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Testing finite activity against infinite activity for jumps, for high frequency observation: an overview

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Our aim is to describe, in a somewhat informal way, a number of preliminary results: In this presentation we use quite strong assumptions, hoping to significantly weaken them in the near future, and we provide no proof, neither empirical or simulation studies, those being not complete by now.

The problem concerns processes having jumps, since it seems more and more clear that models in finance should take jumps into consideration. Traditionally, such models with jumps rely on Poisson or compound processes, as in [11], [3] and [4]. However, more recently some financial models have been proposed, that allow for an infinite number of jumps in finite time intervals, such as the variance gamma model of [10] and [9], the hyperbolic model of [7], the pure jump model of [5] and the finite moment log stable process of [6]. These models can capture both small and frequent jumps, as well as large and infrequent ones. Since the qualitative properties and the mathematical analysis of models with finitely many jumps deeply differ from those of models with infinitely many jumps, it seems appropriate to develop some methods which can discriminate between the two types of models.

In an attempt to bring forth some contribution to these questions, we aim here to develop testing procedures to discriminate empirically between the two situations of finite and infinite number of jumps. This is an important information, which leads to qualitatively very different models. We have a 1-dimensional X observed on a fixed time interval $[0,T]$, at discretely and regularly spaced times $i\Delta_n$. Assuming the observed path has jumps, we want to test whether there are a finite number of jumps or not, two properties commonly referred to as "finite activity" and "infinite activity" for the jump component of X. These properties are in fact satisfied, respectively, on two complementary subsets of the sample space Ω:

$$\begin{aligned}\Omega_T^f &= \{\omega : t \mapsto X_t(\omega) \text{ has finitely many jumps in } [0,T]\}\\ \Omega_T^i &= \{\omega : t \mapsto X_t(\omega) \text{ has infinitely many jumps in } [0,T].\}\end{aligned} \tag{1}$$

We need some assumptions on the process X. However, we wish to keep the solution as

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nonparametric as possible, and in particular we do not want to specify the structure of the volatility, and as little as possible about the structure of the jumps.

1) Description of the Model: The underlying process X is a 1-dimensional Itô semimartingale on some filtered space, which means that its characteristics (B, C, ν) are absolutely continuous with respect to Lebesgue measure. B is the drift, C is the quadratic variation of the continuous martingale part, and ν is the compensator of the jump measure μ of X. In other words, we have

$$B_t(\omega) = \int_0^t b_s(\omega)ds, \quad C_t(\omega) = \int_0^t \sigma_s(\omega)^2 ds, \quad \nu(\omega, dt, dx) = dt\, F_t(\omega, dx), \tag{2}$$

Here b and σ are optional process, and $F = F_t(\omega, dx)$ is a transition measure from $\Omega \times \mathbb{R}_+$ endowed with the predictable σ-field into $\mathbb{R}\backslash\{0\}$.

Equivalent to (2), one may write X as

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int x 1_{\{|x|\leq 1\}}(\mu - \nu)(ds, dx) + \int_0^t \int x 1_{\{|x|>1\}} \mu(ds, dx).$$

where W is a standard Wiener process. Writing X in this way is more customary, but our assumptions mainly rely upon the ingredients coming in (2), and the process σ_t which is the square-root of c_t.

Of course, having (2) is not enough, and we need assumptions on the coefficients. The assumptions on the "continuous part", that is on b and σ, are as follows (for some of the results they could substantially be weakened, but in these notes it seems simpler to state a single set of assumptions):

Assumption 1. The volatility process σ_t is an Itô semimartingale, that is it can be written (necessarily in a unique way) as

$$\sigma_t = \sigma_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + N_t + \sum_{s\leq t} \Delta\sigma_s 1_{\{|\Delta\sigma_s|>1\}}, \tag{3}$$

where N is a local martingale which is orthogonal to the Brownian motion W, and further the compensator of the process $[N, N]_t + \sum_{s\leq t} 1_{\{|\Delta\sigma_s|>1\}}$ is of the form $\int_0^t n_s ds$. We also assume that the processes $\widetilde{b}_t$ and n_t are locally bounded (this means that there is a sequence (T_n) of stopping times, increasing to infinity, and such that those processes, stopped at any time T_n, are bounded), and the processes b_t and $\widetilde{\sigma}_t$ are càdlàg. Furthermore the process σ is "non-degenerate" in the sense where $\int_0^T \sigma_s^2 ds > 0$ a.s. (T is the horizon of our observations).

The non-degeneracy above means that (almost) all possible paths have a non-vanishing contribution from the continuous martingale (or Wiener) part of X. In practice, all models with finite activity for jumps satisfy this property, which is thus not a genuine restriction in the tests we are constructing here.

The assumptions on the jumps are quite restrictive, and in fact the same as in the paper [2] which is concerned with the estimation of the so-called Blumenthal-Getoor index. However, it is likely that these assumptions can be significantly weakened, at least for the test for which the null hypothesis is Ω_T^f:

Assumption 2. There are two constants $0 \leq \beta' < \beta < 2$ such that the Lévy measure $F_t = F_t(\omega, dx)$ is of the form

$$F_t(dx) \;=\; \frac{1}{|x|^{1+\beta}} \left(a_t^{(+)} 1_{\{0<x\leq z_t^{(+)}\}} + a_t^{(-)} 1_{\{-z_t^{(-)}\leq x<0\}} \right) dx + F_t'(dx), \tag{4}$$

where, for some locally bounded process $L_t \geq 1$,

(i) $a_t^{(+)}$, $a_t^{(-)}$, $z_t^{(+)}$ and $z_t^{(-)}$ are nonnegative predictable processes satisfying

$$\frac{1}{L_t} \leq z_t^{(+)} \leq 1, \qquad \frac{1}{L_t} \leq z_t^{(-)} \leq 1, \qquad A_t := a_t^{(+)} + a_t^{(-)} \leq L_t, \tag{5}$$

(ii) $F_t' = F_t'(\omega, dx)$ is a signed measure, whose absolute value $|F_t'|$ satisfies

$$\int (|x|^{\beta'} \wedge 1)\, |F_t'|(dx) \leq L_t. \tag{6}$$

This assumption is satisfied if the discontinuous part of X is a stable process of index $\beta \in (0,2)$: take $z_t^+ = z_t^- = 1$, and a_t^+ and a_t^- are constants, and the residual measure F_t' is the restriction of the Lévy measure to the complement of $[-1,1]$, and (6) is satisfied with any $a \in (0,1)$. When the discontinuous part of X is a tempered stable process the assumption is also satisfied with the same processes as above, but now the residual measure F_t' is not positive in general (although it again satisfies (6) with any $\beta' \in (0,\beta)$.

This assumption also accounts for a stable or tempered stable with time varying intensity (when $z_t^+ = z_t^- = 1$ but a_t^+ and a_t^- are genuine processes). It also accounts for any process of the form

$$Y_t \;=\; Y_0 + \int_0^t w_s\, dX_s,$$

as soon as X satisfies the same assumption and w_t is locally bounded and predictable.

It turns out that, under Assumption 2, the set Ω_T^i of (1) contains the set

$$\Omega_T^{i\beta} \;=\; \{\overline{A}_T > 0\}, \qquad \text{where } \overline{A}_t \;=\; \int_0^t A_s\, ds. \tag{7}$$

This is usually not equal to Ω_T^i, though, so our test will in fact test the null hypothesis Ω_T^f against $\Omega_T^{i\beta}$, or vice-versa, instead of (1).

2 - Testing hypotheses which are subsets of the sample space: Here, we specify the notion of testing when the null and alternative hypotheses are families of possible outcomes. Suppose that we want to test the null hypothesis "we are in a subset Ω_0" of Ω, against the alternative "we are in a subset Ω_1", with of course $\Omega_0 \cap \Omega_1 = \emptyset$. We then construct a critical (rejection) region C_n at stage n, that is when the time lag between observations is Δ_n. This critical region is itself a subset of Ω, which should depend only on the observed values of the process X at stage n. We are not really within the framework of standard statistics, since the two hypotheses are themselves random.

We then take the following as our definition of the asymptotic size, for a given triple of coefficients:

$$a \;=\; \sup\Big(\limsup_n \; \mathbb{P}(C_n \mid A): \; A \in \mathcal{F}, \; A \subset \Omega_0\Big). \tag{8}$$

Here $P(C_n \mid A)$ is the usual conditional probability knowing A, with the convention that it vanishes if $P(A) = 0$. If $P(\Omega_0) = 0$ then $a = 0$, which is a natural convention since in this case we want to reject the assumption whatever the outcome ω is. Note that a features some kind of "uniformity" over all subsets $A \subset \Omega_0$.

As for the asymptotic power, we define it as

$$P \;=\; \inf\Big(\liminf_n \; \mathbb{P}(C_n \mid A): \; A \in \mathcal{F}, \; A \subset \Omega_1, \; \mathbb{P}(A) > 0\Big). \tag{9}$$

Again, this is a number.

Before stating the results, we introduce some notation. First, the observed increments of X are $\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$. We take a sequence u_n of positive numbers, which serve as thresholds and always go to 0. There will be restrictions on this sequence, expressed by the following:

$$u_n/\Delta_n^\rho \;\to\; \infty \tag{10}$$

for some $\rho > 0$: this condition becomes weaker when ρ increases. Two specific values for ρ, in connection with the power $p \geq 2$ which will be used below, are of interest for us:

$$\rho_1(p) \;=\; \frac{p-2}{2p}, \qquad \rho_2(p) \;=\; \frac{2p-4}{11p-10}. \tag{11}$$

These quantities increase when p increases, and $\rho_1(p) > \rho_2(p) > 0$ when $p > 2$. Finally, with any $p > 0$ we associate the increasing processes

$$B(p, u_n, \Delta_n)_t \;=\; \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p \, 1_{\{|\Delta_i^n X| \leq u_n\}} \tag{12}$$

consisting of the sum of the p^{th} power of the increments of X, truncated at level u_n.

3 - The Finite Activity Null Hypothesis: We first set the null hypothesis to be finite activity, that is $\Omega_0 = \Omega_T^f$, whereas the alternative is $\Omega_1 = \Omega_T^i$. We choose an integer $k \geq 2$

and a real $p > 2$. We consider the test statistics, which depends on k and p, and also on the sequence u_n going to 0, and on the terminal time T, as follows:

$$S_n \;=\; \frac{B(p, u_n, k\Delta_n)_T}{B(p, u_n, \Delta_n)_T}.$$

Theorem 1. *a) Under Assumption 1 and if the sequence u_n satisfies (10) with some $\rho < 1/2$, we have*

$$S_n \stackrel{\mathbb{P}}{\longrightarrow} k^{p/2-1} \qquad \text{on the set } \Omega_T^f. \tag{13}$$

b) Under Assumptions 1 and 2, and if the sequence u_n satisfies (10) with $\rho = \rho_1(p)$, we have

$$S_n \stackrel{\mathbb{P}}{\longrightarrow} 1 \qquad \text{on the set } \Omega_T^i \quad (\text{resp. } \Omega_T^{i\beta}). \tag{14}$$

To construct a reasonable test with a given level for finite samples, we need a central limit theorem associated with the convergence in (13), and a standardized version goes as follows ($\stackrel{\mathcal{L}-(s)}{\longrightarrow}$ denotes the stable convergence in law, see for example [8] for this notion):

Theorem 2. *Under Assumption 1, and if the sequence u_n satisfies (10) with some $\rho < 1/2$, we have*

$$(S_n - k^{p/2-1})/\sqrt{V_n} \stackrel{\mathcal{L}-(s)}{\longrightarrow} \mathcal{N}(0,1) \qquad \text{in restriction to } \Omega_T^f,$$

where

$$V_n \;=\; N(p,k)\,\frac{B(2p, u_n, \Delta_n)_T}{(B(p, u_n, \Delta_n)_T)^2},$$

and

$$N(p,k) = \frac{1}{m_{2p}}\,\Big(k^{p-2}(1+k)m_{2p} + k^{p-2}(k-1)m_p^2 - 2k^{p/2-1}m_{k,p}\Big),$$

and $m_{p,k} = \mathbb{E}(|U|^p|U + \sqrt{k-1}\,V|^p)$ and $m_p = \mathbb{E}(|U|^p)$ for U and V two independent $N(0,1)$ variables (with the notation of [1] we have $N(p,k) = M(p,k)m_p^2/m_{2p}$).

We are now ready to exhibit a critical region. Denoting by z_a the a-quantile of $N(0,1)$, that is $\mathbb{P}(U > z_a) = a$ where U is $N(0,1)$, we set

$$C_n \;=\; \{S_n < k^{p/2-1} - z_a\sqrt{V_n}\}. \tag{15}$$

Theorem 3. *Under Assumptions 1 and 2, and if the sequence u_n satisfies (10) with some $\rho < 1/2$, the asymptotic level of the critical region defined by (15) for testing the null Ω_T^f against the alternative $\Omega_T^{i\beta}$ equals a, and the asymptotic power is 1.*

4 - The Infinite Activity Null Hypothesis: In the second case we set the null hypothesis to be infinite activity, that is $\Omega_0 = \Omega_T^{i\beta}$, whereas the alternative is $\Omega_1 = \Omega_T^f$. We choose three reals $\gamma > 1$ and $p' > p > 2$. We define a family of test statistics as follows:

$$S'_n \;=\; \frac{B(p', \gamma u_n, \Delta_n)_T\, B(p, u_n, \Delta_n)_T}{B(p', u_n, \Delta_n)_T\, B(p, \gamma u_n, \Delta_n)_T}.$$

Theorem 4. *Assume Assumptions 1 and 2.*

a) If the sequence u_n satisfies (10) with $\rho = \rho_1(p)$, we have

$$S'_n \xrightarrow{\mathbb{P}} \gamma^{p'-p} \qquad \text{on the set } \Omega_T^{i\beta}. \tag{16}$$

b) If the sequence u_n satisfies (10) with some $\rho \leq 1/2$, we have

$$S'_n \xrightarrow{\mathbb{P}} 1 \qquad \text{on the set } \Omega_T^{f}. \tag{17}$$

The associated central limit theorem is:

Theorem 5. *Under Assumptions 1 and 2 with $\beta' < \beta/2$ and if the sequence u_n satisfies (10) with $\rho = \rho_2(p)$, we have*

$$(S'_n - \gamma^{p'-p})/\sqrt{V'_n} \xrightarrow{\mathcal{L}-(s)} \mathcal{N}(0,1) \qquad \text{in restriction to } \Omega_T^{i\beta}$$

where

$$\begin{aligned} V'_n = \gamma^{2p'-2p}\Big(& \frac{B(2p,u_n,\Delta_n)_T}{(B(p,u_n,\Delta_n)_T)^2} + (1-2\gamma^{-p})\frac{B(2p,\gamma u_n,\Delta_n)_T}{(B(p,\gamma u_n,\Delta_n)_T)^2} \\ & + \frac{B(2p',u_n,\Delta_n)_T}{(B(p',u_n,\Delta_n)_T)^2} + (1-2\gamma^{-p'})\frac{B(2p',\gamma u_n,\Delta_n)_T}{(B(p',\gamma u_n,\Delta_n)_T)^2} \\ & - 2\frac{B(p+p',u_n,\Delta_n)_T}{B(p,u_n,\Delta_n)_T\, B(p',u_n,\Delta_n)_T} \\ & - 2(1-\gamma^{-p}-\gamma^{-p'})\frac{B(p+p',\gamma u_n,\Delta_n)_T}{B(p,\gamma u_n,\Delta_n)_T\, B(p',\gamma u_n,\Delta_n)_T}\Big). \end{aligned}$$

The critical region will therefore be

$$C'_n \;=\; \{S'_n < \gamma^{p'-p} - z_a\sqrt{V'_n}\}. \tag{18}$$

Theorem 6. *Under Assumptions 1 and 2 with $\beta' < \beta/2$, and if the sequence u_n satisfies (10) with $\rho = \rho_2(p)$, the asymptotic level of the critical region defined by (18) for testing the null $\Omega_T^{i\beta}$ against the alternative Ω_T^{f} equals a, and the asymptotic power is* 1.

Under the null hypothesis the rate of convergence is $1/u_n^{\beta/2}$ (contrary to the situation of Theorem 3, where the rate was $1/\sqrt{\Delta_n}$ whatever β and u_n were). So, although asymptotically u_n does not explicitly show in the test itself, one should probably take u_n as small as possible (we have no choice as to β, of course). That is, one should take ρ as large as possible, which in turn results in choosing p as big as possible (recall (11)).

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