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Malliavin calculus applied to mathematical finance and a new formulation of the integration-by-parts

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1 Introduction

In recent years there appear several papers in finance on jump models and on jump-diffusion models using stochastic calculus, after the success of the Black-Scholes model. Indeed, classical [1] and [16] include chapters on jump-diffusions. Recent examples are [17], [22], [10], [11], and [27]. However, fairly restricted types of jump processes have been treated, due to the technical difficulities. For example, [1] and [16] have treated the diffusion + compound Poisson model. The so-called geometric Lévy model $S_t = S_0 e^{Z_t}$, where Z_t denotes a Lévy process (with infinite jumps), has not been included in the previous typical jump models studied in many papers.

Let S_t denote a jump-diffusion given as a solution to SDE which is driven by a Lévy process. We study here as an application of Malliavin calculus of jump type the sensitivity analysis for asset prices. Basic concept is as follows.

$$price = E^Q[(\text{pay-off})].$$

Here *price* means today's ($t = 0$) value of some contingent claim (pay-off) with respect to S_t in future ($t = T$), and Q is a risk neutral probability.

We assume the pay-off depends on some parameter λ. We consider the marginal move of the price with respect to λ by using the integration-by-parts:

$$\frac{\partial}{\partial\lambda}(price)(\lambda) = E^Q[(\text{pay-off}).(\text{weight})(\lambda)].$$

The L.H.S. denotes the marginal move of the asset price with respect to λ, hence it serves to measure the stability of the price. Such quantities qre called Greeks. Some examples of Greeks are *Delta*, *Vega*, *Gamma*, *Rho* and *Theta*. For the precise definition, see below.

The basic framework of this thoery on the Wiener space has been established in [8]. We study in this paper some functionals on the Wiener-Poisson space, and develop a stochastic calculus of variations to achieve the integration-by-parts formula.

2 Jump-diffusion models in closed form

Let $N(dtdz)$ be a Poisson random measure on $[0,T] \times \mathbf{R}$ with the mean measure $dt \cdot \delta_{\{1\}}$, and W_t be a Wiener process on $\mathbf{R}$.

Let Z_t be a simple Lévy process given by

$$Z_t = \sigma_1 W_t + \sigma_2 \tilde{N}_t,$$

where $\tilde{N}_t = N_t - t$.

The price process S_t associated to this Z_t is defined by

$$\frac{dS_t}{S_{t-}} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)d\tilde{N}_t, S_0 = x.$$

Here $r(t), \sigma_1(t), \sigma_2(t)$ are deterministic functions. Then S_t is represented explicitly in closed form

$$S_t = x\exp[\int_0^t \sigma_1(s)dW_s + \int_0^t (r(t) - \sigma_2(s))ds - \frac{1}{2}\int_0^t \sigma_2^2(s)ds] \times \Pi_{k=1}^{N_t}(1 + \sigma_2(T_k))$$

where $T_1, T_2, ...$ are jump times of N_t. cf. [1] (3.2).

More generally, assume that X_t is a jump *semimartingale*, such that it is a solution to a SDE driven by a Lévy process. The price process is defined by

$$\frac{dS_t}{S_{t-}} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)dX_t, S_0 = x.$$

Then S_t is represented also in closed form by

$$S_t = x\exp[\int_0^t \sigma_1(s)dW_s + \int_0^t (r(t) - \frac{1}{2}\sigma_1^2(s))ds + \int_0^t \sigma_2(s)dX_s - \frac{1}{2}\int_0^t \sigma_2^2(s)d[X,X]_s]$$

$$\times\Pi_{s=0}^t((1 + \sigma_2(s)\Delta X_s)\exp(-\sigma_2(s)\Delta X_s + \frac{1}{2}(\sigma_2(s)\Delta X_s)^2)).$$

Note that the product is a infinite product in general.

Let, for example, $F = S_T, T > 0$. If we know explicitly the density of F via closed formulae above, then we can estimate $E[f(F)]$ directly. We may then have closed forms for Greeks for "good" f. This way is called the kernel density estimation method [13]. An example of a such density is the variance gamma distribution [18]. However this is not always the case. For example, there is no explicit formula for the price of American option.

3 Greeks

Let λ be some parameter in S_T given above, and let $F = F^\lambda$ be a functional of S_t. That is, for example, $F = S_T^{(\lambda)}, T > 0$ or $F = \int_0^T S_t^{(\lambda)}dt$. Let f be a a.e. smooth function taking values on $\mathbf{R}$. Then $f(F)$ is a random variable. An example of $f(x)$ is $f_0(x) = (x - K).1_{[K,\infty)}$, or its smooth regularization $f = f_0 * \varphi_\epsilon$, where φ_ϵ is a molifier.

So called *Greeks* associated to $f(F)$ are given as follows.

(1) Delta $= \exp\{-\int_0^T r(t)dt\}\frac{\partial}{\partial x}E[f(F)]$.

Delta is the derivative of the price with respect to the parameter $\lambda = x$ (the initial value of S).

(2) Vega $= \exp\{-\int_0^T r(t)dt\}\frac{\partial}{\partial \sigma_1}E[f(F)]$.

More precisely, for $\epsilon > 0$, let

$$\frac{dS_t^\epsilon}{S_{t-}^\epsilon} = r(t)dt + (\sigma_1(t) + \epsilon\tilde{\sigma}_1(t))dW_t + \sigma_2(t)d\tilde{N}_t, S_0^\epsilon = x.$$

We put

$$C_\epsilon \equiv \exp\{-\int_0^T r(t)dt\}E[f(S_T^\epsilon)].$$

Then Vega $= \frac{\partial C_\epsilon}{\partial \epsilon}|_{\epsilon=0}$. This is a (Fréchet) derivative of S_t with respect to $\sigma_1(.)$ (coefficient of the Wiener process) in the direction $\tilde{\sigma}_1(.)$.

Other Greeks are, for example,

(3) Gamma $= \exp\{-\int_0^T r(t)dt\}\frac{\partial^2}{\partial x^2}E[f(F)]$.

(4) Rho $= \frac{\partial}{\partial r}(\exp\{-\int_0^T r(t)dt\}E[f(F)])$. (The *Rho* is defined similarily as *Vega*.)

(5) Theta $= \frac{\partial}{\partial T}(\exp\{-\int_0^T r(t)dt\}E[f(F)])$.

We remark that these Greeks can be regarded as corresponding (first or second) terms in the asymptotic expansion

$$E[F^\lambda] - E[F] = c_1\lambda + \frac{1}{2}c_2\lambda^2 + \cdots$$

when $\lambda > 0$ is small.

4 Weights

For the calculation of Greeks we can use Malliavin calculus for jump-diffusion processes. In this section we assume that the 1-dimensional process X_t driving the SDE above is given by $X_t = \sigma_1 W_t + \sigma_2 Z_t$, where Z_t is a Lévy process

$$Z_t = bt + \int_0^t \int_{|z|\le 1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|>1} zN(dsdz)$$

whose Lévy measure is given by $\mu(dz)$. We do not assume $\mu(dz)$ is absolutely continuous with respect to the Lebesgue measure. It can even be a discrete measure. (If $\mu = \delta_{\{1\}}$ then Z_t is a Poisson process N_t.) In this case it is not practical to compute Greeks along the closed form expression in general.

Let $F = F^x$ be as in the previous section ($\lambda = x$). For a random variable $G^x \in L^2$ depending on x, we have

$$\frac{\partial}{\partial x}E[G^x f(F)] = E[G^x \partial f(F)\partial_x F] + E[\partial_x G^x \cdot f(F)].$$

If we choose $G^x \equiv 1$,

$$\frac{\partial}{\partial x}E[f(F)] = E[\partial f(F).\partial_x F]. \tag{0}$$

We introduce a gradient operator $D_u, u = (t, z)$, on the Poisson space on $[0, T] \times \mathbf{R}$. We assume the chain rule

$$D_u f(F) = \partial f(F).D_u F \tag{1}$$

and the local operator property

$$D_u(XY) = XD_uY + YD_uX \tag{2}$$

hold for the operator D_u. By the chain rule for the gradient D_u and by the integration by parts, we have

$$\text{R.H.S. of (0)} = E[\frac{D_uf(F)}{D_uF}.\partial_xF] = E[D_uf(F).\frac{\partial_xF}{D_uF}] = E[f(F)\delta(\frac{\partial_xF}{D_uF})]. \tag{3}$$

This leads to the calculation for *Delta*.

Here $\delta(.)$ is the adjoint operator (Skorohod integral) associated to the gradient D_u, and the term $\delta(\cdots)$ is called a *weight* provided that it is square integrable. In practical computation it is important to calculate this *Weight*.

We can proceed the calculation (3) above following the formula

$$\delta(vG) = G\delta(v) - \int_0^T \int D_uGv(u)dt\mu(dz) \tag{4}$$

(cf. [6] Proposition 1).

To compute *Gamma* we need to compute the second derivative

$$\frac{\partial^2}{\partial x^2}E[f(F)] = \frac{\partial}{\partial x}E[f(F).\delta(G)] = E[f(F)\frac{\partial}{\partial x}\delta(G)] + E[f(F)\delta(\delta(G)G)],$$

where $G = \frac{\partial_xF}{D_uF}$.

For the precise framework for this calculation on the Wiener-Poisson space, there seems to exist no decisive set-up up to now (e.g., gradient operator, its adjoint, norms, Sobolev spaces, ...). In the section 7 we present a new framework for this.

5 Finite difference operator and gradient operator on Poisson space

Let $Z_t = \tilde{N}_t$ for simplicity. On the Poisson space we introduce two gradients.

Let $U = [0,T] \times \mathbf{R}$. We choose u in U of the form $u = (t,1)$. Let $F = f(T_1, ..., T_n)$, where $f = f(x_1, ..., x_n)$ is a smooth function and T_k denotes the k-th jump time of N_t. We introduce two gradient of F on U.

We put

$$D_uF = -\sum_{N_t<k\le n} \partial_k f(T_1, ..., T_n). \tag{5}$$

Here ∂_k denotes $\frac{\partial}{\partial x_k}$. This definition is due to Carlen-Pardeux [4].

We introduce a finite difference operator $\tilde{D}$ by

$$\tilde{D}_uF = f(T_1, ..., T_{N_t}, t, T_{N_t+1}, ..., T_{n-1}) - f(T_1, ..., T_n) \tag{6}$$

if $N_t < n$. The above is equivalent to

$$\tilde{D}_u F = f(T_1, ..., T_k, t, T_{k+1}, ..., T_{n-1}) - f(T_1, ..., T_n) \quad (7)$$

if $T_k < t \le T_{k+1}$. This definition is due to Nualart-Vives [21] (see also Picard [23]).

The operator D_u satisfies the properties (1), (2) in Sect. 4, whereas $\tilde{D}_u$ does *not*. Instead we have by the mean value theorem when φ is differentiable :

$$\tilde{D}_u \varphi(F) = \int_0^1 \partial\varphi(F + \theta\tilde{D}_u F) d\theta . \tilde{D}_u F. \quad (8)$$

And also

$$\tilde{D}_u(FG) = F \cdot \tilde{D}_u G + G \cdot \tilde{D}_u F + \tilde{D}_u F \tilde{D}_u G \quad (9)$$

(cf. [21] Lemma 6.1).

The gradient D_u is closable (in $L^2(\Omega, L^2([0,T]))$), and its adjoint is given by

$$\delta(v) = \int_0^T v(t) d\tilde{N}_t - \int_0^T D_u v(t) dt.$$

Further we have

$$E[\int_0^T D_u F \ v \ dt] = E[F\delta(v)]$$

([26] Propositions 7, 8, [19] p.104).

The formula (4) then reads

$$\delta(vG) = G\delta(v) - (v, D_u G) = G\delta(v) + \int_0^T v(t)(\sum_{N_t < k \le n} \partial_k g(T_1, ..., T_n)) dt$$

if $G = g(T_1, ..., T_n)$. Hence, the formula (3) reads

$$\delta(\frac{\partial_x F}{D_u F}) = \partial_x F \delta(\frac{1}{D_u F}) - (\frac{1}{D_u F}, D_u(\partial_x F)).$$

Although, due to (9), $\tilde{D}_u$ does not satisfy the chain rule, we can show the property below between $\tilde{D}_u$ and D_u for which the chain rule holds.

Let $p_n(t) = P(N_t = n) = \frac{t^n}{n!} e^{-t}$ be the density function of T_n. We have then

$$p_n'(t) = p_{n-1}(t) - p_n(t), \ t > 0.$$

Let $T = \infty$. In view of this formula, the formula

$$\frac{d}{du} \int_t^u g(s,u) ds = g(u,u) + \int_t^u \frac{\partial}{\partial u} g(s,u) ds,$$

and due to the fact that the jump times of a Poisson process are uniformly distributed given the number of jumps, we have the following Proposition.

Proposition *Let* $F = f(T_1, ..., T_n)$ *and* $G = g(T_1, ..., T_n) = \varphi(F)$. *That is,* $g = \varphi \circ f$. *Then*

$$E[D_u G / \mathcal{F}_t] = E[\tilde{D}_u G / \mathcal{F}_t].$$

The proof is due to N. Privault. It also follows from the Kabanov formula (cf. [21] Theorem. 6.2). We state the direct proof below in case $n = 5$. Proof for the general case is easy.

Example

Let $G = g(T_n)$. We have the equality directly as follows.

$$E[D_u g(T_n)/\mathcal{F}_t] = -1_{\{N_t<n\}} E[g'(T_n)/\mathcal{F}_t]$$
$$= -\int_t^\infty g'(x) p_{n-1-N_t}(x-t)dx = g(t)p_{n-1-N_t}(0) + \int_t^\infty g(x)p'_{n-1-N_t}(x-t)dx$$
$$= g(t)1_{\{T_{n-1}\le t<T_n\}} + \int_t^\infty g(x)p'_{n-1-N_t}(x-t)dx$$
$$= g(t)1_{\{T_{n-1}\le t<T_n\}} + \int_t^\infty (p_{n-2-N_t}(x-t) - p_{n-1-N_t}(x-t))g(x)dx$$
$$= g(t)1_{\{T_{n-1}\le t<T_n\}} + E[1_{\{T_{n-1}>t\}}g(T_{n-1}) - 1_{\{T_n>t\}}g(T_n)/\mathcal{F}_t]$$
$$= E[1_{\{T_{n-1}>t\}}g(T_{n-1}) + 1_{\{T_{n-1}\le t<T_n\}}g(t) - 1_{\{T_n>t\}}g(T_n)/\mathcal{F}_t]$$
$$= E[1_{\{N_t<n-1\}}(g(T_{n-1}) - g(T_n)) + 1_{\{N_t=n-1\}}(g(t) - g(T_n))/\mathcal{F}_t]$$
$$= E[\tilde{D}_u g(T_n)/\mathcal{F}_t].$$

It is natural that they coincide with each other by the uniqueness of the Clark-Ocone formula for G, as they are the conditional expectation terms (integrands) in the 1-st stochastic integral in the Clark-Ocone expression. Cf. [28]. However we can see it directly in this case.

Proof.

Let $G = g(T_1, T_2, ..., T_5)$.

$$E[D_u G/\mathcal{F}_t] = -\sum_{N_t<k\le 5} E[\partial_k g(T_1, ..., T_5)/\mathcal{F}_t]$$
$$= -\sum_{N_t<k\le 5} \int_0^\infty e^{-(s_5-t)} \int_t^{s_5} \cdots \int_t^{s_{N_t+2}} \partial_k g(T_1, ..., T_{N_t}, s_{N_t+1}, ..., s_5) ds_{N_t+1} ... ds_5$$
$$= -\sum_{k=N_t+2}^{5} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \cdots \frac{\partial}{\partial s_k} \int_t^{s_k} \cdots \int_t^{s_{N_t+2}} g(T_1, ..., T_{N_t}, s_{N_t+1}, ..., s_5) ds_{N_t+1} ... ds_5$$
$$+ \sum_{k=N_t+2}^{5} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \cdots \int_t^{s_k} \cdots \int_t^{s_{N_t+2}} g(T_1, ..., T_{N_t}, s_{N_t+1}, s_k, s_k, s_{k+1}, ..., s_5) ds_{N_t+1} ... d\hat{s}_{k-1} ... ds_5$$
$$-1_{\{N_t<5\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \cdots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_{N_t+1}} g(T_1, ..., T_{N_t}, s_{N_t+1}, ..., s_5) ds_{N_t+1} ... ds_5$$
$$= -1_{\{N_t<4\}} \int_t^\infty e^{-(s_5-t)} \int_t^{s_5} \cdots \int_t^{s_{N_t+2}} \frac{\partial}{\partial s_5} g(T_1, ..., T_{N_t}, s_{N_t+1}, ..., s_5) ds_{N_t+1} ... ds_5$$

$$-\sum_{k=N_t+2}^{4}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{k+1}}\frac{\partial}{\partial s_k}\int_t^{s_k}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$+\sum_{k=N_t+2}^{5}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_k}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},s_k,s_k,s_{k+1},...,s_5)ds_{N_t+1}...d\hat{s}_{k-1}...ds_5$$

$$-1_{\{N_t<5\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}}\frac{\partial}{\partial s_{N_t+1}}g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$=-1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\frac{\partial}{\partial s_5}\int_t^{s_5}\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$-\sum_{k=N_t+2}^{4}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_{k-1},s_{k+1},s_{k+1},...,s_5)ds_{N_t+1}...d\hat{s}_k...ds_5$$

$$+\sum_{k=N_t+2}^{5}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_k}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},s_k,s_k,s_{k+1},...,s_5)ds_{N_t+1}...d\hat{s}_{k-1}...ds_5$$

$$-1_{\{N_t<5\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}}\frac{\partial}{\partial s_{N_t+1}}g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$=-1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$+1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+2},s_{N_t+2},...,s_5)ds_{N_t+1}...ds_5$$

$$-1_{\{N_t<5\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}}\frac{\partial}{\partial s_{N_t+1}}g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$=-1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$+1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+2},s_{N_t+2},...,s_5)ds_{N_t+1}...ds_5$$

$$-1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+2},s_{N_t+2},...,s_5)ds_{N_t+1}...ds_5$$

$$+1_{\{N_t<4\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},t,s_{N_t+2},...,s_5)ds_{N_t+2}...ds_5$$

$$-1_{\{5=N_t+1\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5} g(T_1,...,T_{n-1},s_5)ds_5$$

$$+1_{\{5=N_t+1\}}g(T_1,...,T_{n-1},t)$$

$$=-1_{\{N_t<5\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},s_{N_t+1},...,s_5)ds_{N_t+1}...ds_5$$

$$+1_{\{N_t<5\}}\int_t^\infty e^{-(s_5-t)}\int_t^{s_5}\cdots\int_t^{s_{N_t+2}} g(T_1,...,T_{N_t},t,_{N_t+2},...,s_5)ds_{N_t+2}...ds_5$$

$$=E[\tilde{D}_uG/\mathcal{F}_t].$$

The operators D_u and δ can be extended to the case

$$Z_t = \sum_{k=1}^{m} \tilde{N}_k(t),$$

where N_k's are independent Poisson processes, by composing a direct sum of independent Poisson spaces (cf. [19] p.103). On the other hand, for the adjoint of $\tilde{D}_u$, see the section 7.

6 Integration-by-parts setting by Bismut

In this section we state the integration-by-parts formula by using Bismut perturbation. We sketch the idea below in case $d = 1$. We assume in this section that $\mu(dz) = g(z)dz$, where $g(z)$ is a smooth function on $\mathbf{R}$ having compact support.

Let v be a bounded predictable process on $[0, +\infty)$ to $\mathbf{R}$. We consider the perturbation

$$\theta^\lambda : z \mapsto z + \lambda\nu(z)v, \quad \lambda \in \mathbf{R}.$$

Here $\nu(z)$ is a smooth function which is $O(z^2)$ near $z = 0$. Let $N^\lambda(dsdz)$ be the Poisson random measure defined by

$$\int_0^t \int \phi(z)N^\lambda(dsdz) = \int_0^t \int \phi(\theta^\lambda(z))N(dsdz), \phi \in C_0^\infty(\mathbf{R}).$$

We put $Z_s^\lambda = \int_0^t \int zN^\lambda(dudz)$, and denote by P^λ its law . Set $\Lambda^\lambda(z) = \{1+\lambda\nu'(z)v\}\frac{g(\theta^\lambda(z))}{g(z)}$, and

$$U_t^\lambda = \exp[\{\int_0^t \int \log \Lambda^\lambda(z)N(dsdz_j) - \int_0^t ds \int(\Lambda^\lambda(z) - 1)g(z)dz].$$

Then Z_t^λ is a martingale, and P^λ has the derivative

$$\frac{dP^\lambda}{dP} = U_t^\lambda \quad on \quad \mathcal{F}_t.$$

where $\mathcal{F}_t$ denotes the $\sigma-$field generated by Z_t (cf. [2] Theorem 6-16, Bismut [3], (2.34)).

Consider the perturbed process F_s^λ which is defined by a SDE driven by Z^λ in place of Z. Then $E^P[f(F_t)] = E^{P^\lambda}[f(F_t^\lambda)] = E^P[f(F_t^\lambda)U_t^\lambda]$, and we have $0 = \frac{\partial}{\partial\lambda}E^P[f(F_t^\lambda)U_t^\lambda], f \in C_0^\infty(\mathbf{R})$. By the chain rule, for $|\lambda|$ small, we have

$$\frac{\partial f}{\partial\lambda}(F_t^\lambda) = D_x f(F_t^\lambda) \cdot \frac{\partial F_t^\lambda}{\partial\lambda}, \quad f \in C_0^\infty(\mathbf{R}).$$

We have for $\lambda = 0$

$$E^P[D_x f(F_t) \cdot \frac{\partial F_t^\lambda}{\partial\lambda}|_{\lambda=0}] = -E^P[f(F_t)\frac{\partial}{\partial\lambda}U_t^\lambda|_{\lambda=0}].$$

By Corollary 6-17 of [2], we may differentiate U_t^λ with respect to λ, to obtain

$$R_t \equiv \frac{\partial}{\partial\lambda}U_t^\lambda|_{\lambda=0} = \int_0^t \int \frac{\text{div } \{g(\cdot)v\nu(\cdot)\}(z)}{g(z)}\{N(dsdz) - dsg(z)dz\}.$$

Next we compute $H_t^\lambda \equiv \frac{\partial F_t^\lambda}{\partial \lambda}$. F_t^λ is differentiable a.s. for $|\lambda|$ small, and its derivative at $\lambda = 0, H_t = H_t^0$ is obtained explicitly as the solution of a SDE (cf. [2] Theorem 6-24). We put $DH_t = \frac{\partial}{\partial \lambda} H_t^\lambda|_{\lambda=0}$, where $\frac{\partial}{\partial \lambda} H_t^\lambda$ is the second Fréchet derivative of F_t^λ defined as in [2] Theorem 6-44. Then $\frac{\partial}{\partial \lambda} H_t^{\lambda,-1}|_{\lambda=0} = -H_t^{-1} DH_t H_t^{-1}$. Here $\frac{\partial}{\partial \lambda} H_t^{\lambda,-1}$ is defined by $< \frac{\partial}{\partial \lambda} H_t^{\lambda,-1}, e >=$ trace $[e' \mapsto < -H_t^{\lambda,-1}(\frac{\partial}{\partial \lambda} H_t^\lambda \cdot e') H_t^{\lambda,-1}, e >], e \in \mathbf{R}$.

We carry out the integration-by-parts procedure for $G_t^\lambda = f(F_t^\lambda) H_t^{\lambda,-1}$. Recall we have $E[G_t^0] = E[G_t^\lambda \cdot U_t^\lambda]$. Taking the Fréchet derivation $\frac{\partial}{\partial \lambda}|_{\lambda=0}$ for both sides yields

$$0 = E[D_x f(F_t) H_t^{-1} H_t] + E[f(F_t) \frac{\partial}{\partial \lambda} H_t^{\lambda,-1}|_{\lambda=0}] + E[f(F_t) H_t^{-1} \cdot R_t].$$

This yields

$$E[D_x f(F_t)] = E[f(F_t) \mathcal{A}_t^{(1)}]$$

where

$$\mathcal{A}_t^{(1)} = \{H_t^{-1} DH_t H_t^{-1} - H_t^{-1} R_t\}.$$

This is the integration-by-parts formula in Bismut setting. We can calculte $H_t^{-1} DH_t H_t^{-1}$ explicitly.

7 New integration-by-parts setting for jump diffusion

This is a joint work with Prof. H. Kunita. [12]

From the gradient-adjoint formula to the integration-by-parts formula for $f(F)$, there are several attempts. Here we recall one which is based on Picard's method.

In this section, let $N(dtdz)$ be a Poisson random measure on $[0, T] \times \mathbf{R}^m$ and W_t be a Wiener process on $\mathbf{R}^m$, $m \geq 1$.

Let T_0 be a positive number and let $T = [0, T_0]$. Let Ω_1 be the set of all continuous maps $\omega_1 : T \to \mathbf{R}^m$ such that $\omega_1(0) = 0$ and let $\mathcal{F}_1$ be the smallest σ-field of Ω_1 with respect to which $\{w_1(t), t \in [0, T]\}$ are measurable. Let P_1 be a probability measure on $(\Omega_1, \mathcal{F}_1)$ such that $W(t) := \omega_1(t)$ is a standard 1-dimensional Brownian motion.

Set

$$\varphi(\rho) = \int_{|z| \leq \rho} |z|^2 \mu(dz). \tag{10}$$

We say that the measure μ satisfies an *order condition* if there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho \to 0} \frac{\varphi(\rho)}{\rho^\alpha} > 0. \tag{11}$$

Note that Lévy measures with finite mass do not satisfy the order condition, because $\liminf_{\rho \to 0} \frac{\varphi(\rho)}{\rho^\alpha} = 0$ holds for any $\alpha \in (0, 2)$ then. On the other hand, Lévy measures of stable laws with exponent β satisfies the order condition with $\alpha = 2 - \beta$.

Let T_0 be a positive number and let $T = [0, T_0]$. Let Ω_1 be the set of all continuous maps $\omega_1 : T \to \mathbf{R}^m$ such that $\omega_1(0) = 0$ and let $\mathcal{F}_1$ be the smallest σ-field of Ω_1 with respect to which $\{w_1(t), t \in [0,T]\}$ are measurable. Let P_1 be a probability measure on $(\Omega_1, \mathcal{F}_1)$ such that $W(t) := \omega_1(t)$ is a standard 1-dimensional Brownian motion.

Let Ω_2 be the set of all integer valued measures on $U = T \times \mathbf{R}^m$ such that $\omega_2(T \times \{0\}) = 0$ and let $\mathcal{F}_2$ be the smallest σ-field of Ω_2 with respect to which $\{w_2(E);\ E$ are Borel sets in $U\}$ are measurable. Let P_2 be a probability measure on $(\Omega_2, \mathcal{F}_2)$ such that $N(dtdz) := \omega_2(dtdz)$ is a Poisson random measure with intensity measure $\hat{N}(dtdz) := dt\mu(dz)$, where μ is a Lévy measure.

Let $H = L^2(T; \mathbf{R}^m)$. For $h_l \in H$, we set

$$W(h_l) = \int_T h_l(s) dW_s.$$

We denote by $\mathcal{S}_1$ the collection of random variables X written as

$$X = f(W(h_1), \cdots, W(h_{n_1})),$$

where $f(x_1, ..., x_{n_1})$ is bounded $\mathcal{B}(\mathbf{R}^{n_1})$ measurable, smooth in $(x_1, ..., x_{n_1})$, $n_1 \in \mathbf{N}$. The Malliavin-Shigekawa's derivative of X (with respect to the first variable ω_1) is an 1-dimensional row vector stochastic process given by

$$D_t X = \sum_l \frac{\partial f}{\partial x_l}(W(h_1), ..., W(h_n)) h_l(t). \tag{12}$$

Next, we shall introduce difference operators $\tilde{D}_u, u \in U$, acting on the Poisson space. For each $u = (t,z) = (t, z_1) \in U$, we define a map $\varepsilon_u^- : \Omega_2 \to \Omega_2$ by $\varepsilon_u^- \omega_2(A) = \omega_2(A \cap \{u\}^c)$, and $\varepsilon_u^+ : \Omega_2 \to \Omega_2$ by $\varepsilon_u^+ \omega_2(A) = \omega_2(A \cap \{u\}^c) + 1_A(u)$. (These are extended to Ω by setting $\varepsilon_u^\pm(\omega_1, \omega_2) = (\omega_1, \varepsilon_u^\pm \omega_2)$) It holds $\varepsilon_u^- \omega = \omega$ a.s. P for any u since $\omega_2(\{u\}) = 0$ holds for almost all ω_2 for any u. The difference operators $\tilde{D}_u$ for a $\mathcal{F}_2$-measurable random variable X is defined after Picard [23] by

$$\tilde{D}_u X = X \circ \varepsilon_u^+ - X. \tag{13}$$

Let $\mathbf{u} = (u^1, ..., u^k) = ((t_1, z^1), ..., (t_k, z^k)) = (\mathbf{t}, \mathbf{z})$. We set $|\mathbf{u}| = |\mathbf{z}| = \max_{1 \le i \le k} |z^i|$ and $\gamma(\mathbf{u}) = |z^1| \cdots |z^k|$. We define $\varepsilon_{\mathbf{u}}^+ = \varepsilon_{u_1}^+ \circ \cdots \circ \varepsilon_{u_k}^+$ and $\tilde{D}_{\mathbf{u}} = \tilde{D}_{\mathbf{u}}^k = \tilde{D}_{u_1} \cdots \tilde{D}_{u_k}$. Further for $\mathbf{z} = (z^1, ..., z^k)$ where $z^i \in \mathbf{R}^m$, we set $\partial_{\mathbf{z}} g = \partial_{z^1} \cdots \partial_{z^k} g$. It is an k-vector function.

Let $\mathcal{S}_2$ be the collection of random variables X written as

$$X = f(N(\varphi_1), \cdots, N(\varphi_{n_2})),$$

where $f(x_1, ..., x_{n_2})$ is bounded $\mathcal{B}(\mathbf{R}^{n_2})$ measurable, smooth in $(x_1, ..., x_{n_2})$, $n_2 \in \mathbf{N}$.

Let $\mathcal{S} = \mathcal{S}_1 \otimes \mathcal{S}_2$. Spaces $\mathcal{S}_1, \mathcal{S}_2$ are identified with $\mathcal{S}_1 \otimes 1, 1 \otimes \mathcal{S}_2$ respectively. The space $\mathcal{S}$ is the linear span of functionals X such that

$$X = \sum_{i+j=k} X_1^{(i)} X_2^{(j)}, k \in \mathbf{N},$$

where $X_1^{(i)} = f_1^{(i)}(W(h_1), ..., W(h_i))$ and $X_2^{(j)} = f_2^{(j)}(N(\varphi_1), ..., N(\varphi_j))$. Here $f_1^{(i)}$ and $f_2^{(j)}$ are bounded $\mathcal{B}(\mathbf{R}^i)$ $(\mathcal{B}(\mathbf{R}^j))$ measurable, smooth functions of i (j) variables, respectively.

The adjoint $\tilde{\delta}$ of the operators $\tilde{D} = (\tilde{D}_u)_{u \in U}$ is defined as follows. Let $Z_u = Z_{t,z}$ be an $\mathcal{F}$-measurable random field, integrable with respect to $\tilde{N} = N - \hat{N}$, i.e.,

$$E[\int_U |Z_u \circ \varepsilon_u^-|(N + \hat{N})(du)] < \infty.$$

We set

$$\tilde{\delta}(Z) = \int_U Z_u \circ \varepsilon_u^- \tilde{N}(du). \tag{14}$$

It is known that this operator satisfies the adjoint property:

$$E[X\tilde{\delta}(Z)] = E\left[\int_U \tilde{D}_u X Z_u \hat{N}(du)\right], \tag{15}$$

for any bounded $\mathcal{F}$-measurable random variable X. ([23], Lemma 1.4).

We shall next introduce linear maps Q and $\tilde{Q}_\rho$ by

$$QY = \int_T (D_t F) D_t Y dt, \tag{16}$$

$$\tilde{Q}_\rho Y = \frac{1}{\varphi(\rho)} \int_{A(\rho)} (\tilde{D}_u F) \tilde{D}_u Y \hat{N}(du). \tag{17}$$

Lemma *The adjoints of Q and $\tilde{Q}_\rho$ exist and are equal to*

$$Q^* X = \delta((DF)^T X), \tag{18}$$

$$\tilde{Q}_\rho^* X = \tilde{\delta}_\rho((\tilde{D}F)^T X), \tag{19}$$

respectively, where

$$\tilde{\delta}_\rho(Z) = \frac{1}{\varphi(\rho)} \tilde{\delta}(Z 1_{A(\rho)}) = \frac{1}{\varphi(\rho)} \int_{A(\rho)} Z_u \circ \varepsilon_u^- \tilde{N}(du). \tag{20}$$

Let $f(x)$ be a C^2-function with bounded derivatives. We claim a modified formula of integration by parts. Note that $D_t(f(F)) = f(F) D_t F = (D_t F)\partial f(F)$. Then we get

$$Qf(F) = \int_T (D_t F) D_t(f(F)) dt = R\partial f(F). \tag{21}$$

Concerning the difference operator $\tilde{D}_u$, we have by the mean value theorem,

$$\tilde{D}_u(f(G)) = (\tilde{D}_u G)^T \int_0^1 \partial f(G + \theta \tilde{D}_u G) d\theta, \tag{22}$$

for a random variable G on the Poisson space. This implies

$$\tilde{Q}_\rho f(F) = \tilde{R}_\rho \partial f(F) \tag{23}$$
$$+ \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F (\tilde{D}_u F)^T \left(\int_0^1 \{\partial f(F + \theta \tilde{D}_u F) - \partial f(F)\} d\theta\right) \hat{N}(du).$$

Here

$$\tilde{R}_\rho = \frac{1}{\varphi(\rho)} \int_{A(\rho)} \tilde{D}_u F(\tilde{D}_u F)^T \hat{N}(du).$$

Sum up (21) and (23) and then take the inner product of this with $S_\rho X$. Its expectation yields the following.

Proposition [12] *(Analogue of the formula of integration by parts) For any X we have*

$$E[(X, \partial f(F))] = E[(Q + \tilde{Q}_\rho)^*(S_\rho X) f(F)] \tag{24}$$

$$-\frac{1}{\varphi(\rho)} E\left[\left(X, S_\rho \int_{A(\rho)} \tilde{D}_u F(\tilde{D}_u F)^T \left(\int_0^1 \{\partial f(F + \theta \tilde{D}_u F) - \partial f(F)\} d\theta\right) \hat{N}(du)\right)\right].$$

Here $S_\rho = (R + \tilde{R}_\rho)^{-1}$.

Remark. If there is no Poisson part in (15), then the formula is written as

$$E[(X, \partial f(F))] = E[Q^*(R^{-1}X) f(F)] = E[\delta((R^{-1}X, DF)) f(F)]. \tag{25}$$

On the other hand, if $\tilde{R}_\rho$ is not zero or equivalently $\tilde{Q}_\rho$ is not zero, we have a remaining term (the last term of (15)). We have this term even if Z_t is a simple Poisson process N_t or its sums. However, if we take $f(x) = e^{i(w,x)}, w \in \mathbf{R}^d \setminus \{0\}$, we have $\partial f(x) = ie^{i(w,x)} w$ and

$$e^{i(w,F+\theta \tilde{D}_u F)} - e^{i(w,F)} = e^{i(1-\theta)(w,F)} \tilde{D}_u (e^{i(w,\theta F)}).$$

Hence we have an expression of the integration-by-parts for the functional

$$E[(X, w) \partial_x (e^{i(w,F)})] = E[(Q^* + \tilde{Q}_\rho^* + R_{\rho,w}^*) S_\rho X \cdot e^{i(w,F)}], \quad \forall w. \tag{26}$$

Here

$$R_{\rho,w}^* Y = -\frac{i}{\varphi(\rho)} \int_0^1 \left(\tilde{\delta}(e^{i(1-\theta)(w,F)} \chi_\rho \tilde{D}F(\tilde{D}F)^T Y), e^{i(\theta-1)(w,F)} w\right) d\theta.$$

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