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Economic growth under two stochastic perturbations

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1 Introduction

Study of economic growth has long history of almost 600 years after A. Smith and T. Malthus. The main thesis of economic growth theory is to answer the following question: *Why some countries are so rich and the others are so poor?*

In nowadays economic growth theories, *Neo-classical growth model* plays the fundamental part. This model was developed by the works of R. Solow, 1956, 1957.

After Solow's work, R. Lucus (1988), D. Romer (1986), G. Mankiw (1992), and etc refined Solow model by importing advances of technology or human factor. In this paper, we discuss Solow model with stochastic perturbations.

We will start from the original Solow model. An economy in the model is considered in the following setting.

Assumption 1.1. (i) *The economy is an isolated island in where many labors live. There is a social planner, who governs all economic.*

(ii) *There is one good. At time t, production $Y(t)$ of the good depends on two factors, capital $K(t)$ and labor $L(t)$. The good can be either consumed or invested as capital.*

(iii) *The social planner saves a constant fraction $s \in (0,1)$ of production, to be added to the economy's capital stock, and distributes the remaining fraction uniformly across the labors of the economy.* $\diamond$

In what follows, we introduce the following standard signatures in economic theory:

$$\begin{aligned} &Y(t) = \text{output at time } t, && K(t) = \text{capital stock at time } t, \\ &I(t) = \text{investment at time } t, && C(t) = \text{consumption at time } t, \\ &L(t) = \text{the number of labors at time } t. && \end{aligned} \tag{1.1}$$

From Assumption 1.1 (+ a little assumptions), the following condition is derived:

Condition 1.2. (i) *The economy is* Keynes system, *that is*

$$I(t) + C(t) = Y(t).$$

(ii) *The output of the good production is given by* the production function $F : [0,\infty) \times [0,\infty) \to [0,\infty)$, *that is*

$$Y(t) = F(K(t), L(t)). \tag{1.2}$$

(iii) *Capital depreciates at a fixed rate $\lambda \in [0,1]$, that is*

$$K'(t) = I(t) - \lambda K(t).$$

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(iv) *Saving rate $s \in (0,1)$ is constant, that is*

$$Y(t) = s\,Y(t) + C(t).$$

(v) *The population of labors increases in a constant rate n:*

$$L'(t) = n\,L(t). \quad \diamond \tag{1.3}$$

In addition, we assume that the production function F in (1.2) is neo classical, i.e. the following condition is fulfilled.

Condition 1.3. *The production function F is a strictly concave C^2 class function with $F(0,L) = 0 = F(K,0)$. Moreover F satisfies:*

$$0 < \lim_{K\to 0} \partial_K F(K,L), \qquad 0 < \lim_{L\to 0} \partial_L F(K,L) = \infty, \tag{1.4}$$

$$\lim_{K\to\infty} \partial_K F(K,L) = 0, \qquad \lim_{L\to\infty} \partial_L F(K,L) = 0. \tag{1.5}$$

$$F(a\,K,\ a\,L) = a\ F(K,L) \quad \textit{for } \forall a > 0; \quad \text{CRS (constant returns to scale) property.} \quad \diamond \tag{1.6}$$

Remark 1.4. (i) In many prior papers, (1.4) is replaced by the stronger Inada condition, that is

$$\lim_{K\to 0} \partial_K F(K,L) = \infty, \qquad \lim_{L\to 0} \partial_L F(K,L) = \infty. \tag{1.7}$$

But Inada condition (1.7) concludes*2 that

> huge growth of the production are derived from small increment in the capital only if the population of labors is sufficiently large.

Therefore some economists ([7] et.al.) assert that Inada condition is not adequate to assume, and we assume Condition 1.3 only.

(ii) A production function F is called DRS (decreasing returns to scales) if the following inequality holds:

$$F(a\,K,\ a\,L) < a\ F(K,L) \quad \text{for } \forall a > 0.$$

It is called IRS (increasing return to scales) if the following inequality holds:

$$F(a\,K,\ a\,L) > a\ F(K,L) \quad \text{for } \forall a > 0. \quad \diamond$$

Example 1.5. (i) The Cobb-Douglas production function is a typical example fulfilling Condition 1.3 and Inada condition (1.7), that is

$$F(K,L) = K^\alpha L^{1-\alpha} \quad \text{with a constant } 0 < \alpha < 1. \tag{1.8}$$

(ii) Let c be a positive constant. A modified Cobb-Douglas production function

$$F(K,L) \equiv (K + cL)^\alpha L^{1-\alpha} - c^\alpha L \tag{1.9}$$

is an example which fulfills Condition 1.3 without Inada condition (1.7). $\diamond$

We introduce the per ca-pita measurements, that is

$$\begin{aligned} y(t) &\equiv Y(t)/L(t) \quad \text{(per ca-pita GDP)},\\ k(t) &\equiv K(t)/L(t) \quad \text{(per ca-pita capital stock)}, \end{aligned} \tag{1.10}$$

By CRS condition in Condition 1.3, (ii),

$$y(t) = \frac{Y(t)}{L(t)} = \frac{F(K(t),L(t))}{L(t)} = F\Big(\frac{K(t)}{L(t)},1\Big) \equiv f(k(t)) \tag{1.11}$$

We also call this f as a production function. By definition (1.11) of f and Condition 1.3,

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*2 For instant, see [7] or [13, §4].

Condition 1.6. *A production function $f : [0,\infty) \to [0,\infty)$ is a strictly concave C^2 class function with $f(0) = 0$. Moreover f satisfies*

(1.12a) $$\lim_{k\to\infty} f'(k) = 0,$$

(1.12b) $$\exists \lim_{k\to 0} f'(k) > 0,$$

where $\lim_{k\to 0} f'(k)$ *may be infinity.* ◇

Remark 1.7. If F satisfies Inada condition (1.7), then (1.12b) is replaced by the following condition

$$\lim_{k\to 0} f'(k) = \infty. \quad \diamond$$

Example 1.8. (i) If F is the Cobb-Douglas production function (1.8), then

$$f(k) = k^\alpha.$$

(ii) If F is the Cobb-Douglas production function (1.9), then

$$f(k) = (k + c)^\alpha - c^\alpha. \quad \diamond$$

Combining the equations in Condition 1.2, we derive ODE for the capital stock $K(t)$:

(1.13) $$\begin{aligned} K'(t) &= \{Y(t) - C(t)\} - \lambda K(t) = s\,Y(t) - \lambda K(t) \\ &= s\,F(K(t), L(t)) - \lambda K(t). \end{aligned}$$

By a simple calculation,

$$k'(t) = \left(\frac{K(t)}{L(t)}\right)' = \frac{K'(t)}{L(t)} - \frac{K(t)}{L(t)} \cdot \frac{L'(t)}{L(t)}.$$

Now (1.13) and (1.3) give the dynamics of capital stock in per ca-pita measurement:

(1.14) $$\text{(Solow equation)} \quad k'(t) = s\,f(k(t)) - (\lambda + n)\,k(t),$$

where $s \in (0,1)$ is saving rate, $\lambda \in [0,1]$ is capital depreciating rate, and n is population growth rate.

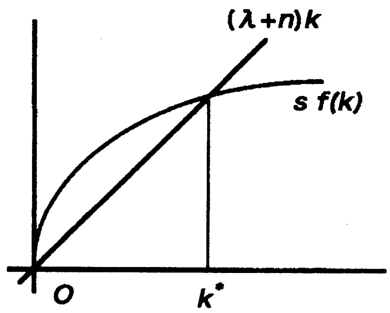

Fig. 1.1 The state of golden age k^*

Owing to Condition 1.6, if $\lim_{k\to 0} f'(k) > (\lambda + n)/s$, then there exists a unique solution k^* to

(1.15) $$s\,f(k) - (\lambda + n)\,k = 0, \quad k > 0,$$

and it is a stable fixed point of Solow equation (1.14).

Proposition 1.9. *If* $\lim_{k\to 0} f'(k) > (\lambda + n)/s$, *then there exists a unique point* $k^* > 0$ *which solves* (1.15). *We call* k^* *as* the state of golden age, *since*

$$\lim_{t\to\infty} k(t) = k^* \quad \textit{for any} \quad k(0) > 0. \quad \diamond$$

2 Verification of Solow model

We shall compare the result in Proposition 1.9 with a statics in the real economy between 1980 and 1997 which will be shown in Fig. 2.1.

Growth rate of per ca-pita GDP is $y'(t)/y(t)$. From (1.11) and Solow equation (1.14), we have

$$\frac{y'(t)}{y(t)} = \frac{f'(k(t))}{f(k(t))} k'(t) = f'(k(t)) \left\{ s - (\lambda + n) \frac{k(t)}{f(k(t))} \right\}. \tag{2.1}$$

Suppose that $f'(0) > (\lambda + n)/s$. Then owing to Condition 1.6, we know that the right hand side of (2.1) behaves as

$$f'(k) \left\{ s - (\lambda + n) \frac{k}{f(k)} \right\} \quad \begin{cases} > 0 & \text{and monotonously decreases in } k \text{ if } 0 < k < k^*, \\ = 0 & \text{if } k = k^*, \\ < 0 & \text{if } k > k^*. \end{cases}$$

From the above arguments, it follows that:

Proposition 2.1. (i) *If $k(0)$ is small (i.e. poor countries), growth rate of per ca-pita GDP, $y'(t)/y(t)$, should be bigger than the case of large $k(0)$ (i.e. rich countries).*

(ii) *If $k(t)$ is near to the golden age k^*, growth rate of per ca-pita GDP should be very small.* ◇

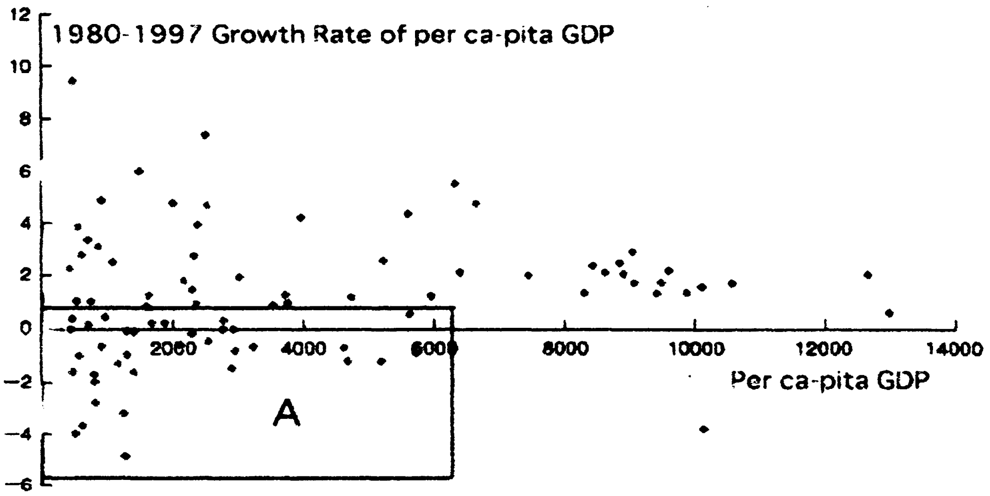

Fig. 2.1 Per ca-pita GDP and its growth rate (*Source-book: World Development Indicators*)

Fact 2.2. (i) *Economies of some countries are much conform to Solow model* (2.1), *what are Japan and Korean for instance.*

(ii) *USA must have reached the golden age, but he firmly maintains 2 % growth rate of per ca-pita GDP during this 100 years. This fact contradicts to the conclusion* (ii) *in Proposition 2.1.*

(iii) *There exists many such countries as in the area* A *of Fig. 2.1, what are counter examples against the conclusion* (i) *in Proposition 2.1. In particular, it is hard to account negative growth rates in the area* A. ◇

In order to overcome the difficulties stated in Fact 2.2 (iii), many economists make various attempts to approve Solow model.

I. R. Lucus (1988), D. Romer (1986), and etc imported **advances in technology** $A(t)$ into Solow model. For instant, Lucus considered Harrod type production function

$$Y(t) = F\big(K(t), A(t)\, L(t)\big), \tag{2.2}$$

where

$$A(t) \equiv A_0 \exp\{gt\}, \quad g \text{ is a non-negative constant} \tag{2.3}$$

is an advances in technology for each labor.

While some economists*3 consider Hicks type production function

$$Y(t) = A(t)\, F(K(t), L(t)), \tag{2.4}$$

in place of (2.2) with the same $A(t)$ as in (2.3).

II. H. Uzawa (1965), G. Mankiw(1992), and etc introduced **human factor** $H(t)$. For instant, Mankiw introduced a production function

$$Y(t) = F(K(t), L(t), H(t))$$

with a human factor $H(t)$, and he implied a simultaneous equations

$$\begin{aligned} k'(t) &= s_k\, y(t) - (n - \lambda_k)k(t), \\ h'(t) &= s_h\, y(t) - (n - \lambda_h)k(t), \end{aligned}$$

where s_k is a constant saving rate to capital stock. s_h is a constant saving rate to human capital stock, and λ_k, λ_h are constant depreciating rates.

III. After adjusting factor g in (2.3) and so on, they have obtained prosperous theories against to the conclusion (ii) in Proposition 2.1. However their theories are not sufficient to defeat negative growth rates stated in the conclusion (iii) in Proposition 2.1.

3 Solow equation under two random factors

Apart form Lucus and etc., some economists tried to randomize Solow equation. In particular, R. Merton*4(1975) has shifted the population growth equation (1.3) on to a SDE

$$dL(t,w) = n\, L(t,w)\, dt + \sigma_1\, L(t,w)\, dB_1(t,w), \tag{3.1}$$

where n and σ_1 are positive constants and $\{B_1(t,w)\}$ is a one dimensional Brownian motion. Applying Itô's formula to (1.13), Merton has obtained a SDE which accounts per ca-pita capital stock $\{k(t,w)\}$ as a diffusion process in $(0,\infty)$.

After Merton, Cho and Cooley (2001) replaced (2.3) by the diffusion process $A(t,w)$ which is a solution of the following SDE

$$dA(t,w) = g\, A(t,w)\, dt + \sigma_2\, A(t,w)\, dB_2(t,w) \tag{3.2}$$

where g and σ_2 are positive constants and $\{B_2(t,w)\}$ is a one dimensional Brownian motion.

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*3 Solow himself, E. Denison, D. Jorgenson, and etc..

*4 He is famous as a founder of Mathematical Finace.

In this note, we randomize both of the population growth $L(t,w)$ and advances in technology $A(t,w)$ by SDE's (3.1) and (3.2) where $\{B_1(t,w)\}$ and $\{B_2(t,w)\}$ are independent one dimensional Brownian motions. Then we consider an economy with the Harrod type production function (2.2).

Following Merton, we apply Itô's formula to a modified capital stock (per ca-pita capital stock with advances in technology)

$$x(t,w) \equiv \frac{K(t,w)}{A(t,w)\,L(t,w)}. \tag{3.3}$$

Then we have

$$d\Big(\frac{K(t,w)}{A(t,w)\,L(t,w)}\Big) = \frac{K'(t,w)}{A(t,w)\,L(t,w)} - \frac{K(t,w)}{A(t,w)\,L(t,w)}\Big\{\frac{dA(t,w)}{A(t,w)} + \frac{dL(t,w)}{L(t,w)}\Big\}$$
$$+ \frac{K(t,w)}{A(t,w)\,L(t,w)}\Big\{\frac{2(\sigma_1\,A(t,w))^2}{2\,(A(t,w))^2}\,dt + \frac{2(\sigma_2\,L(t,w))^2}{2\,(L(t,w))^2}\,dt\Big\}.$$

Since F is CRS (Condition 1.3 (ii)), we see that

$$\begin{aligned} dx(t,w) &= \{s\,f(x(t,w)) - (\lambda + n + g - {\sigma_1}^2 - {\sigma_2}^2)\,x(t,w)\}\,dt \\ &\quad + x(t,w)\{\sigma_1\,dB_1(t,w) + \sigma_2\,dB_2(t,w)\} \end{aligned} \tag{3.4}$$

Here we introduce a new one dimensional Brownian motion

$$W(t,w) \equiv \frac{-1}{\sigma}(\sigma_1\,B_1(t,w) + \sigma_2\,B_2(t,w)) \quad \text{with } \sigma \equiv \sqrt{{\sigma_1}^2 + {\sigma_2}^2} \tag{3.5}$$

and obtain a randomized Solow equation:

(3.6) **the stochastic Solow equation**

$$dx(t,w) = \{s\,f(x(t,w)) - (\lambda + n + g - \sigma^2)\,x(t,w)\}dt + \sigma\,x(t,w)\,dW(t,w).$$

Remark 3.1. For later use, we also show an explicit representation of $A(t,w)\,L(t,w)$.

$$A(t,w)\,L(t,w) = A(0)\,L(0)\ \exp\{\sigma\,W(t) + (n + g - \frac{\sigma^2}{2})\,t\}, \quad t \geq 0. \ \diamond \tag{3.7}$$

4 Economy driven by the stochastic Solow equation

First we shall study precise behaviors of $\{x(t,w)\}$ and its growth rate.
Applying Appendix §A, I, we classify the boundaries 0 and ∞ for $\{x(t,w)\}$. The boundaries are the natural boundary but they are either finite or infinite according to the value of

$$\theta \equiv \lambda + n + g - \frac{\sigma^2}{2}. \tag{4.1}$$

A direct calculation derives the following result.

Lemma 4.1. (i) $\{x(t,w)\}$ in (3.6) is a diffusion process in the interval $(0,\infty)$ with such boundary points as

	$\theta < 0$	$0 \leq \theta \leq s\,f'(0)$	$s\,f'(0) < \theta$
0	infinite natural		finite natural
∞	finite natural	infinite natural	

(ii) Corresponding to $\{x(t,w)\}$, the speed measure density $m(x)$ is define as in Appendix A, I, and it behaves as follow[*5]:

	$\theta=0$	$0<\theta<s\,f'(0)$	$s\,f'(0)\le\theta$
$m(x)$	unbounded	bounded	unbounded

◇

Now we are in the position to classify the asymptotics of $\{x(t,w)\}$ according to Appendix A:

Theorem 4.2. *Let the production function f satisfy Condition 1.6, and define a constant θ by* (4.1). *Then asymptotics of $x(t,w)$ are as follows:*

θ	$\theta<0$	$\theta=0$	$0<\theta<s\,f'(0)$	$\theta=s\,f'(0)$	$s\,f'(0)<\theta$
$x(t,w)$	$\to\infty$ *a.s.*	null recurrent	positive recurrent	null recurrent	$\to 0$ *a.s.*

Here 'positive recurrent' *means that the process is a recurrent diffusion in* $(0,\infty)$ *with an invariant probability measure. While* 'null recurrent' *means that the process is recurrent but its invariant measure is not probability one.* ◇

Remark 4.3. If $\{x(t,w)\}$ is null recurrent, then it converges to a boundary in Césaro's sense, that is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T x(t,w)\,dt=\begin{cases}\infty & a.s. \quad \text{if } \theta=0\\ 0 & a.s. \quad \text{if } \theta=s\,f'(0).\end{cases} \tag{4.2}$$ ◇

Next we discuss about the growth rate of the modified capital stock $x(t,w)$. In the non-random case, the growth rate of per ca-pita capital stock $k(t)$ is defined as

$$\frac{k'(t)}{k(t)}=\Big(\log k(t)\Big)'. \tag{4.3}$$

But in our case, the modified capital stock $x(t,w)$ is a diffusion process and $x'(t,w)$ has no sense. So we should consider an average growth rate in time[*6]

$$\rho(T,w)\equiv\frac{\log x(T,w)-\log x(0)}{T}=\frac{1}{T}\int_0^T d(\log x(t,w)), \tag{4.4}$$

instead of the instant growth rate (4.3).

Applying Ito's formula to $\rho(t,w)$, we easily obtain

$$\rho(T,w)=\frac{1}{T}\int_0^T\frac{s\,f(x(t,w))}{x(t,w)}\,dt-\theta-\sigma\frac{W(T)}{T}. \tag{4.5}$$

From Theorem 4.2 and Appendix A, we can compute the first term in the right hand side of (4.5) (see Appendix B).

Proposition 4.4. *Let the production function f satisfy Condition 1.6, and define a constant θ by* (4.1). *Then asymptotics of $\rho(T,w)$ is as follows:*

θ	$\theta<0$	$0\le\theta\le s\,f'(0)$	$s\,f'(0)<\theta$
$\rho(T,w)$	$\to-\theta$ *a.s.*	$\to 0$ *a.s.*	$\to s\,f'(0)-\theta$ *a.s.*

◇

[*5] It suggests properties of an invariant measure to $\{x(t,w)\}$.
[*6] This converges to the Lyapunov index of $\{x(t,w)\}$ as $t\to\infty$.

Our $x(t,w)$ is the modified capital stock (3.3), it is not easy to see behaviors of the total capital stock $K(t,w)$ itself from Proposition 4.4. So we show its asymptotics. Since $K(t,w) = A(t,w)\,L(t,w)\,x(t,w)$, it holds that

$$\lim_{t\to\infty} \frac{\log K(t,w) - \log K(0)}{t} = \lim_{t\to\infty} \rho(t,w) + n + g - \frac{\sigma^2}{2} \quad a.s.. \tag{4.6}$$

Now the following theorem is obtained from Proposition 4.4 and (4.6).

Theorem 4.5. Let the production function f satisfy Condition 1.6, and a constant θ be defined by (4.1). On the $(\lambda,\ n+g-{\sigma_2}^2/2)$ plain, we define domains A through E as in Fig. 4.1.

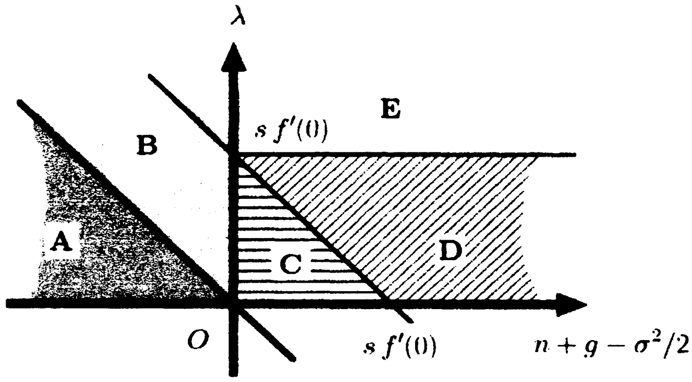


Fig. 4.1 The domains on $(\lambda,\ n+g-{\sigma_2}^2/2)$ plain

Then the following are asymptotics of the modified capital stock $\{x(t,w)\}$ and the total capital stock $\{K(t,w)\}$:

$(\lambda,\ n+g-{\sigma_2}^2/2)$	$\in$ A	$\in$ B	$\in$ C
$\{x(t,w)\}$	$\to \infty$ $a.s.$	recurrent	recurrent
Mean growth rate of $\{K(t,w)\}$	$\to -\lambda$ $a.s.$ negative growth	$\to n+g-\sigma^2/2$ $a.s.$ negative growth	$\to n+g-\sigma^2/2$ $a.s.$ positive growth
$(\lambda,\ n+g-{\sigma_2}^2/2)$	$\in$ D	$\in$ E	
$\{x(t,w)\}$	$\to 0$ $a.s.$	$\to 0$ $a.s.$	
Mean growth rate of $\{K(t,w)\}$	$\to s\,f'(0)-\lambda$ $a.s.$ positive growth	$\to s\,f'(0)-\lambda$ $a.s.$ negative growth	

◇

5 Economic growth and economic indexes

In this section we investigate precise relation between asymptotics of the per ca-pita GDP $y(t,w)$, the total GDP $Y(t,w)$, and θ in (4.1), assuming that F is the Cobb-Douglas production function (1.8).

$Y(t,w)$ and $y(t,w)$ are given by

$$Y(t,w) = F\big(K(t,w), A(t,w)\,L(t,w)\big) = A(t,w)\,L(t,w)\ f\big(x(t,w)\big),$$
$$y(t,w) = \frac{Y(t,w)}{L(t,w)} = A(t,w)\ f\big(x(t,w)\big).$$

So if the production function F is of Cobb-Douglas type (1.8), then it holds that

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{\log Y(t,w) - \log Y(0)}{t} = \alpha \lim_{t\to\infty} \rho(t,w) + n + g - \frac{\sigma_1{}^2 + \sigma_2{}^2}{2} \quad a.s., \\
&\lim_{t\to\infty} \frac{\log y(t,w) - \log y(0)}{t} = \alpha \lim_{t\to\infty} \rho(t,w) + g - \frac{\sigma_2{}^2}{2} \quad a.s.. \quad \diamond
\end{aligned} \tag{5.1}
$$

From Proposition 4.4 and (5.1), we have the following main theorem.

Theorem 5.1. *Suppose that the production function f is of Cobb-Douglas type. On the $(n - \sigma_1{}^2/2,\ g - \sigma_2{}^2/2)$ plain, we define domains* A *through* F *as in Fig. 5.1.*

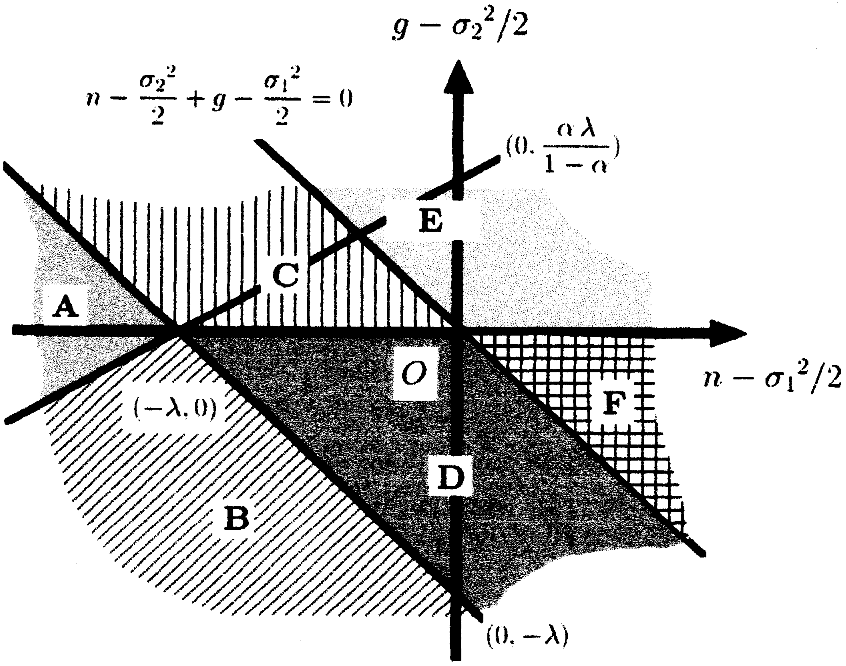


Fig. 5.1 The domains on the $(n - \sigma_1{}^2/2,\ g - \sigma_2{}^2/2)$ plain

Then the following are asymptotics of the per ca-pita GDP $\{y(t,w)\}$ and the total GDP $\{Y(t,w)\}$:

$(g - \sigma_1{}^2/2,\ n - \sigma_2{}^2/2)$	$\in$ A	$\in$ B	$\in$ C
Mean growth rate of $\{Y(t,w)\}$	$\to \gamma_1$ *a.s.* negative growth	$\to \gamma_1$ *a.s.* negative growth	$\to \gamma_2$ *a.s.* negative growth
Mean growth rate of $\{y(t,w)\}$	$\to \gamma_3$ *a.s.* positive growth	$\to \gamma_3$ *a.s.* negative growth	$\to \gamma_4$ *a.s.* positive growth

$(g - \sigma_1{}^2/2,\ n - \sigma_2{}^2/2)$	$\in$ D	$\in$ E	$\in$ F
Mean growth rate of $\{Y(t,w)\}$	$\to \gamma_2$ *a.s.* negative growth	$\to \gamma_2$ *a.s.* positive growth	$\to \gamma_2$ *a.s.* positive growth
Mean growth rate of $\{y(t,w)\}$	$\to \gamma_4$ *a.s.* negative growth	$\to\to \gamma_4$ *a.s.* positive growth	$\to \gamma_4$ *a.s.* negative growth

where

$$\gamma_1 \equiv (1-\alpha)\,(g+n-\frac{\sigma_1{}^2+\sigma_2{}^2}{2})-\alpha\,\lambda,$$
$$\gamma_2 \equiv n+g-\frac{\sigma_1{}^2+\sigma_2{}^2}{2},$$
$$\gamma_3 \equiv (1-\alpha)\,(g-\frac{\sigma_2{}^2}{2})-\alpha\,(n-\frac{\sigma_1{}^2}{2}+\lambda),$$
$$\gamma_4 \equiv g-\frac{\sigma_2{}^2}{2}. \quad \diamond$$

Remark 5.2. (i) Under the stochastic Solow equation (3.6), there is no such state of 'golden age' as in Proposition 1.9.

(ii) If the economic indexes $(g-\sigma_1{}^2/2,\ n-\sigma_2{}^2/2)$ are there on one of the domains B, D, and F of Fig. 5.1, then growth rates of those countries should be negative with probability one.

(iii) There exists only one domain E on the $(g-\sigma_1{}^2/2,\ n-\sigma_2{}^2/2)$ plain of Fig. 5.1 such that both rate of the total and the per ca-pita GDP grow up positively. $\diamond$

Appendix A One dimensional diffusion process with boundaries

I. We shall review behaviors of the diffusion process $\{x(t,w)\}$ defined by SDE (3.6).

Fix an arbitrary point $k_0 \in (0,\infty)$, and define

$$\textbf{(the scale function)}\quad S(x) \equiv \int_{x_0}^{x} \varphi(y)\,dy,$$
$$\textbf{(the speed measure density)}\quad m(x) \equiv \frac{1}{\sigma^2\,x^2\cdot\varphi(x)},$$

where

$$\varphi(y) \equiv \exp\{-2\int_{x_0}^{y} \frac{sf(\xi)-(\lambda+n+g-\sigma^2)\,\xi}{\sigma^2\xi^2}\,d\xi\},\quad y>0.$$

Using the scale function S and the speed measure $m(k)\,dk$, Feller (1954) and Itô-McKean (1965) classified boundaries of a one dimensional diffusion into five types, that is

a regular boundary, an entrance, an exit, an infinite natural, and a finite natural.

II. For $\{x(t,w)\}$ given by SDE (3.6), its boundary points are 0 and ∞, and both are natural boundaries. In this case, asymptotic behaviors is already known, Nishioka (1976).

Case 1. *Both are infinite natural:*
$\{x(t,w)\}$ is *recurrent* on the interval $(0,\infty)$, and density function of an invariant measure is

(Appendix A.1)
$$\mu(x) \equiv \frac{m(x)}{C} = \frac{1}{C}\cdot\frac{1}{\sigma^2\,x^2\,\varphi(x)}.$$

Here the constant C is

(Appendix A.2)
$$C \equiv \begin{cases} \int_0^\infty m(x)\,dx & \text{if the integral is finite,} \\ 1 & \text{otherwise.} \end{cases}$$

In addition, the following Ergodic Theorem holds:

Maruyama-Tanaka (1957): *If functions g, h are integrable with respect to $\mu(y)\,dy$, then*

(Appendix A.3)
$$\lim_{T\to\infty} \frac{\displaystyle\int_0^T g(x(t,w))\,dt}{\displaystyle\int_0^T h(x(t,w))\,dt} = \frac{\displaystyle\int_0^\infty g(y)\,\mu(y)\,dy}{\displaystyle\int_0^\infty h(y)\,\mu(y)\,dy} \quad a.s.,$$

where the denominator in the right hand side must not vanish.

Case 2. *One is finite natural and the other is infinite natural:*
(i) $\{x(t,w)\}$ cannot reach boundaries within a finite time, almost surely.
(ii)
$$\lim_{t\to\infty} x(t,w) = \begin{cases} 0 \quad a.s. & \text{if 0 is finite natural} \\ & \quad \text{and } \infty \text{ is infinite natural,} \\ \infty \quad a.s. & \text{if 0 is infinite natural} \\ & \quad \text{and } \infty \text{ is finite natural.} \end{cases}$$

Case 3. *Both are finite natural:*
The statement (i) in Case 2 is true, but

$$\mathbf{P}_x\,[\lim_{t\to\infty} x(t,w) = 0] = \frac{S(\infty) - S(x)}{S(\infty) - S(0)},$$
$$\mathbf{P}_x\,[\lim_{t\to\infty} x(t,w) = \infty] = \frac{S(x) - S(0)}{S(\infty) - S(0)}.$$

Appendix B Sketch to the proof of Proposition 4.4

First remember SDE (4.5) that is

(Appendix B.1)
$$\rho(T,w) = \frac{s}{T}\int_0^T \frac{f(x(t,w))}{x(t,w)}\,dt - (\lambda + n + g - \frac{\sigma^2}{2}) - \frac{\sigma}{T}W(T,w).$$

Here note that

(Appendix B.2)
$$\lim_{T\to\infty} \frac{W(T)}{T} = 0 \quad a.s.$$

from the iterated law of large number for a Brownian motion.
Step 1. Let $\theta < 0$. By Theorem 4.2, $x(t,w) \to \infty$ $a.s.$ if $\theta < 0$. Since f satisfies Condition 1.6,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{s\,f(x(t,w))}{x(t,w)}\,dt = \lim_{x\to\infty} \frac{s\,f(x)}{x} = 0 \quad a.s..$$

This and (Appendix B.2) derive that $\rho(T,w) \to -\theta$ $a.s.$ if $\theta < 0$.
Step 2. Let $0 < \theta < s\,f'(0)$. In this case, there exists an invariant probability measure whose density is $\mu(x)$ of (Appendix A.1).

We shall calculate the first term on the right hand side of (Appendix B.1). Put $\beta \equiv 2(\lambda+n+g)/\sigma^2-2$.

$$\text{The first term} = s\int_0^\infty dx\,\frac{f(x)}{x}\,\frac{C}{\sigma^2\,x^{2+\beta}}\,\exp\left\{\frac{2s}{\sigma^2}\int_{x_0}^{x}\frac{f(\xi)}{\xi^2}\,d\xi\right\}$$

$$= \frac{s\,C\,\sigma^2}{2s}\int_0^\infty dx\,\frac{1}{\sigma^2\,x^{1+\beta}}\left(\exp\left\{\frac{2s}{\sigma^2}\int_{x_0}^{x}\frac{f(\xi)}{\xi^2}\,d\xi\right\}\right)'$$

$$= \frac{C}{2}\cdot\frac{1}{x^{1+\beta}}\exp\left\{\frac{2s}{\sigma^2}\int_{x_0}^{x}\frac{f(\xi)}{\xi^2}\,d\xi\right\}\Bigg|_{x=0}^{\infty}$$

$$+\frac{\sigma^2\,C}{2}(1+\beta)\int_0^\infty dk\,\frac{1}{\sigma^2\,k^{2+\beta}}\exp\left\{\frac{2s}{\sigma^2}\int_{k_0}^{k}\frac{f(\xi)}{\xi^2}\,d\xi\right\}$$

$$= \frac{\sigma^2\,C}{2}\,(1+\beta)\,\frac{1}{C} = \lambda+n+g-\frac{\sigma^2}{2} = \theta,$$

where we used that

$$\frac{s\,f(\xi)}{\xi} \simeq s\,f'(0) > \theta \quad \text{if } \xi \text{ is sufficiently small.}$$

Now we have proved that $\lim_{T\to\infty}\rho(T) = 0$.

Step 3. We shall investigate asymptotics of $\rho(t,w)$ when $\theta = 0$.
In this case, an invariant measure $\mu(x)\,dx$ is not finite, that is

$$\int_L^\infty \mu(x)\,dx \sim \int_L^\infty \frac{1}{\sigma^2\,x}\,dx = \infty \quad \text{for large } L.$$

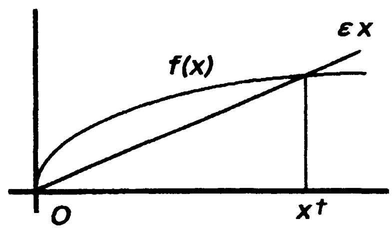
Moreover the function $f(x)/x$ may not be integrable.

Fix a sufficiently small $\varepsilon > 0$, and define a function h as

$$h(x) \equiv \varepsilon x, \quad x \geq 0.$$

Since $s\,f'(0) > 0$, we can find a unique point $x^\dagger > 0$ such that

$$f(x) = h(x), \quad x > 0.$$



We define new functions $\tilde f$ and $\tilde h$ by

$$\tilde f(x) \equiv \begin{cases} f(x) & 0 \leq x < x^\dagger \\ h(x) & x^\dagger \leq x, \end{cases}$$

$$\tilde h(x) \equiv \tilde f(x) - h(x) = \begin{cases} f(x) - \varepsilon x & 0 \leq x < x^\dagger \\ 0 & x^\dagger \leq x. \end{cases}$$

Here $\tilde h(x)/x$ is integrable with respect to $\mu(x)\,dx$, since $\mu(x) \sim x^N$ for small x with any $N > 0$.

We easily see that

$$T \geq \int_0^T I_{(0,L)}(x(t,w))\,dt, \qquad \int_0^\infty I_{(0,L)}(x)\,\mu(x)\,dx < \infty,$$

for arbitrary $L > 0$. By Ergodic Theorem (Appendix A.3), the next inequality holds with probability one:

$$\begin{aligned} 0 &\leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{\tilde{h}(x(t,w))}{x(t,w)}\,dt \\ &\leq \lim_{T\to\infty} \frac{\displaystyle\int_0^T \frac{\tilde{h}(x(t,w))}{x(t,w)}\,dt}{\displaystyle\int_0^T I_{(0,L)}(x(t,w))\,dt} = \frac{\displaystyle\int_0^\infty \frac{\tilde{h}(x)}{x}\,\mu(x)\,dx}{\displaystyle\int_0^\infty I_{(0,L)}(x)\,\mu(x)\,dx}. \end{aligned}$$

Note that $\int_0^\infty \mu(x)\,dx = \infty$, and let $L \to \infty$. Then we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\tilde{h}(x(t,w))}{x(t,w)}\,dt = 0 \quad a.s..$$

With respect to h, remark that

$$\frac{1}{T}\int_0^T \frac{h(x(t,w))}{x(t,w)}dt = \frac{1}{T}\int_0^T \varepsilon\ dt = \varepsilon.$$

From this and the previous calculation,

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\tilde{f}(x(t,w))}{x(t,w)}\,dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\tilde{h}(x(t,w))}{x(t,w)}\,dt + \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{h(x(t,w))}{x(t,w)}\,dt = \varepsilon. \end{aligned}$$

The definition of $\tilde{f}$ implies that $0 \leq f \leq \tilde{f}$, and we have

$$\begin{aligned} 0 &\leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{f(x(t,w))}{x(t,w)}\,dt \\ &\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\tilde{f}(x(t,w))}{x(t,w)}\,dt = \varepsilon \quad a.s.. \end{aligned}$$

Here $\varepsilon > 0$ is arbitrary. Let $\varepsilon \downarrow 0$ and we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{f(x(t))}{x(t)}\,dt = 0 \quad a.s..$$

Note that our assumption is $\theta = 0$. Now we have

$$\lim_{T\to\infty} \rho(T) = 0 - \theta = 0 \quad a.s.,$$

by an analogous way as in *Step 1*.

We shall omit the remained proof. □

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