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Some remarks on approximate sampling theorems: estimate with Besov norm and an application

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Abstract

This is an informal report based on a joint work with Prof. Stéphane Jaffard. Let φ be a bump function which permits an exact interpolation of functions on $\mathbb{Z}$ such as the sinc function or the modified spline functions [12]. We define the sampling approximation of a given function f by $S_N(f,\varphi)(x) := \sum_{k\in\mathbb{Z}} f(k/N)\,\varphi(Nx-k)$. Then under suitable conditions for φ, we have a sharp asymptotic error estimate measured by the Besov norm. A possible application is to data analysis of Hölder continuous functions.

1 Introduction

The famous Shannon sampling theorem gives an exact reproducing formula for band limited analytic functions using the sinc function. In this note we would like to report on the error estimate of the sampling approximation for a not sufficiently regular function with more general classes of sampling functions measured in the Besov norm. A possible application is to data analysis of a sample path, namely a method of getting an upper estimate of the order of regularity α of a Hölder continuous function by measuring values at only dyadic points.

2 sampling function φ

One type of sampling functions is defined by slight modification of the sinc function (let us call it of type I) and another type is defined by some modification of the cardinal B-splines (let us call it of type II).

Namely, a function of type I is a product of the sinc function $\sin \pi x/\pi x$ and a smooth function of Schwartz class which is equal to 1 and whose Fourier transform has a small support at a neighborhood of the origine $\xi = 0$, i.e. $\{\xi \mid |\xi| < \delta\pi\}$, $\delta < 1/2$, for example.

On the other hand, a function of type II is a continuous function with a compact support with respect to the x-variable which also vanishes at $x \in \mathbb{Z}$ except at 0. Also this function should satisfy the so-called Strang-Fix condition, i.e. its Fourier transform vanishes at $2\pi k$ $(k \in \mathbb{Z}\setminus\{0\})$ at least up to the second order at $2\pi k$ $(k \in \mathbb{Z}\setminus\{0\})$ and it decays at ∞ with an order faster than $O(|\xi|^{\beta}, \beta < -1)$. The typical classical example is the cardinal B-spline of order 1 (=degree 2) which also satisfies the two scaling relation. This sampling function can be used in the case where the regularity index s in the definition of Besov space is not so large (i.e. $0 < s < 2$), which we shall suppose in the sequel for uniformity of presentation. See [12] for another examples.

Remark 1 *Sampling functions φ of both types enable the exact interpolation, i.e., they vanish at all integer points except at 0.*

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Now, as in the abstract, let us define the sampling approximation of a given function f by $S_N(f,\varphi)(x) := \sum_{k\in\mathbb{Z}} f(k/N)\,\varphi(Nx-k)$. Then note that $S_N(f,\varphi)(k) = f(k)$ for any $k \in \mathbb{Z}$ due to Remark 1.

3 Error estimate in $L^p(\mathbb{R})$ norm

Let φ be any function described above and let $B^s_{p,q}$ be the Besov space with $0 < s, 1 \leq p, q \leq \infty$. Then we have the following error estimate on the sampling approximation.

Theorem : $\|S_N(f,\varphi) - f\|_{L^p(\mathbb{R})} \leq C\,N^{-1/p}\|f\|_{B^{1/p}_{p,1}(\mathbb{R})}$, $1 < p < \infty$.

A slightly modified statement is given by the following estimate.

$$\|S_N(f,\varphi) - f\|_{L^p(\mathbb{R})} \leq CN^{-s}\|f\|_{B^s_{p,\infty}(\mathbb{R})},$$

where $1/p < s, 1 < p < \infty$, and C is a constant which depends only on φ, p and s.

The ingredients of the proof of Theorem are the following:

(i) Poisson summation formula

(ii) a suitable decomposition of f into the low frequency part and the high frequency part

(iii) L^p boundedness result of the Fourier multiplier (Marcinkiewicz, Hörmander) in the case for φ of type II

(iv) Parseval's identity for Fourier transform

(v) Interpolation theorem of linear operators

Remark 2 (i) *The asymptotic estimate is a kind of generalization of an estimate in [11] where we obtained it for Sobolev spaces on the whole real line inspired from [9].*

(ii) *The usefulness of the Poisson summation formula for the regular sampling theorem was suggested to one of the authors by Prof. P. Malliavin.*

4 An application to data analysis

Let $f(x)$ be a kind of α-Hölder continuous function $(0 < \alpha < 1)$ like the stock price curve defined on $\mathbb{R}$ and let us consider the case where the function $f(x)$ locally belongs to $B^s_{p,\infty}(\mathbb{R}), s = \alpha + 1/p$. Then our problem is to find the information on α from the observed value of $f(x)$ only at dyadic points $k/2^j, k = 0, 1, \cdots 2^j - 1,\ j = 1, \cdots$, i.e. from the sampling approximation $f_j(x) = \sum_{k=0,1,\cdots,2^j-1} f(k/2^j)\,\varphi(2^j x - k)$, which is similar to $S_{2^j}(f,\varphi)\,(x)$ studied above with $N = 2^j$.

Then thanks to Theorem, we have

Proposition 4.1

$$\|f_j - f\|_{L^p[0,1]} \leq C\,2^{-js}\,\|f\|_{B^s_{p,\infty}}, \quad j = 1, 2, \cdots,$$

Therefore, $\|f_j - f_{j-1}\|_{L^p[0,1]}$ has also the same estimate.

Now let us choose $\varphi(x) = \mathbf{N}_2(x)$ the translated cardinal B-spline of order 1, i.e. $\mathbf{N}_2(x) = \mathbb{I}_{[0,1]} * \mathbb{I}_{[0,1]}\,(x+1)$, which is a sampling function of the type II. If we use this particular φ, we have the following estimate.

Corollary 4.1 *Let $c_{j,k} = f(k/2^j)$. Then, for a function $f \in B^s_{p,\infty}(\mathbb{R}), s = \alpha + 1/p$, we have the upper estimate:*

$$\alpha \leq -\limsup_{j\to\infty} \frac{1}{jp} \log_2\Big(\sum_k \mid c_{j,2k+1} - \frac{1}{2}(c_{j-1,k} + c_{j-1,k+1}) \mid^p\Big)$$

For the proof, we use the two scale relation, $\mathbf{N}_2(x) = \frac{1}{2}\mathbf{N}_2(2x-1) + \mathbf{N}_2(2x) + \frac{1}{2}\mathbf{N}_2(2x+1)$, and then we have $(f_j - f_{j-1})(x) = \sum_k \{c_{j,2k+1} - \frac{1}{2}(c_{j-1,k} + c_{j-1,k+1})\}\, \mathbf{N}_2(2^j x - 2k - 1)$, from which follows $\|f_j - f_{j-1}\|^p_{L^p[0,1]} \simeq \sum_k |c_{j,2k+1} - \frac{1}{2}(c_{j-1,k} + c_{j-1,k+1})|^p \|\, \mathbf{N}_2\|^p_{L^p[0,1]} 2^{-j}$, and the rest of the computation is elementary.

Remark 3 *In view of the equivalence result of Theorem 3.4 in p.362 [5] (possibly slightly different), related to Ciesielski's results of 1973, we may have actually the equivalence :*

$$\|f\|_{B^{\alpha+1/p}_{p,q}(\mathbb{R})} \simeq \Big(\sum_{j=0}^{\infty} 2^{j\alpha q}\Big(\sum_k |c_{j,2k+1} - \frac{1}{2}(c_{j-1,k} + c_{j-1,k+1})|^p\Big)^{q/p}\Big)^{1/q}, (c_{-1,k} = 0)\,.$$

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