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Styles of Greek arithmetic reasoning

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1. On the Concept of Style of Thinking

The concept of style of scientific thinking seemingly appeared in the middle 20th century, possibly by the founders of modern theoretical physics, Max Born (1882-1970), Wolfgang Pauli (1900-1958) and others, who understood by that term a complex of distinctive features of classical or non-classical fundamental physical theories. In particular, Max Born, in his 37th Guthrie Lecture, delivered 13th March 1953, notes

> I think that there are general attitudes of the mind which change very slowly and constitute definite philosophical periods with characteristic ideas in all branches of human activities, science included. Pauli, in a recent letter to me, has used the expression 'styles', styles of thinking, styles not only in art, but also in science. Adopting this term, I maintain that physical theory has its styles and that its principles derive from this fact a kind of stability. They are, so to speak, *relatively a priori* with respect to that period. If you are aware of the style of your own time you can make some cautious predictions. You can at least reject ideas which are foreign to the style of your time. [1]

Born further advances his thought, by distinguishing the Greek style, the style of the Christian era, the Galileo-Newton style and the new style "commenced in 1900, when Planck published his radiation formula and the idea of the quantum of energy."[2]

Felix Klein, in his *Vorlesungen*[3], noticed a difference between two styles of mathematical thinking: the intuitive and the formalist styles. A mathematician thinking in intuitive manner strives to penetrate into the essence of a problem, gain an insight of its solution and then to state and prove a theorem. Proof is secondary for him, in comparison to the intuitive insight. On the other hand, the main task of a mathematician thinking in formalist manner is to prove a theorem in minute detail and provide alternative proofs, in order to ascertain the mathematical fact.

In the mid 60s, the concept of style of scientific thinking or other similar terms, such as Thomas S. Kuhn's "paradigms,"[4] Imre Lakatos' "scientific research programs,"[5] etc. started to be used in history and methodology of science. However, the concept of style was established in history of science in Alistair Cameron Crombie's monumental work *Styles of Scientific Thinking in the European Tradition*[6]. He distinguished in the history of classical European science a taxonomy of six styles "distinguished by their objects of inquiry and their method of argument: (1) postulational; (2) the experimental style; (3)

[1] Max Born, "The Conceptual Situation in Physics and the Prospects of its Future Development," 1953 *Proceedings of the Physical Society. Section A*, **66**, 501. Print publication: Issue 6 (1 June 1953).
[2] *Ibid.* 502.
[3] Klein, Felix, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, Teil II, Verlag von Julius Springer, Berlin, 1927.
[4] Thomas S. Kuhn, *The Structure of Scientific Revolutions*, 1st. ed., Chicago: Univ. of Chicago Pr., 1962.
[5] Imre Lakatos, *The Methodology of Scientific Research Programmes: Philosophical Papers* Volume 1. Cambridge: Cambridge University Press, 1978.
[6]) Alistair Cameron Crombie, *Styles of Scientific Thinking in the European Tradition: The History of Argument and Explanation Especially in the Mathematical and Biomedical Sciences and Arts.* London: Gerald Duckworth & Company, 1995.

hypothetical modelling; (4) taxonomy; (5) probabilistic and statistical analysis; (6) historical derivation. Each style defined the questions to be put to its subject-matter, and those questions yielded answers within that style[7].

In modern history of mathematics, the concept of "style" of mathematical thinking is commonly used as a looser and more general notion of "method," including certain crank-handle aspects of mathematics, such as mathematical patterns, commitment to a rule or standard, mathematical activity, cognitive attitudes, modes of understanding, etc. A mathematical style does not generate either a discovery or a demonstration; it is not easily definable, for it may be a "French-school style" or a "Gottingen's style," or a "seventeenth-century style", a methodologically defined style (such as, for instance, "experimental" style) or a style defined by a problem.

Crombie ascribes to the Greeks the achievement of the postulational style, as exemplified in Euclid's *Elements*. In our paper, we present two historical types of styles of Greek mathematical thinking that can hardly be called postulational. Notably, the (Neo-) Pythagorean style of arithmetic thinking that can be characterised as a style of developing number theory by *genetic constructions (definitions)*[8], and the Euclidean style of arithmetic, which can be characterised as *genetic*[9], in the sense that number theory is developed from below by effective proofs.

2. On the style of Pythagorean arithmetic reasoning

The style of arithmetic in the Neo-Pythagoreans' treatises[10] is strikingly different from that of the Euclidean *Elements*. Namely, it is characterised by the absence of proof in the Euclidean sense and lack of mathematical sophistication that has led certain historians to consider this type of mathematics as a feature of decadence of mathematics in this period[11]. The alleged absence of originality in these works has also given grounds to believe that "the arithmetic presented in these works derives substantially from an ancient, primitive stage of Pythagorean arithmetic" and to use them as "an index of the character of arithmetic science in the 5th century."[12]

The style of Neo-Pythagorean arithmetic reasoning has the following characteristics:

(1) *Arithmetical reasoning is conducted over a 3-dimensional "domain" that extends indefinitely in the direction of increase.* Number is *conventionally* designated as a "suite" (or schematic pattern) that is a finite sequence of signs (alphas) standing unbounded in the direction of increase and bounded below by the *monas* in the direction of decrease. The resulting configuration possesses internal structure and proper order and serves as a pattern exemplifying the mode of construction of the kind of number considered in each case. Thus, number depends on the monad and, according to Theon "there are infinitely many monads."

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[7] *Ibid.* 1, xi.
[8] Vandoulakis, I.M "A Genetic Interpretation of Neo-Pythagorean Arithmetic," *Oriens - Occidens Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales* [to appear].
[9] Vandoulakis, I.M. "Was Euclid's Approach to Arithmetic Axiomatic?" *Oriens - Occidens Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales*, 2 (1998), 141-181.
[10] Neo-Pythagorean arithmetic survived in the texts of Nicomachus of Gerasa (2nd century AD), Theon of Smyrna (2nd century AD), Iamblichus, Domninos of Larissa (5th century AD), Asclepius of Tralles, and Ioannes Philoponus (5th-6th century AD).
[11] Tannery, Paul(us). *La Géométrie Grecque.* Histoire générale de la géométrie élémentaire. Paris, 1887, 11-12.
Heath, Thomas L. *A History of Greek Mathematics.* vol. 1-2. Oxford. Clarendon Press, 1921, 1, 97-99.
[12] Knorr, Wilbur Richard. *The Evolution of the Euclidean Elements.* Reidel Publ. Co. 1975, 132.

The linear structure of numbers is violated in the case of *plane,* and *solid numbers* that have two-, or three-dimensional geometrical configurations, respectively. Plane (solid, respectively) numbers are represented by a specific schematic pattern in the form of (plane or spatial) "region," consisting of signs of lower rank (linear or plane, respectively). In this way, the "domain" over which arithmetical reasoning is conducted is stratified, depending on the dimension of the respective configuration, i.e. its combinatorial complexity.

The stratification of the "domain" makes necessary the introduction of different number-generating operations at each level. The linear "domain" is generated by the use of the iterative process of adding a monad. The plane "domain" is generated by the use of the "gnomon."

(2) *The starting-point of Neo-Pythagorean arithmetic is a single object,* i.e. the monad, denoted by an alpha, and taken to be a d e s i g n a t e d object (not a number), that is

$$\mathcal{L} = \{\alpha\}.$$

Over this set, an iterative procedure of attaching an alpha is admitted. Numbers are then defined as suites of the form

$$k \rightleftharpoons \langle \alpha, \alpha, \alpha, \ldots \alpha \rangle$$

where $\rightleftharpoons$ means that k is an abbreviation for the suite

$$\alpha, \alpha, \alpha, \ldots \alpha$$

consisting of k signs.

Further, the "natural suite" is introduced as the finite sequence of the form

$$<1, 2, 3, \ldots, k>$$

and the various kinds of numbers, such as the even-times even, the even-times odd numbers etc., are then specified as suites constructed according to certain rules[13].

(3) *Arithmetic is developed by genetic constructions of various configurations.* The act of the establishment of the next step in the process of genetic construction is implemented by operations of combinatorial character, and its implementation entails a modification in the preceding state of the "region."

Accordingly, Neo-Pythagorean arithmetic is a *visual theory of counting* over a distinctive combinatorial "domain," insofar as it concerns genetic constructions of various finite schematic patterns, that is suites and (plane or spatial) configurations. All these configurations are the result of concrete completed genetic constructions, which possess the following features:

(i) They begin from the same initial object, i.e. the monad;[14]

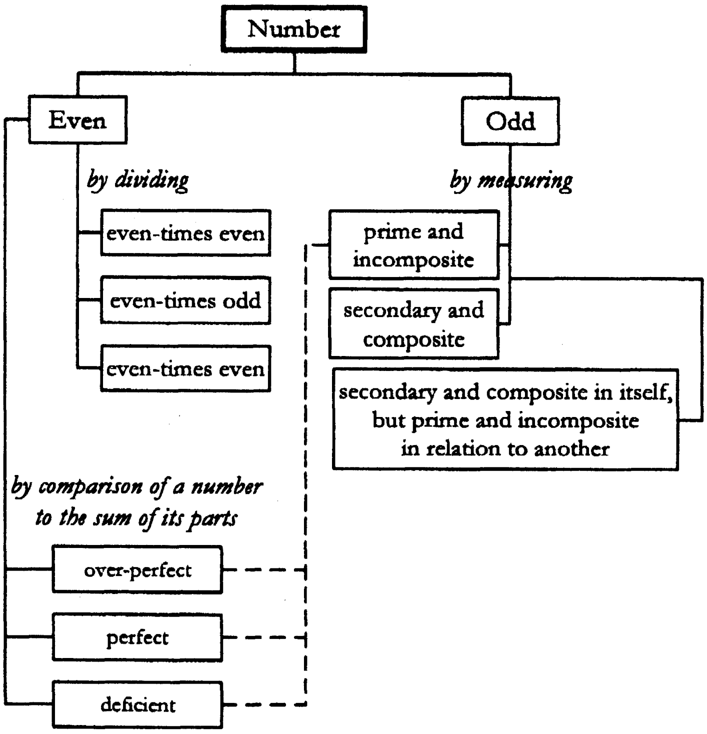
[13] The natural suite should not be confused with the natural series; the former is a finite constructional element and serves as pattern exemplifying the mode of construction of the kind of number considered in each case. For further details, see Vandoulakis, I.M "A Genetic Interpretation of Neo-Pythagorean Arithmetic," *Oriens - Occidens Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales* [to appear].

[14] There are constructions that begin not directly from the unit, but from a natural suite. However, in these cases the construction of that suite, beginning from the unit, is obvious or has been realised previously and is reasonably omitted.

(ii) Provided that the result of the application of certain iterative operations (addition by a unit, application of the gnomon or other derivative operations) generate numbers of the same kind, new numbers are constructed;

(iii) It is stated that the method of generation of a particular kind of number can generate all the numbers of the kind required.

The first two clauses enable one to construct new numbers out of given ones. The third clause (sometimes omitted by the Neo-Pythagorean authors) says that the method of construction described by the first two clauses "exhausts" all the numbers that have to be constructed.[15]



Scheme 1. Classification of numbers in Book I of Nicomachus' "Introduction to Arithmetic"

The objects thus introduced in Neo-Pythagorean arithmetic by genetic constructions are infinite sequences, usually incompletely exemplified by a natural suite or certain (finite) combinatorial configuration. Insofar as the objects discussed (suites and combinatorial configurations) are always finite instances and never considered in their completion, what is implicitly involved here is the *abstraction of potential infinity* that allows reasoning about however long genetic processes. The realizability of the

[15] Special attention should be paid on the formulation of the third clause. In modern genetic definitions, the analogous clause is usually phrased as follows:

(iii*) There are no other objects except those generated by the application of the first two clauses.

Thus the clauses (i), (ii), and (iii*) are taken to define a set of objects that are specified by means of the clauses (i) and (ii). The possibility for an object to posses the defining property is based on clause (iii*). This clause implicitly assumes the universe of all objects as given beforehand. In virtue of the law of excluded middle, every object of the universe either possesses the defining property or does not. Then clause (iii*) is taken to mean that there are no objects in the set determined by the defining property except those that have been constructed in the way described by (i) and (ii).

However, this does not seem to be the way of thinking of the ancient authors, for whom the totality of all numbers was not considered as given beforehand. It could not be for one more reason: the statement "a number such that ..." (in the sense of its effective construction) was for the Neo-Pythagorean arithmeticians an "experimental" fact, whereas the statement "there are no numbers such that ..." is not an "experimental" fact. Accordingly, clause (iii) is never phrased by the Neo-Pythagoreans in the form of negative existential sentence.

genetic constructions is taken to be potential. The natural suites, for instance, are taken to be extendible "as far as you wish," and the process of construction of successive combinatorial configurations can continue *ad infinitum*.

Moreover, the objects introduced in Neo-Pythagorean arithmetic are not only (potentially) infinite, but also intrinsically associated with the rule for their genetic construction from the monad. In this sense, we can say that the Neo-Pythagorean approach to arithmetic is not purely extensional. Definitions of arithmetical concepts are reduced to the demonstration how a definite combinatorial rule works, when one passes from a number to its successor, in the process of construction of the considered kind of number. Arithmetic theorems are thus ultimately reduced to the demonstration that the transition from a number to its successor follows a definite combinatorial rule.

(4) *The Finitary Principle.* It is obvious that the method outlined above is not grounded upon the idea of proof (in the strict sense of the word). In the context of Neo-Pythagorean arithmetic, numbers are conceived as given and any statement about them asserts something, which is confirmed in each instance by simple combinatorial means. If, for example, two numbers are given, it is sufficient to *confirm* by the construction of the corresponding configuration (or by observations over the exemplary suite), whether what has been stated about these numbers is correct or not. The demonstration is performed by inspection over a finite fragment of a usually (potential) infinite object. In this sense, we can say that the statements of Neo-Pythagorean arithmetic have *finitary* meaning.

(5) *Mental experiments.* Arithmetical reasoning is conducted in the form of mental experiments over concrete objects of combinatorial character. Any assertion about numbers utters a law, which can be confirmed in each case by pure combinatorial means. For any given concrete numbers, it sufficient to *verify* by the construction of the corresponding configuration whether the law uttered holds for the considered numbers. Thus, such type of arithmetical reasoning admits the representation by (configurations of) letters as intented interpretation for the effective confirmation (*deixis*) of the arithmetical statements and develops in accordance with its combinatorial model.

(6) *Genetic construction (demonstration) vs. proof.* Therefore, the foundation of Neo-Pythagorean arithmetic is not proof, in the style of Euclidean *Elements*, and the works of other mathematicians of the classical antiquity, but the idea of genetic construction (demonstration), by means of which the correctness of arithmetical statements is confirmed. This type of arithmetical reasoning about given numbers can be realized *without assumptions of axiomatic character*. The confirmation of arithmetical statements is realizable by a specific 'experiment.'

In this context, arithmetical reasoning is conducted as theoretical *visual* reasoning concerning the possibility to carry out certain genetic constructions over a domain of concrete objects. These objects are introduced by genetic constructions and represent infinite sequences that are illustrated incompletely by means of a finite suite or configuration.

(7) *Generality.* In the expositions of Neo-Pythagorean arithmetic, one does not find enunciation of universal theorems, that is formal statements beginning with quantificational words of the type "all," "every," etc. The problem primarily concerns the case when a general property is asserted of an infinite (in the sense described above) domain of objects. In these cases the general property is established by induction, that is by means of certain constructions of such a character that the possibility to repeat a similar reasoning so that to implement the corresponding construction for *any* other given number of the same kind is evident. On this ground one can conclude that whatever number of the kind might have been given, it is possible to confirm (by analogous line of reasoning) that this number has the property in question.

The use of such a rule of generality is necessary for the development of Neo-Pythagorean arithmetic, in absence of specific quantification. In this way, the statements of Neo-Pythagorean arithmetic can be understood as general declarations of our capability to implement certain constructions for *any* given number of certain kind. The belief that we can implement the required construction for any given number (combinatorial pattern) might have been rooted in the experience gained from the realization of such constructions. As a result of this kind of "experimentation" it becomes clear how one has to proceed in each case, i.e. at each case it is clear what we have to do. In this sense, the statements established in Neo-Pythagorean arithmetic are *general*, that is they establish that for any number of certain kind the outcome of the construction, according to a definite rule, is a number possessing the required property.

(8) *Negation.* It is obvious from the evidence of arithmetical reasoning discussed so far that Neo-Pythagorean arithmetic is about affirmative sentences stating something 'positive' that can be confirmed by means of the construction of the corresponding configuration. We do not meet in Neo-Pythagorean arithmetic any kind of 'negative' sentences, that is statements asserting existence of a number specified by a negative property (or lacking a property), or statements asserting impossibility of a construction. Such statements could not have a straightforward "experimental" character within Neo-Pythagorean arithmetic.

This preconception against negation is also evidenced by the Pythagorean philosopher Philolaus, who declares that

> "the nature of number and harmony does not permit falsity, because it is not peculiar to it. .... Truth is peculiar ... to the kind of number" (Stobeus; D/K/ Vors. 44 B11)

Thus, the lack of negative statements in the canonical exposition of Neo-Pythagorean arithmetic seems to a doctrinal characteristic of the Pythagorean mathematical tradition. This feature lends to this kind of arithmetic certain peculiar semantic characteristics, namely all sentences in Neo-Pythagorean arithmetic are affirmative statements asserting the possibility of certain constructions and, thereby, the Neo-Pythagorean arithmetic contains only such "experimental" truths that describe the actual state of affairs. From a modern point of view, this kind of arithmetic represents a *positive* (negationless), *finitary fragment* of Peano arithmetic.

3. On the style of Euclidean arithmetic reasoning

Euclid's style of arithmetical reasoning is also not postulational. The lack of specific arithmetical axioms in Book VII has puzzled historians of mathematics[16]. In our view, it is hardly possible to ascribe to the Greeks a conscious undertaking to axiomatize arithmetic. The view that associates the beginnings of the axiomatization of arithmetic with the works of Grassman[17], Dedekind[18] and Peano[19] seems to be more plausible[20]. In this case, the following questions can be stated: why arithmetic was

[16] Mueller, Ian. *Philosophy of Mathematics and Deductive Structure in Euclid's "Elements".* MIT, 1981, 59.
Artman, Benno. "Euclid's *Elements* and its Prehistory" Mueller, Ian. ΠΕΡΙ ΤΩΝ ΜΑΘΗΜΑΤΩΝ *(Peri ton Mathematon).* Edmonton. Alberta: Academic Printing & Publishing, 1992 [*Apeiron* **XXIV** no 4 (December 1991)], 32.
Jean-Louis Gardies, "Sur l'axiomatique de l'arithmétique euclidienne," *Oriens - Occidens Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales,* 2 (1998), 125-140.

[17] Grassman H. *Lehrbuch der Arithmetik für höhere Lehranstalten.* Berlin. 1861.

[18] Dedekind J.W.R. *Was sind und was sollen die Zahlen?* Brunswick. 1888.

[19] Peano Giuseppe. *Arithmeticas principia, novo methodo exposita.* Torino. 1889.

[20] Demidov, Sergej S. "Sur l'histoire de la méthode axiomatique." *Actes du XIIe Congress International d'Histoire des Sciences,* Paris. t. IV, 1971, 45-47.

axiomatized so late? How Euclidean arithmetic is constructed? The first question is conclusively answered by Yanovskaja[21]. Her major and quite conclusive argument is that "algorithms in arithmetic have absolute character, whereas in geometry we have to do with reducibility algorithms"[22]. The answer to the second question is, in our view, the following: Euclidean arithmetic is constructed not axiomatically, but by *effective proofs*[23].

(1) *The "domain" of Euclid'd "Elements," Book VI.* The Euclidean number – *arithmos* – has the following formal structure:

$$A \rightleftharpoons \{aE\}_{a\geq 2}$$

where E designates the unit and a is the number of times (multitude) that E is repeated to obtain the number A, denoted by a segment.

Euclid constructs his arithmetic for the numbers-*arithmoi*, that is for the numbers designated as segments, while the arithmetic of multitudes is taken for granted. Thus, arithmetic is constructed as formal theory of numbers-*arithmoi*, while the concept of multitude or iteration number has a specific metatheoretical character[24].

The concepts "equal," "less," "greater," to which today are ascribed a purely quantitative meaning, in Euclid seems to be also associated with the geometric notion of relative position, but also applied to multitudes when Euclid compares two sets of numbers-*arithmoi.*

(2) *Generality.* Euclid sometimes uses quantificational words applied to numbers-*arithmoi*, although such expressions are very rare. The most common way by which Euclid expresses generality is to speak about *arithmos* without article. Thus, most enunciations in Euclid's arithmetical Books state some property about numbers, where *arithmos* is used without article. However, when he proceeds to the *ekthesis* of the theorem he makes a number of important linguistic-logical operations:

a. designation of numbers-*arithmoi* is introduced by means of a segment named by one (or two) letter(s);

Demidov, Sergej S. "On Axiomatic and Genetic Construction of Mathematical Theories." Hintikka, Jaakko, Gruender, David, and Agazzi, Evandro (Eds). 1981. *Theory Change, Ancient Axiomatics, and Galileo's Methodology. Proceedings of the 1978 Pisa Conference on the History and Philosophy of Science.* Dordrecht: Reidel Publ. Co., Vol. 1, 215-222.
Medvedev, Fedor A. "On the Role of Axiomatic Method in the Development of Ancient Mathematics." Hintikka, Jaakko, Gruender, David, and Agazzi, Evandro (Eds). 1981. *Theory Change, Ancient Axiomatics, and Galileo's Methodology. Proceedings of the 1978 Pisa Conference on the History and Philosophy of Science.* Dordrecht: Reidel Publ. Co., Vol. 1, 223- 225.

21 Yanovskaya S. A. "From the history of axiomatics." *Report in the 3rd All-Union Congress of Mathematics, Section of History of Mathematics*, 26 June 1956, II, 105 [in Russian].
Yanovskaya S. A. "From the history of axiomatics." *Istoriko-matematičeskie issledovaniya*, 1958, 63-96 (Expanded version of the preceding report) [in Russian].

22 *Ibid.*, 64, see also Bychkov, Sergej S. "Geometry and the axiomatic method", *Istoriko-matematičeskie issledovaniya* **1 (36)** (1996) 2, 195-204 [in Russian].

23 Vandoulakis I.M 1998. "Was Euclid's Approach to Arithmetic Axiomatic?" *Oriens - Occidens* Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales, **2**, 141-181.

24 Bashmakova, Izabella G. "The Arithmetic Books of Euclid's *Elements*". *Istoriko-matematičeskie issledovaniya* **I** (1948), 321 [in Russian].
Vitrac, Bernard. *De Quelques Questions Touchant au Traitement de la Proportionnalité dans les Éléments d'Euclide.* Paris, 1993, I, 118.

b. the name of the number-*arithmos* (that is the letter) is now used with definite article standing before it.

In this way, general statements about numbers are interpreted as statements about an arbitrary given (indicated) number. Quantification is not formally expressed by means of variables and quantifiers ranging over them, but using the ordinary expressive means of natural language. In virtue of the instantiation described above the process of proof takes places actually with an arbitrary given number. This "rule of specification" is considered inversible, although Euclid applies the inverse rule very rarely in the arithmetical Books. The degree of generality thus attained is no higher than generality expressible by free variables ranging over numbers.

(3) *Fundamental concepts.* The basic undefined concept of Euclidean arithmetic is that of *to measure (katametrein)*, which underlies most of the kinds of numbers defined by Euclid. The concept "a number B measures a number A" can be interpreted as follows:

$$B \text{ measures } A \rightleftharpoons (B < A) \,\&\, (A = nB),$$

that is A is obtained by n repetitions of B.

After the introduction of several arithmetical concepts, such as "part," "multiple," "parts," and others, the concept of *proportional numbers* is introduced as four-place predicate over numbers[25]

$$\text{Proportional}(A,B,C,D) \rightleftharpoons \begin{cases} \{[(C < A) \,\&\, (A = nC) \,\&\, (D < B) \,\&\, (B = nD)] \vee \\ [(A < C) \,\&\, (C = mA) \,\&\, (B < D) \,\&\, (D = mB)] \vee \\ \{(\exists X)[(A = mX) \,\&\, (B = nX) \,\&\, (m > n > 1)] \,\&\, \\ (\exists Y)[(C = mY) \,\&\, (D = nY) \,\&\, (m > n > 1)]\}\} \end{cases}$$.[26]

(3) *Implicit assumptions concerning reasoning over infinite processes.* In the proofs of Proposition 1 and 2, exposing the process of *anthyphairesis*, Euclid uses implicitly the following implicit assumptions:

i. *The least number principle*: the set of multiples nB, such that $nB \geq A$, has least element n_0, such that $n_0 B \geq A$, yet $(n_0 - 1)B < A$.
ii. *The infinite descent principle*: the process of *anthyphairesis* will terminate in a finite number of steps, that is the chain $A > B > B_1 > B_2 > \ldots > B_k > \ldots$ is finite.
iii. If X measures A and B, then X measures $A \pm B$, that is if $A = mX$, $B = nX$, then $A \pm B = (m \pm n)X$.

The first is equivalent to the principle of mathematical induction if the following axiom is added: every number (except the unit) has a predecessor. The second assumption is equivalent to the principle of mathematical induction and is used in Proposition 31

(VII, 31) $(\forall A)\{\text{Composite}(A) \rightarrow (\exists B)[\text{Prime}(B) \,\&\, (B < A) \,\&\, (A = nB)]\}$[27]

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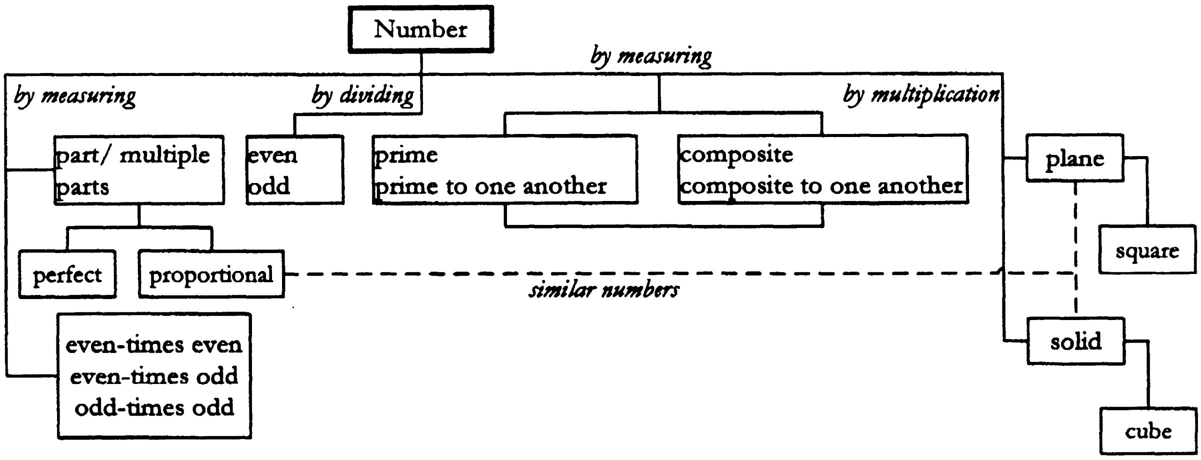
[25] Thomas L. Heath (Ed.). *The Thirteen Books of Euclid's Elements.* 1908, 1926, 1956. Translated from the text of Heiberg with introduction and commentary. Vols. 1-3. Cambridge; Dover, New York, 278 Def. 20.

[26] For the justification of this explication, see Vandoulakis, I.M. "Was Euclid's Approach to Arithmetic Axiomatic?" *Oriens - Occidens Cahiers du Centre d'histoire des Sciences et des philosophies arabes et Médiévales*, 2 (1998), 141-181.

[27] From this proposition follows directly Proposition 32, that any number either is prime or is measured by some prime number. From (31) and (32) follows, that any composite number is a product of primes in one way only. However, Euclid does not make the additional step neither to formulate, nor to prove this proposition. See, for instance,

However, the use of these principles has always *finitary* character in Euclid.



Scheme 2. Kinds of Numbers defined in Euclid's Book VII

(4) *Introduction of entities of higher complexity.* In Propositions 20-22, Euclid uses the "class" of all pairs that "have the same ratio." Each such class is uniquely associated with one pair of numbers, namely the least pair of numbers that have the same ratio. Euclid gives an effective procedure for finding such a least pair.

First, assuming the *existence* of such a pair, Euclid proves that the least pair should possess the following properties:

a) Let (A_0, B_0) be the least pair of numbers, which have the same ratio, and (A, B) any other pair of numbers, which have the same ratio. Then $A = mA_0$ and $B = mB_0$ (Proposition 20);
b) Prime to one another numbers are the least of those which have the same ratio with them (Proposition 21[28]) and, conversely, the least pairs of numbers which have the same ratio with them are prime to one another (Proposition 22).

Further, Euclid proves a number of theorems needed for the effective construction of the least pair of numbers having the same ratio, which is done in Proposition 33. The later proposition in combination with Proposition 21 gives the uniqueness of the least pair.

(5) *The finitary principle and the use of effective procedures.* Thus, Euclidean arithmetic is constructed from below, beginning from the unit. Further, a number of arithmetical concepts are introduced in the Definitions of Book VII. From these, the concepts of part, multiple, parts, proportionality, and prime numbers are not defined effectively. However, they become effective in virtue of Propositions 1, 2, and 3 that provide an effective procedure for any numbers to find their common measure. In this way, the proofs of the Propositions 4-19 should be considered as effective either. In particular, some of them are conducted by manipulations of equality-type relations between numbers-*arithmoi*, multitudes and part (parts).

Knorr, Wilbur Richard. "Problems in the interpretation of Greek number theory: Euclid and the Fundamental theorem of arithmetic" *Stud. Hist. Phil. Sci.* 7 (1976) 3, 353-368.
Hendy M.D. "Euclid and the Fundamental Theorem of Arithmetic" *Historia Mathematica* 2(1975), 189-191.
Mullin A.A. "Mathematico-philosophical remarks on new theorems analogous to the fundamental theorem of arithmetic" *Notre Dame Journal of Formal Logic* 6(1965), 218-222.

[28] We should note that in the proof of Proposition 21, Euclid uses the so called least number principle, that is he assumes the existence of a least pair of those which have the same ratio.

The introduction of more complex objects is realised through the comparison of these objects and the establishment of an equality-type relation between them. In particular, ratios are introduced through the comparison of ratios ("sameness"), in Propositions VII, 20-22. Yet, Euclid confines himself solely in the statement of the sameness of ratios, without trying to make the additional step to define ratios themselves, based on their sameness. In other words, Euclid nowhere reaches a *definition by abstraction* of the concept of ratio.

Instead, Euclid gives again in Proposition VII 33 an effective procedure for finding the least pair from the numbers that have the same ratio. This pair is unique and characterises the whole class of pairs that have the same ratio. Euclid also provides an effective construction of the least common multiple and the least number out of given parts.

Therefore all propositions that involve existence of numbers appear, in the context of Euclid's arithmetic, associated with some effective procedure for finding the required number. It is constructed without assumptions of axiomatic character. It lacks the concept of absolute number or any elaborated concept of equality. Instead, it is constructed as 'formal' theory of *arithmoi*. The concept of number-segment (*arithmos*) is some kind of 'formalisation' of the metalinguistic concept of 'multitude' (*plethos*). The only "principle" (*arkhe* – starting point) of Euclid's system of arithmetic is the unit (*monas*). Assuming the existence of units in nature, that is in an ontological sense that is never applied in arithmetical reasoning itself, he proceeds to the introduction of new kinds of numbers by means of effective procedures.

Nowhere the author of the *Elements* makes use of the assumption that the numbers form a fixed universe of discourse that is given beforehand. Hence he never postulates or proves existence of numbers having a certain property, but always 'constructs' the required numbers by means of effective procedures. Existence of numbers is never deduced by strong indirect arguments. The use of *reductio ad absurdum* relies on a specific propositional form of the law of excluded middle and applies to decidable arithmetical predicates. Moreover, Euclid seems to avoid the law of excluded middle in the arithmetical proofs. All propositions of the form $P(A) \vee \neg P(A)$ are proved by consideration of each part of the disjunction separately.

Only the potential infinite is used in Euclid's arithmetical reasoning. The actual infinite is never used, even in the most sophisticated cases that involve reasoning over infinite processes, such as the use of infinite descent, the least number principle and mathematical induction. The arithmetical operations are always applied on finite objects. All these enable us to characterise Euclidean arithmetic as arithmetic of the *finite mind.*

Furthermore, the approach adopted by Euclid does not need any special predicate logic. Euclid's arithmetic can be characterised as a finitary fragment of classical arithmetic; hence it does not necessarily presuppose the full force of first-order predicate logic. In our view, Euclid's approach was just the most "natural" way to develop arithmetic.

4. In Lieu of Conclusion

In historiography of Greek mathematics, number theory is commonly treated within the prevailing postulational style of thinking. This has led to the recognition that Greek number theory of the Euclidean *Elements* fall short from the postulational ideal, as exemplified in the other Books of the *Elements* and to underestimation of the number theory of the (Neo-) Pythagorean authors. The dominant historiographical viewpoint prevented historians to search for alternative approaches and recognise another style of reasoning in Greek number theory. In our view, the postulational style was not universal for Greek mathematics. It appropriately applies to Greek geometry, yet not to number theory, to which the Greeks have developed a different approach.

The Greek-European postulational style of mathematical thinking is usually contrasted to the Chinese style of thinking that makes use of concrete mathematical objects and practical algorithmic procedures. However, the recognition of the genetic style of Greek arithmetic enables us to reformulate the question of comparison between the two mainstream traditions. In comparison to the Chinese style, the objects of Greek number theory, that is the numbers (*arithmoi*), are not concrete, but "given", that is finitary entities, while the demonstrative procedures are not practical, but "effective."