<table>
<thead>
<tr>
<th>Title</th>
<th>On a $J_1$-convergence theorem for stochastic processes on $D[0, \infty)$ having monotone sample paths and its applications (The 8th Workshop on Stochastic Numerics)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Textversion</td>
<td>publisher</td>
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Kyoto University
On a \( J_1 \)-convergence theorem for stochastic processes on \( D[0, \infty) \) having monotone sample paths and its applications

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Abstract

In this paper, we first show that Bingham’s assertion on weak convergence under \( J_1 \)-topology of stochastic processes with monotone sample paths is incorrect by giving a counter example and then remark that his assertion is valid if the limit process is continuous. Furthermore, we give a simple sufficient condition for weak convergence under \( J_1 \)-topology of stochastic processes with monotone sample paths. We apply the result to the weak convergence under \( J_1 \)-topology of hitting time processes of 1-dimensional generalized diffusion processes.

1 Introduction

Let \( D[0, \infty) \) be the class of functions defined on \([0, \infty)\) to \( \mathbb{R} \) which have left limits on \((0, \infty)\) and are right continuous on \([0, \infty)\). Bingham asserted in [2] that a sequence of random variables \( \{X_n\} \in D[0, \infty) \), which has monotone sample paths, converges weakly to an \( X_\infty \in D[0, \infty) \) in \( J_1 \)-topology if \( X_\infty \) is continuous in probability and \( X_n \) converges to \( X_\infty \) in finite dimensional distributions sense.

In this paper, we remark that his proof contains an error and give an example which shows that his statement does not hold. We also remark that his conclusion is valid if we change the continuity in probability assumption for \( X_\infty \) to continuity of all sample functions.

We, then, give an additional condition so that the conclusion of Bingham’s assertion holds and apply the result to the weak convergence in \( J_1 \)-topology of additive processes and sums of independent random variables. The former result is then applied directly to weak convergences under \( J_1 \)-topology of hitting time processes of 1-dimensional generalized diffusion processes to selfsimilar additive processes.

Define

\[
\Delta(c, X, T) = \sup_{0 \leq t-c < t_1 < t < t_2 < t+c \leq T} \min(|X(t_1) - X(t)|, |X(t_2) - X(t)|).
\]

Stone [7] proved the following theorem.

Theorem 1.1 (Stone) If \( \{X_n : n = 1, 2, \ldots\} \) are random variables in \( D[0, \infty) \), \( X_n \) converges to a random variable \( X_\infty \) in \( D[0, \infty) \) as \( n \to \infty \) weakly under the \( J_1 \)-topology on \( D[0, \infty) \) if and only if

(a) The finite-dimensional distributions of \( X_n \) converge to those of \( X_\infty \) as \( n \to \infty \)

(b) \( \lim_{c \to 0} \limsup_{n \to \infty} P(\Delta(c, X_n, T) > \epsilon) = 0 \) for each \( \epsilon > 0 \) and each \( 0 < T < \infty \).

Bingham’s assertion in next section reduces the proof to the above Stone’s theorem.
2 On Bingham's assertion

Bingham asserted the following assertion as Theorem 3 in [2].

**Bingham's assertion**: Let \( \{X_n\}_{n \geq 1} \) be a sequence of stochastic processes whose path-functions lie in \( D[0, \infty) \). If

(i) the finite-dimensional distributions of \( X_n \) converge to those of \( X_\infty \),

(ii) the process \( X_\infty \) is continuous in probability and

(iii) the processes \( X_n \) have monotone path-functions,

then \( X_n \) converges to \( X_\infty \) weakly under the \( J_1 \)-topology on \( D[0, \infty) \).

He defines the quantity

\[
D(k, X, T) = \max\{|X(rT/k) - X((r-1)T/k)| : r = 1, 2, \ldots, k\}.
\]

He asserts that, by stochastic continuity of \( X_\infty \),

\[
P(\Delta(Tk^{-1}, X_\infty, T) > \epsilon) \to 0 \quad \text{as} \quad k \to \infty
\]  

for each \( \epsilon > 0 \) and \( 0 < T < \infty \) (for this convergence, the stochastic continuity assumption is not necessary if \( X_\infty \) is in \( D([0, \infty)) \), because \( \Delta(Tk^{-1}, X, T) \to 0 \) as \( k \to \infty \) for \( X \in D([0, \infty)) \)). Then he asserts that the inequalities

\[
\frac{1}{2} \Delta(2Tk^{-1}, X, T) \leq D(k, X, T) \leq \Delta(Tk^{-1}, X, T)
\]  

holds and using these inequalities, he shows the validity of the condition (b) in Stone's theorem. But, the second inequality in (2) does not hold even for the following simple function:

\[
x(t) = \begin{cases} 
0, & 0 \leq t < a, \\
1, & a \leq t \leq T,
\end{cases}
\]

where \( 0 < a < T \). For this function, \( D(k, x, T) = 1 \) while \( \Delta(Tk^{-1}, x, T) = 0 \) for large \( k \). One may consider, instead of (1),

\[
P(D(k, X_\infty, T) > \epsilon) \to 0 \quad \text{as} \quad k \to \infty
\]  

for each \( \epsilon > 0 \) and \( 0 < T < \infty \). But this does not hold even for a simple stochastic process. Let \( X(t) \) be a Poisson process with parameter \( \lambda \) and choose \( \epsilon \in (0,1) \). Then,

\[
P(D(k, X_\infty, T) \leq \epsilon) = \prod_{r=1}^{k} P(X(rT/k) - X((r-1)T/k) \leq \epsilon) = e^{-\lambda T}.
\]

Moreover, the above assertion does not hold under the assumptions in the theorem. The following is a counter example: Let \( T = 1 \). Let

\[
X_\infty(t, \omega) = 2\mathbb{I}_{[\xi(\omega), 1]}(t), \\
X_n(t, \omega) = \mathbb{I}_{[\xi(\omega) - \frac{1}{n}, 1]}(t) + \mathbb{I}_{[\xi(\omega) + \frac{1}{n}, 1]}(t),
\]

where \( \xi(\omega) \) is a random variable uniformly distributed on \([\frac{1}{3}, \frac{2}{3}]\). Then \( X_\infty(t) \) and \( X_n(t) \) are non-decreasing and right continuous with left limits. Hence, \( X_\infty \) is stochastically continuous. \( X_n(t) \) converges in finite dimensional distributions, but does not converge in \( J_1 \)-topology. The last assertion is immediate from the fact that \( \Delta(c, X_n, T) = 1 \) for \( \frac{1}{n} < c \). This example is stated as Problem 12.5 in [1].
Remark 2.1. If we substitute the condition "continuous in probability" by "continuous with probability 1", then the above theorem is valid since in this case, $D(k, X_\infty(\omega), T) \to 0$ as $n \to \infty$ with probability 1 and hence (3) holds as $k \to \infty$ for each $\epsilon > 0$ and $0 < T < \infty$. By the assumption (i), $\lim_{n\to\infty} P(D(k, X_n, T) > 2\epsilon) \leq P(D(k, X_\infty, T) > \epsilon)$. Using this inequality, the convergence in (3) and the left inequality in (2), we have the weak convergence under the $J_1$-topology.

If the above Bingham's assertion were true, then the statement would be very useful. In fact, many papers use Bingham's "Theorem 3". For example, in Corollary 3.5 and Corollary 3.11 in [5] and Corollary 3.4 in [4]. Fortunately, sample paths of the limit processes in Corollary 3.11 in [5] and Corollary 3.4 in [4] are continuous, the conclusion is valid by the above remark. In Corollary 3.5 in [5], limit processes are purely discontinuous. Hence the result in Remark 2.1 can not be applied.

In the following section, we investigate what additional condition is needed so that weak convergence under $J_1$-topology holds. The result is described as Theorem 3.1, in the following section. The conclusion of Corollary 3.5 in [5] is valid by Theorem 3.3, which is an application of Theorem 3.1.

3 Main results

Lemma 3.1 Let $x \in D[0, 1]$ with non-decreasing path. Let

$$x_1(t) = \sum_{0 \leq s \leq t, x(s) - x(s-) > \epsilon} (x(s) - x(s-))$$

and let $x_2(t) = x(t) - x_1(s)$. Assume that $x_1(t)$ has at most one jump in a sub interval $(r, t]$ in $[0, 1]$. Then, for $r < s < t$,

$$(x(s) - x(r)) \land (x(t) - x(s)) \leq x_2(t) - x_2(r).$$

Proof First assume that there is no jump greater than $\epsilon$ in $(r, t]$. Then, for $r \leq s \leq t$,

$$(x(s) - x(r)) \land (x(t) - x(s)) \leq (x(t) - x(r))/2 = (x_2(t) - x_2(r))/2.$$  

Next, assume that $r < s_0 \leq t$ be the jumping point of $x_1(t)$. We divide into three cases.

(1) $x(s_0) - \epsilon < \frac{1}{2}(x(t) - x(r)) < x(s_0)$,
(2) $x(s_0) \leq \frac{1}{2}(x(t) - x(r))$ and
(3) $\frac{1}{2}(x(t) - x(r)) \leq x(s_0)$.

If (1) holds, then

$$(x(s) - x(r)) \land (x(t) - x(s)) \leq (x(s_0) - x(r)) \lor (x(t) - x(s_0)) \leq x_2(t) - x_2(r).$$

Now, assume that the case (2) or (3). Then

$$(x(s) - x(r)) \land (x(t) - x(s)) \leq (x(t) - x(r))/2 \leq x_2(t) - x_2(r).$$

$\Box$

Let $X$ be a random variable in $D[0, \infty)$. For each $\epsilon > 0$,

$$X^{1, \epsilon}(t) = \sum_{0 < \epsilon \leq t, X(s) - X(s-) > \epsilon} (X(s) - X(s-))$$

and $X^{2, \epsilon}(t) = X(t) - X^{1, \epsilon}(t)$. Let $\Omega^1(\epsilon, X, T)$ be a set of $\omega$ such that $X(t, \omega)$ has at most one jump greater than $\epsilon$ in any semi closed interval $(r, r + \delta]$ contained in $(0, T]$, where $\delta > 0$.  

111
Theorem 3.1 Suppose that \( \{X_n\} \) is a sequence of stochastic processes in \( D[0, \infty) \) with non-decreasing sample paths which converges in finite dimensional distributions sense to a stochastic process \( X_\infty \in D[0, \infty) \) as \( n \to \infty \). If

(i) for each \( \epsilon > 0 \), \( X_n^{2,\epsilon} \) converges to \( X_\infty^{2,\epsilon} \) as \( n \to \infty \) in finite dimensional distributions sense and

(ii) for each \( \epsilon, T > 0 \), \( \lim_{\delta \downarrow 0} \limsup_{n \to \infty} P(\Omega_{T/\delta}(X_n, T)) = 1 \),

then \( X_n \) converges to \( X_\infty \) weakly in \( J_1 \)-topology on \( D[0, \infty) \).

Proof Let \( X \) be a random variable in \( D[0, \infty) \) and let \( \omega \in \Omega_{T/k}(X, T) \). If \( \frac{2\ell-1}{2k}T \leq s \leq \frac{2\ell+1}{2k}T \) and \( s - \frac{T}{2k} < r < s < t < s + \frac{T}{2k} \) for \( \ell = 1, \ldots, k-1 \), then, by Lemma 3.1,

\[
(X(t) - X(s)) \wedge (X(s) - X(r) \leq X^2(\frac{\ell+1}{k}T) - X^2(\frac{\ell-1}{k}T).
\]

Hence

\[
\Delta(\frac{1}{2k}, X, T) \leq \max_{1 \leq \ell \leq k-1} \{X^2(\frac{\ell+1}{k}T) - X^2(\frac{\ell-1}{k}T)\}.
\]

Denote the right hand side of the above inequality \( E(k, X^2) \). We have

\[
\limsup_{n \to \infty} P(E(k, X_n^{2,\epsilon}) > 2\epsilon) \leq P(E(k, X_\infty^{2,\epsilon}) > 2\epsilon)
\]

by (i). If \( k \) is sufficiently large, then \( E(k, X_\infty^{2,\epsilon}(\omega)) < \frac{3}{2}\epsilon \). Hence

\[
\lim_{k \to \infty} E(k, X_\infty^{2,\epsilon}(\omega)) \vee \frac{3}{2}\epsilon - \frac{3}{2}\epsilon = 0.
\]

Then,

\[
P(E(k, X_\infty^{2,\epsilon}) > 2\epsilon) \leq P(E(k, X_\infty^{2,\epsilon}) \vee \frac{3}{2}\epsilon > 2\epsilon)
\leq P(E(k, X_\infty^{2,\epsilon}) \vee \frac{3}{2}\epsilon - \frac{3}{2}\epsilon > \frac{1}{2}\epsilon) \to 0 \text{ as } k \to \infty.
\]

This shows that

\[
P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon) \leq P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon : \Omega_{T/k}(X_n, T)) + P(\Omega_{T/k}(X_n, T))
\leq P(E(k, X_n^{2,\epsilon}) > 2\epsilon) + P(\Omega_{T/k}(X_n, T))
\]

By assumption (ii) and (4), we have, for each \( T > 0 \) and \( \epsilon > 0 \),

\[
\lim_{k \to \infty} \limsup_{n \to \infty} P(\Delta(\frac{1}{2k}, X_n, T) > 2\epsilon) = 0.
\]

Appealing to Stone's theorem, we have the conclusion. \( \square \)

We say that a stochastic process \( X \) is an additive process in law if it has independent increments, \( X(0) = 0 \) a.s. and it is stochastically continuous. Moreover, if the sample paths are contained in \( D[0, \infty) \) with probability one, then we call \( X \) simply additive process. See [3] for detail.

Theorem 3.2 Let \( \{X_n\} \) be a sequence of additive processes having non-decreasing sample paths. If \( X_n \) converges to a stochastically continuous process \( X_\infty \) in finite dimensional distributions sense, then there is an additive process \( X \) in \( D[0, \infty) \) with non-decreasing sample paths which is equivalent to \( X_\infty \) and \( X_n \) converges to \( X \) weakly under \( J_1 \)-topology as \( n \to \infty \).
Proof If the finite dimensional distributions of $X_n$ converge to the finite dimensional
distributions of $X_\infty$, then $X_\infty$ is an additive process in law. It has a modification $X$ whose
sample paths are contained in $[0, \infty)$ and non-decreasing. For the convergence of $X_n$ to $X_\infty$
in finite dimensional distributions sense, it is necessary and sufficient that for every $t \geq 0$
the distribution of $X_n(t)$ converges to the distribution of $X_\infty(t)$. Let
\[ Ee^{-\varepsilon X_n(t)} = \exp \left( -\gamma(t) + \int_0^\infty (e^{-\varepsilon x} - 1)\nu(t, dx) \right) \]
and
\[ Ee^{-\varepsilon X_\infty(t)} = \exp \left( -\gamma(t) + \int_0^\infty (e^{-\varepsilon x} - 1)\nu(t, dx) \right). \]
For the convergence of the distribution of $X_n(t)$ to the distribution of $X_\infty(t)$, it is necessary
and sufficient that the Lévy measure $\nu(t, A)$ converges to the Lévy measure $\nu_\infty(t, A)$ for
every continuity Borel set $A$ of $\nu_\infty$ which is outside of a neighbourhood of 0 and $t \geq 0$ and
$\gamma(t) \rightarrow \gamma(t)_\infty$ ([3]). This shows that for any $\varepsilon > 0$, $X_n^{2,\varepsilon}$ converges to $X_\infty^{2,\varepsilon}$ as $n \rightarrow \infty$
in finite dimensional distributions sense since $\nu_\infty^{2,\varepsilon}(t, \cdot) = \nu(t, \cdot \cap (0, \varepsilon))$. Let $s_\ell = \frac{\ell}{k}T$ for
$l = 0, 1, \ldots, k$ and $t_\ell = \frac{2\ell+1}{2k}T$ for $\ell = 0, 1, \ldots, k - 1$. If the number of jumps of $X_n^{1,\varepsilon}(t, \omega)$
contained in each interval $(s_\ell, s_{\ell+1}]$, $\ell = 0, 1, \ldots, k - 1$ and $(t_\ell, t_{\ell+1}]$, $\ell = 0, 1, \ldots, k - 2$
are almost one, then $\omega \in \Omega_{T/k}(X_n, T)$. Probability that the number of jumps of $X_n^{1,\varepsilon}(t, \omega)$
contained in each intervals $(s_\ell, s_{\ell+1}]$, $\ell = 0, 1, \ldots, k - 1$ are almost one is
\[ \prod_{\ell=0}^{k-1} e^{-\nu((s_\ell, s_{\ell+1}] \cup (\varepsilon, \infty))} (1 + \nu((s_\ell, s_{\ell+1}], (\varepsilon, \infty))) \]
By the convergence of $X_n$ to $X_\infty$ in finite dimensional distributions sense,
\[ \liminf_{n \rightarrow \infty} e^{-\nu((0, T], (\varepsilon, \infty))} \prod_{\ell=0}^{k-1} (1 + \nu((s_\ell, s_{\ell+1}], (\varepsilon, \infty))) \geq 1 \]
We have
\[ e^{-\nu((0, T], (\varepsilon, \infty))} \prod_{\ell=0}^{k-1} (1 + \nu((s_\ell, s_{\ell+1}], (\varepsilon, \infty))) \geq \exp \left[ -\nu((0, T], (\varepsilon, \infty)) + \nu((0, T], (\varepsilon, \infty)) \left( 1 - \max_{0 \leq \ell \leq k-1} \frac{1}{2} \nu((s_\ell, s_{\ell+1}], (\varepsilon, \infty)) \right) \right] \]
by the inequality $\log(1 + x) \geq x - \frac{1}{2}x^2$ for $x \geq 0$. Since $X_\infty$ is stochastically continuous,
$\nu_\infty(t, \cdot)$ is continuous in $t$ and hence $\max_{0 \leq \ell \leq k-1} \nu((s_\ell, s_{\ell+1}], (\varepsilon, \infty)) \rightarrow 0$ as $k \rightarrow \infty$.
The right hand side of the above inequality tends to 1 as $k \rightarrow \infty$. Similarly, we have that
the probability that the number of jumps of $X_n^{1,\varepsilon}(t, \omega)$ contained in each intervals $(t_\ell, t_{\ell+1}]$, $\ell = 0, 1, \ldots, k - 2$
are almost one tends to 1 as $n \rightarrow \infty$ and then $k \rightarrow \infty$. Hence
\[ \lim \limsup_{k \rightarrow \infty} P(\Omega_{T/k}(X_n, T)) = 1. \]
By appealing to Theorem 3.1 we have the weak converge under $J_1$-topology of $X_n \rightarrow X$. □
Skorohod [6] obtains the same result for general Lévy processes.

**Theorem 3.3** Let for each $n \geq 1$, $\{t_{n,k}\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive numbers such that $\lim_{k \to \infty} t_{n,k} = \infty$ and $\lim_{n \to \infty} \sup_{k \geq 1} (t_{n,k+1} - t_{n,k}) = 0$. Let for each $n \geq 1$, $\{Y_{n,k}\}_{k=1}^{\infty}$ be a sequence of independent nonnegative random variables such that for each $\epsilon > 0$,

$$\lim_{n \to \infty} \max_{k \geq 1} P(Y_{n,k} > \epsilon) = 0.$$ 

If there is a measure $\nu(t)$ on $(0, \infty)$ continuous in $t > 0$ such that for each $t > 0$,

$$\lim_{n \to \infty} \sum_{t_{n,k} \leq t} P(Y_{n,k} > x) = \nu(t, (x, \infty))$$

at every continuity point $x > 0$ of $\nu(t)$ and

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \sum_{t_{n,k} \leq t} \int_{0 < x < \epsilon} x^2 P(Y_{n,k} \in dx) - \left( \int_{0 < x < \epsilon} x P(Y_{n,k} \in dx) \right)^2 = 0,$$

then $X_n(t) = \sum_{t_{n,k} \leq t} Y_{n,k}$ converges weakly under $J_1$-topology to an additive process $\{X(t)\}$ with nondecreasing sample paths such that

$$E(e^{-\theta X(t)}) = \exp(-\gamma(t) + \int_{0}^{\infty} (e^{-\theta x} - 1) \nu(t, dx))$$

**Proof** Convergence of finite dimensional distributions of $X_n(t)$ to those of $X(t)$ is straightforward by the convergence of the distribution of triangular array of independent random variables to a infinitely divisible distribution. We show weak convergence under $J_1$-topology applying Theorem 3.1. Let $T > 0$ and let $s_\ell = \ell T$ for $\ell = 0, 1, \ldots, k$ and $t_\ell = \frac{2\ell+1}{2k}T$ for $\ell = 0, 1, \ldots, k-1$. If the number of jumps of $X_n^\epsilon(t, \omega)$ contained in each interval $(s_\ell, s_{\ell+1}]$, $\ell = 0, 1, \ldots, k-1$ and $(t_\ell, t_{\ell+1}]$, $\ell = 0, 1, \ldots, k-2$ are atmost one, then $\omega \in \Omega_{T/k}^\epsilon(X_n, T)$.

The probability that there is no jump of $X_n(t)$ greater than $\epsilon$ in the interval $(s_\ell, s_{\ell+1}]$ is $\prod_{s_\ell < t_{n,j} \leq s_{\ell+1}} P_{n,j}([0, \epsilon])$ and the probability that there is one jump of $X_n(t)$ in $(s_\ell, s_{\ell+1}]$ is

$$\sum_{s_\ell < t_{n,j} \leq s_{\ell+1}} P_{n,j}((\epsilon, \infty)) \times \prod_{k \neq j, t_{n,k} \in (s_\ell, s_{\ell+1}]} P_{n,k}([0, \epsilon]).$$

The probability that there is atmost one jump of $X_n(t)$ in each $(s_\ell, s_{\ell+1}]$ is

$$\prod_{\ell=0}^{k-1} \left( \prod_{s_\ell < t_{n,j} \leq s_{\ell+1}} (1 - P_{n,j}((\epsilon, \infty))) \right) \left( 1 + \sum_{s_\ell < t_{n,j} \leq s_{\ell+1}} P_{n,j}((\epsilon, \infty))/P_{n,j}([0, \epsilon]) \right).$$

Now, we use the inequalities $\log(1 + x) \geq x - \frac{1}{2} x^2$ for $x > 0$ and $\log(1 - x) \geq -x - x^2$ for $0 \leq x \leq \frac{1}{2}$. We have
\begin{align*}
\geq & \exp \left( \sum_{\ell=0}^{k-1} \left( \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} \log(1 - P_{n,j}((\epsilon, \infty))) \right) + \log \left( 1 + \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon, \infty))/P_{n,j}([0, \epsilon]) \right) \right) \\
\geq & \exp \left( \sum_{\ell=0}^{k-1} \left\{ \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} (-P_{n,j}((\epsilon, \infty)) - \{P_{n,j}((\epsilon, \infty))\}^2) \right\} \right)
+ \frac{1}{\max_i P_{n,i}([0, \epsilon])} \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon, \infty))(1 - \frac{1}{2\min_{i} P_{n,i}([0, \epsilon])} \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} P_{n,i}((\epsilon, \infty))) \right] \\
\geq & \exp \left[ \sum_{\ell=0}^{k-1} \left\{ \sum_{s_{\ell}<t_{n,j}\leq s_{\ell+1}} P_{n,j}((\epsilon, \infty))(1 + \max_i P_{n,i}((\epsilon, \infty))) \right\} \right] \\
& \lim_{\delta \to 0} \limsup_{n \to \infty} P(\Omega_{\delta}(X_n, T)) \\
& \text{Here } \nu((s, t], \cdot) = \nu(t, \cdot) - \nu(s, \cdot). \text{ By the continuity of } \nu(t, \cdot) \text{ in } t, \text{ the above quantity tends to } 1 \text{ as } k \to \infty. \text{ In the same way we have that the probability that } X_n(t) \text{ has at most one jump greater than } \epsilon \text{ in each semiclosed interval } (t_{\ell}, t_{\ell+1}], \ell = 0, \ldots, k - 2, \text{ tends to } 1 \text{ as } n \to \infty \text{ and then } k \to \infty. \text{ Hence } \lim_{n \to \infty} \limsup_{n \to \infty} P(\Omega_{\delta}(X_n, T)) \text{ converges of } X_n^{2,\epsilon}(t) \text{ to } X^{2,\epsilon} \text{ for each } \epsilon > 0 \text{ in finite dimensional distributions sense is automatic from the convergence in finite distributions sense of } X_n \text{ to } X. \\
\text{The above theorem guarantees the validity of Corollary 3.5 in [5].}
\end{align*}

**Theorem 3.4** Let for each } n \geq 1, \{Y_{n,i}\}_{i \geq 1} \text{ is a sequence of independent nonnegative random variables and let } Y_n(t) = k \text{ if } \sum_{j<k} Y_{n,j} < t \leq \sum_{j<k} Y_{n,j}.

If \{X_n(t) = c(n)Y_n(nt)\} \text{ converges to } \{X(t)\} \text{ in finite dimensional distributions sense with } c(n) \text{ satisfying } \lim_{n \to \infty} c(n) = 0, \text{ then } X_n \text{ converges to } X \text{ weakly under } J_1\text{-topology as } n \to \infty.

**Proof** Since } X_n^{2,\epsilon}(t) = X_n(t) \text{ for } \epsilon > c(n), \text{ the processes } \{X_n\} \text{ satisfies the condition (i) of Theorem 3.1 and hence there is no jump greater than } \epsilon. \text{ This means that the condition (ii) is also satisfied. Hence } X_n \text{ converges weakly under } J_1\text{-topology.
4 Limit theorems for hitting times of generalized diffusions

Let $m$ be a non-decreasing, right continuous function defined on $[-\infty, \infty]$ to $[-\infty, \infty]$, which has left limit on $(0, \infty)$, satisfying $m(-\infty) = -\infty$, $m(\infty) = \infty$, $m(0-) = 0$. We set

$$
\ell_1 = \sup\{x < 0 : m(x) = -\infty\}
$$
$$
\ell_2 = \inf\{x > 0 : m(x) = \infty\}
$$

We denote by $E_m$ the support of the measure induced by $m$ restricted to $(\ell_1, \ell_2)$. There naturally corresponds a strong Markov process $\{X_t\}$, called $1$-dimensional generalized diffusion process on $E_m$, whose formal infinitesimal generator is $\frac{d}{dm}\frac{d}{dx}$ to $m$ by changing the time of the Brownian motion (see [8]). The measure $m(dx)$ is called the speed measure of $\{X_t\}$. Denote the hitting time of $x$ for $\{X_t\}$ by $\tau_x$. The hitting time $\tau_x$ can be regarded as a stochastic process with time parameter $x \in E_m$ with independent increments. Moreover, this process can be extended to $x \in (\ell_1, \ell_2)$. This extended process is called generalized hitting time process in [8]. In [8], limit distribution of $\tau_x$ when the process starts at the origin and $x$ tends to $\ell_2$. Some of results in [8] are concerned with convergence of finite dimensional distributions. Since generalized hitting time process has non-decreasing sample paths, by Theorem 3.2, finite dimensional distributions leads to $J_1$-convergence.

Following [8], we introduce assumptions for asymptotics of $m$.

(C$_\gamma$) $(0 \leq \gamma < 1)$: $\ell_2 = \infty$ and $m(x) \sim x^{\gamma/(1-\gamma)}L(x)/(1 - \gamma)$ as $x \to \infty$ with a function $L$ slowly varying at $\infty$.

(C$_1$) $\ell_2 = \infty$ and there is a function $s(x)$ regularly varying at $\infty$ and differentiable for large $x$ such that the derivative $s'(s)$ is positive and monotone and

$$
m(s(x)) \sim e^x L(e^x) as x \to \infty
$$

with a function $L$ slowly varying at $\infty$.

(C$_\gamma$) $(1 < \gamma < \infty)$: $\ell_2 < \infty$ and $m(x) \sim (\ell_2 - x)^{\gamma/(1-\gamma)}K(\ell_2 - x)/(\gamma - 1)$ as $x \to \ell_2$ with a function $K$ slowly varying at $0$.

(C$_\infty$) : $\ell_2 < \infty$ and $m(x) \sim (\ell_2 - x)^{-1}K(\ell_2 - x)^{-1}$ as $x \to \ell_2$ with a function $K$ slowly varying at $0$.

Define $a(x) = \int_0^x m(y)dy$.

**Theorem 4.1** Assume that $\ell_1 = -\infty$, (C$_\gamma$) holds for some $0 \leq \gamma < 1$ and

$$
m(-x)/m(x) \to c \in [0, \infty) as x \to \infty.
$$

Then, the normalized generalized hitting time process $\{\tau_x/a(x)\}$ with any starting point converges weakly under $J_1$-topology on $D[0, \infty)$ to a selfsimilar additive process $\{\tau^0_x\}$ with exponent $(1 - \gamma)^{-1}$ as $x \to \infty$. The process $\{\tau^0_x\}$ is the hitting time process of the generalized diffusion process with speed measure

$$
m^0(x) = \begin{cases} x^{\gamma/(1-\gamma)}/(1 - \gamma), & x \geq 0, \\ -c|x|^{\gamma/(1-\gamma)}/(1 - \gamma), & x < 0. \end{cases}
$$

starting at the origin
Theorem 4.2 Assume that $\ell_1 = -\infty$. Let $\tilde{m}(x) = -m((-x)+) + m(0)$ and $\tilde{a}(x) = \int_0^\infty |m(-y)|dy$. Assume that $\tilde{m}$ satisfies (C_{\gamma}) for some $0 \leq \gamma < 1$ and

$$a(x)/\tilde{a}(x) \to 0 \text{ as } x \to \infty.$$ 

Then, the normalized generalized hitting time process $\{\tau_x/\tilde{a}(x)\}$ with any starting point converges weakly under $J_1$-topology on $D[0,\infty)$ to a selfsimilar additive process $\{\tau^0_t\}$ with exponent $(1-\gamma)^{-1}$ as $n \to \infty$. The process $\{\tau^0_t\}$ is the hitting time process of the generalized diffusion process with speed measure

$$m^0(x) = \begin{cases} 
0, & x \geq 0, \\
-c|x|^{\gamma/(1-\gamma)}/(1-\gamma), & x < 0,
\end{cases}$$

starting at the origin.

Theorem 4.3 Assume that $\ell_1 = -\infty$, (C_1) holds and

$$x|m(-tx)|/a(x) \to 0 \text{ as } x \to \infty.$$ 

Then, the normalized generalized hitting time process $\{\tau_x/a(s(x))\}$ with any starting point converges weakly under $J_1$-topology on $D[0,\infty)$ to a process $\{\tau^0_t\}$ as $x \to \infty$. The process $\{\tau^0_t\}$ is the hitting time process of the generalized diffusion process with speed measure $m^0(t) = e^t$, $t \in \mathbb{R}$ and starting point $-\infty$. $\{\tau^0_t\}_{t \geq 0}$ is a selfsimilar additive process with exponent 1.

Theorem 4.4 Assume that $\ell_1 = -\infty$, (C_\gamma) holds for some $1 < \gamma < \infty$. Then, the normalized generalized hitting time process $\{\tau_{t_2-y|t|}/a(t_2-y)\}_{t<0}$ with any starting point converges weakly under $J_1$-topology on $D[0,\infty)$ to a process $\{\tau^0_t\}$ as $y \downarrow 0$. The process $\{\tau^0_t\}$ is the hitting time process of the generalized diffusion process with speed measure

$$m^0(x) = \begin{cases} 
|x|^{\gamma/(1-\gamma)}/(\gamma-1), & x < 0, \\
\infty, & x > 0,
\end{cases}$$

and starting point $-\infty$. $\{\tau^0_{t-1}\}_{t \geq 0}$ is a selfsimilar additive process with exponent $(\gamma - 1)^{-1}$.

Theorem 4.1 corresponds to Theorem 2, Theorem 4.2 corresponds to Theorem 3, Theorem 4.3 corresponds to Theorem 4 and Theorem 4.4 corresponds to Theorem 6 in [8], respectively.

References

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