On Laplace transforms of certain special functions

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1   Introduction

In this note we consider a Laplace transform of the following multiple integral

$$g(d; n; y) = c\pi^{d/2} \frac{1}{y^{\{n(d+1)-d\}/2+1}} \exp\{-\frac{a_n^2}{y}\}$$

$$= \int \cdots \int_{0 \le v_1 + \cdots + v_{n-1} \le 1; 0 \le v_j} \exp\{\Sigma_{j=1}^{n-1}(a_n^2 - a_j^2)v_j/y\}$$

$$\left[v_1 \cdots v_{n-1}(1 - \Sigma_{j=1}^{n-1}v_j)\right]^{(d+1)/2-1} dv_1 \cdots dv_{n-1} \qquad (1)$$

where c is a constant. This multiple integral is coming from the normed product of multidimensional Cauchy densities, namely

$$f(d; a_1, \cdots a_n; x) = \frac{c_1}{\pi_{j=1}^n(a_j^2 + |x|^2)^{(d+1)/2}},$$

where c_1 is the normalised constant and $0 < a_j$; $a_j \ne a_k$ if $j \ne k$, and $x = (x_1, \cdots, x_d) \in \mathbf{R}^d$. We note that the density function $g(d; 1; u)$ can be expressed by a confluent hypergeometric function,

$$g(d; 1; u) = \frac{c\pi^{d/2}\Gamma((d+1)/2)^2}{\Gamma(d+1)}$$

$$\cdot \frac{1}{u^{d/2+2}} \exp\{-\frac{b^2}{u}\} {}_1F_1(\frac{d+1}{2}; d+1; \frac{(b^2 - a^2)}{u}) \qquad (2)$$

for all dimension d. For the dimension $d = 3$ we obtain

$$g(3; 1; u) = \frac{c\pi^{3/2}}{(b^2 - a^2)^2}\left[\frac{1}{u^{3/2}}(\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\})\right.$$

$$\left. - \frac{2}{(b^2 - a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})\right] \qquad (3)$$

and for the dimension $d = 5$ we obtain

$$g(5; 1; u) = \frac{c\pi^{5/2}}{(b^2 - a^2)^3}\Big[\frac{2!}{u^{3/2}}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})$$

$$-\frac{2 \cdot 3!}{(b^2 - a^2)u^{1/2}}(\exp\{-\frac{a^2}{u}\} + \exp\{-\frac{b^2}{u}\})$$

$$+\frac{4!u^{1/2}}{(b^2 - a^2)^2}(\exp\{-\frac{a^2}{u}\} - \exp\{-\frac{b^2}{u}\})\Big]. \qquad (4)$$

2 The Laplace transforms of the case $n = 1$

Let us denote the Laplace transform of the functions $g(d; 1; u)$ by $\eta(d; 1; s)$ and let us take a branch of the value $\eta(d; +0) = 1$. We obtain

$$\eta(3; 1; s) = \frac{c\pi^2}{(b^2 - a^2)^2}\Big\{(\frac{1}{a}\exp\{-2a\sqrt{s}\} + \frac{1}{b}\exp\{-2b\sqrt{s}\})$$

$$-\frac{2}{(b^2 - a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} - \exp\{-2b\sqrt{s}\})\Big\}, \qquad (5)$$

for the dimension $d = 3$ and for the dimension $d = 5$ we obtain

$$\eta(5; 1; s) = \frac{c\pi^3}{(b^2 - a^2)^3}[2!(\frac{1}{a}\exp\{-2a\sqrt{s}\} - \frac{1}{b}\exp\{-2b\sqrt{s}\})$$

$$-\frac{2 \cdot 3!}{(b^2 - a^2)\sqrt{s}}(\exp\{-2a\sqrt{s}\} + \exp\{-2b\sqrt{s}\})$$

$$+\frac{4!}{(b^2 - a^2)^2 s}\{a\,\exp\{-2a\sqrt{s}\}(1 + \frac{1}{2a\sqrt{s}})$$

$$-b\,\exp\{-2b\sqrt{s}\}(1 + \frac{1}{2b\sqrt{s}})\}]. \qquad (6)$$

Making use of the formula

$$K_\nu(z) = \frac{1}{2}(\frac{1}{2}z)^\nu \int_0^\infty \exp\{-t - \frac{z^2}{4t}\}\,\frac{dt}{t^{\nu+1}}, \qquad (7)$$

we obtain Laplace transforms for the odd dimensional case,

$$\eta(d; 1; s) = \frac{c\pi^{d/2}}{\{(b^2 - a^2)^{l+1}}\Sigma_{j=0}^l\frac{(-1)^{2l-j}(l + j)!}{(b^2 - a^2)^j}\binom{l}{j}2^{j+1/2} \cdot$$

$$\Big\{a^{2j-1}\frac{K_{(j-1)+1/2}(2a\sqrt{s})}{(2a\sqrt{s})^{j-1+1/2}} + (-1)^{l+1+j}b^{2j-1}\frac{K_{(j-1)+1/2}(2b\sqrt{s})}{(2b\sqrt{s})^{j-1+1/2}}\Big\}, \qquad (8)$$

where

$$K_{n+1/2}(z) = (\frac{\pi}{2z})^{1/2}\exp\{-z\}\Sigma_{r=0}^n\frac{(n + r)!}{r!(n - r)!(2z)^r} \qquad (9)$$

(cf. G. Watson [7]).

3 Nonzero region of the Laplace transform

In general, we obtain

$$\eta(d; 1; s) = c\pi^{l+1} \int_0^1 \frac{v^l(1-v)^l}{(b^2 - (b^2 - a^2)v)^{(d+2)/2}}$$

$$\exp\left\{-2(b^2 - (b^2 - a^2)v)^{1/2}\sqrt{s}\right\}$$

$$\Sigma_{r=0}^{l+1} \frac{(l+1+r)!}{r!(l+1-r)!2^r}\left\{(b^2 - (b^2 - a^2)v)^{1/2}\sqrt{s}\right\}^{l+1-r} dv \qquad (10)$$

for the odd dimension $d = 2l + 1$ and we see that $\eta(d; 1; s) \neq 0$ in a neighborhood of the origin except at the origin and $\eta(d; 1; s)$ is bounded in a neighborhood of the origin. From the above Laplace transforms (8) we can see that the Laplace transforms $\eta(d; 1; s)$ do not vanish in a neighborhood of the infinity point.

4 The Laplace transform for the general case

Let us obtain the Laplace transform of $g(d; n; y)$ for the odd dimension. The following equality holds:

$$\exp\{-a_n^2 t\} \cdot t^{nl+n-1}$$

$$\cdot \int \cdots \int_{0 \le v_1 + \cdots + v_{n-1} \le 1; 0 \le v_j} \exp\left\{\Sigma_{j=1}^{n-1}\left(a_n^2 - a_j^2\right)tv_j\right\}$$

$$\cdot\left[v_1 \cdots v_{n-1}\left(1 - \Sigma_{j=1}^{n-1}v_j\right)\right]^l dv_1 \cdots dv_{n-1}$$

$$= (-1)^{(l+1)(n-1)}\Sigma_{r=1}^n \exp\{-a_r^2 t\}$$

$$\cdot\left[\Sigma_{j_{n-1}=0}^l \frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!} \frac{1}{B(r; n, n-1)^{l+j_{n-1}+1}}\right.$$

$$\cdot\Sigma_{j_{n-2}=0}^{l-j_{n-1}} \frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!} \frac{1}{B(r; n, n-2)^{l+j_{n-2}+1}}$$

$$\cdot\Sigma_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}} \frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!}$$

$$\cdot\frac{1}{B(r; n, n-3)^{l+j_{n-3}+1}} \cdots$$

$$\cdot\Sigma_{j_1=0}^{l-j_{n-1}-j_{n-2}\cdots-j_2} \frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!}$$

$$\left.\cdot\frac{1}{B(r; n, 1)^{l+j_1+1}}t^{l-j_1-\cdots-j_{n-2}-j_{n-1}}\right] \qquad (11)$$

where $B(r; n, 1) = a_r^2 - a_{r+1}^2, \cdots, B(r; n, n-r) = a_r^2 - a_n^2, B(r; n, n-r+1) = a_r^2 - a_1^2, \cdots, B(r; n, n-1) = a_r^2 - a_{r-1}^2$.

Theorem 1 *The Laplace transform is given in the following form.*

$$\eta(d;n;s) = \int_0^\infty \exp\{-sy\}\, g(d;n;y)dy$$

$$= c_1 \pi^{l+1/2}(-1)^{(l+1)(n-1)}$$

$$\Sigma_{r=1}^n \Big[\Sigma_{j_{n-1}=0}^l \frac{l!(l+j_{n-1})!}{j_{n-1}!(l-j_{n-1}+1)!} \frac{1}{B(r;n,n-1)^{l+j_{n-1}+1}}$$

$$\Sigma_{j_{n-2}=0}^{l-j_{n-1}} \frac{(l-j_{n-1})!(l+j_{n-2})!}{j_{n-2}!(l-j_{n-1}-j_{n-2})!} \frac{1}{B(r;n,n-2)^{l+j_{n-2}+1}}$$

$$\Sigma_{j_{n-3}=0}^{l-j_{n-1}-j_{n-2}} \frac{(l-j_{n-1}-j_{n-2})!(l+j_{n-3})!}{j_{n-3}!(l-j_{n-1}-j_{n-2}-j_{n-3})!}$$

$$\frac{1}{B(r;n,n-3)^{l+j_{n-3}+1}} \cdots$$

$$\Sigma_{j_1=0}^{l-j_{n-1}-j_{n-2}-\cdots-j_2} \frac{(l-j_{n-1}-j_{n-2}-\cdots-j_2)!(l+j_1)!}{j_1!(l-j_{n-1}-j_{n-2}-\cdots-j_2-j_1)!}$$

$$\frac{1}{B(r;n,1)^{l+j_1+1}} 2\Big(\frac{2}{2a_r\sqrt{s}}\Big)^{\Sigma_{p=1}^{n-1} j_p - 1/2} K_{\Sigma_{p=1}^{n-1} j_p - 1/2}\Big(2a_r\sqrt{s}\Big)\Big] \qquad (12)$$

Here $K_{\Sigma_{p=1}^{n-1} j_p - 1/2}\Big(2a_r\sqrt{s}\Big)$ *is given by (9).*

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