x \vDash \gamma \vDash \hat{G}_{\gamma} \).

Hence by the iteration lemma for semiproperness and lemmas on stages (please see [M] for an account), there exists \( (\beta, y, (\delta^\sigma_n | n < \omega) \subseteq V[G_{\gamma}] \) such that

- \( \gamma < \beta < \alpha^* \).
- \( y \leq x \) in \( P_{\alpha^*} \).
- \( \delta^\sigma_n \leq n < \omega \) are stages for \( y \).
- \( y \vDash \beta^{-1} \vDash \hat{\delta}^\sigma_0 = \beta^\sigma \).
- For all \( n < \omega, \hat{\delta}^\sigma_0 \leq \delta^\sigma_n \) (a step ahead).
- \( y \vDash \gamma \in G_{\gamma} \).
- \( \gamma \vDash \alpha^* \) is \( (P_{\alpha^*}, N_0, \lambda_{\alpha}) \)-semi-generic
- \( \beta, y, (\delta^\sigma_n | n < \omega) \in N_1 \).

Then for any \( a \leq w \) in \( P_{\alpha^*} \) which \( d \) decides the values of \( \beta, y \) and \( \langle \delta^\sigma_n | n < \omega \rangle \), we may consider \( \gamma, \beta, d, y(\delta^\sigma_n | n < \omega) \) satisfying

- \( \gamma < \beta < \alpha^* \).
- \( d \in P_{\beta}, y \vDash \beta \in P_{\beta} \) and \( d \leq y \vDash y \).
- \( y \vDash \gamma \vDash \hat{G}_{\gamma}, y(\beta) \subseteq \hat{N}_1 \) and \( \langle \hat{N}_n | 1 \leq n < \omega \rangle \) is an \( \epsilon \)-chain in \( H^V_{\theta(G_{\gamma})} \) and \( \langle \hat{N}_n \cap \omega_1 | 1 \leq n < \omega \rangle \in \mathcal{F}^V[G_{\gamma}] \).

Now we apply the hypothesis of induction at \( \beta \). Hence there exists \( b, \langle \hat{m}(\beta, n) | n < \omega \rangle \) such that

- \( b \in P_{\beta}, \hat{b} \vDash \gamma = d \) and \( b \leq \gamma \vDash \beta \).
- \( b \vDash \hat{m}(\beta, n) | n < \omega \) is a sequence of strictly increasing natural numbers such that \( 1 \leq \hat{m}(n) \) and \( \hat{N}_{\hat{m}(n)}[G_{\gamma}] \cap \omega_1 = \hat{N}_{\hat{m}(n)} \cap \omega_1 \).

And so

- \( b \vDash \hat{m}(\beta, n) | n < \omega \) is an \( \epsilon \)-chain in \( H^V_{\theta(G_{\beta})} \) and \( \langle \hat{N}_{\hat{m}(n)}[G_{\gamma}] | n < \omega \rangle \subseteq \mathcal{F}^V[G_{\beta}] \).

Since there exists $d$ as above predense many below $w$, we may construct $T_{k+1}$ and

$$(\tau \mapsto (\alpha^\tau, \alpha^\tau, p^\tau, (\delta_n^\tau \mid n < \omega), (M_n^\tau \mid n < \omega), (m(\tau, n) \mid n < \omega)) \mid \tau \in T_{k+1}),$$

where the correspondences are $\alpha^\tau = \beta, \alpha^\tau = b, p^\tau = y, (\delta_n^\tau \mid n < \omega) = (\delta_n^\tau \mid n < \omega), (m(\tau, n) \mid n < \omega) = (\bar{m}(n) \mid n < \omega)$ and $(M_n^\tau \mid n < \omega) = (M_m(n) \mid n < \omega)|G_{k,d}$. This completes the construction.

Let $q$ be a fusion of the tree representation $T$. Let $G_{\alpha}$ be $P_{\alpha}$-generic over $V$ with $q \in G_{\alpha}$. Let us calculate $(i_n \mid n < \omega)$ from the generic cofinal path through $T$ so that for all $k < \omega, (i_n \mid n < k) \in T_k$ and $a^{(i_n \mid n < k)} \in G_{\alpha,i_n \mid n < k}$.

Let

$$M_n = M_n[G_{\alpha}],$$

$$a^k = a^{(i_n \mid n < k)}, p^k = p^{(i_n \mid n < k)},$$

$$\delta_m^k = \delta_m^{i_n \mid n < k}, \ M_m^k = M_m^{i_n \mid n < k}, M_m^k = M_m^k[G_{\alpha}], \ m(k, n) = m((i_0, \cdots, i_k), n)[G_{\alpha+k}].$$

Then

$$a^k \in G_{\alpha^k}, \ p^k \in G_{\alpha^k}.$$ 

$M_0 = \omega_1, M_0[G_{\alpha^0}]$, 

$$M_{m(0,m(1,\cdots,m(k,0),\cdots))}[G_{\alpha_0}] \cdots [G_{\alpha+\alpha+1}] = M_{0}^{k+1},$$

$$M_{m(0,m(1,\cdots,m(k,0),\cdots))}[G_{\alpha_0}] \cdots [G_{\alpha+\alpha+1}] = M_{0}^{k+1}[G_{\alpha+\alpha+1}].$$

Hence

$$M_{m(0,m(1,\cdots,m(k,0),\cdots))}[G_{\alpha_0}] \subseteq M_0^{k+1}[G_{\alpha+\alpha+1}] \supseteq_{\omega_1} M_0^{k+1} \supseteq_{\omega_1} M_{m(0,m(1,\cdots,m(k,0),\cdots))}. $$

So

$$M_{m(0,m(1,\cdots,m(k,0),\cdots))}[G_{\alpha_0}] \cap \omega_1 \leq M_0^{k+1}[G_{\alpha+\alpha+1}] \cap \omega_1 = M_0^{k+1} \cap \omega_1 = M_{m(0,m(1,\cdots,m(k,0),\cdots))} \cap \omega_1.$$ 

Note that $m(0,m(1,\cdots,m(k,0)\cdots))$ strictly increase.

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\square
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References


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