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Some subordination criteria concerning Sălăgean operator

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Abstract

Applying Sălăgean operator, for the class $\mathcal{A}$ of analytic functions $f(z)$ in the open unit disk $U$ which are normalized by $f(0) = f'(0) - 1 = 0$, the generalization of an analytic function to discuss the starlikeness is considered. Furthermore, from the subordination criteria for Janowski functions generalized by some complex parameters, some interesting subordination criteria for $f(z) \in \mathcal{A}$ are given.

1 Introduction, definition and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

\[(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \]

which are analytic in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

\[(1.2) \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \]

which are analytic in $U$. If $p(z) \in \mathcal{P}$ satisfies $\text{Re}(p(z)) > 0 \ (z \in U)$, then we say that $p(z)$ is the Carathéodory function (cf. [1]).

A function $f(z) \in \mathcal{A}$ is said to be starlike of order $\alpha$ in $U$ if it satisfies

\[(1.3) \quad \text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U) \]

for some $\alpha \ (0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are starlike of order $\alpha$ in $U$.

Similarly, if $f(z) \in \mathcal{A}$ satisfies the following inequality

\[(1.4) \quad \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U) \]
for some $\alpha$ ($0 \leq \alpha < 1$), then $f(z)$ is said to be convex of order $\alpha$ in $\mathbb{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which are convex of order $\alpha$ in $\mathbb{U}$. As usual, in the present investigation, we write

$$S^*(0) \equiv S^* \quad \text{and} \quad \mathcal{K}(0) \equiv \mathcal{K}.$$  

The classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [7].

By the familiar principle of differential subordination between analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ satisfying the following conditions:

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if $g(z)$ is univalent in $\mathbb{U}$, then it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $p(z) \in \mathcal{P}$, we introduce the following function

$$(1.5) \quad p(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1)$$

which has been investigated by Janowski [3]. Thus, the function $p(z)$ given by (1.5) is said to be the Janowski function. Here, for some $A$ and $B$ ($-1 < B < A \leq 1$), the function $p(z)$ given by (1.5) is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by

$$\left| p(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2}.$$ 

Thus, it is clear that

$$(1.6) \quad \text{Re}(p(z)) > \frac{1 - A}{1 - B} \geq 0 \quad (z \in \mathbb{U}).$$

Also, if we take $B = -1$ in (1.5), then we see that

$$(1.7) \quad p(z) = \frac{1 + Az}{1 - z} \quad (-1 < A \leq 1)$$

is analytic and univalent in $\mathbb{U}$ and the domain $p(\mathbb{U})$ is the right half-plane satisfying

$$(1.8) \quad \text{Re}(p(z)) > \frac{1}{2} (1 - A) \geq 0.$$
Hence, we see that the Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

And, as the generalization of Janowski function, Kuroki, Owa and Srivastava [2] have discussed the function
\[ p(z) = \frac{1 + Az}{1 + Bz} \]
for some complex parameters $A$ and $B$ which satisfy one of following conditions
\[
\begin{align*}
(i) & \quad |B| < 1, A \neq B, \text{ and } \Re(1 - AB) \geq |A - B| \\
(ii) & \quad |B| = 1, A \neq B, |A| \leq 1, \text{ and } 1 - A\overline{B} > 0.
\end{align*}
\]
First, for some complex numbers $A$ and $B$ which satisfy the following condition
\[
(i) \quad |B| < 1, A \neq B, \text{ and } \Re(1 - AB) \geq |A - B|,
\]
the function $p(z) = \frac{1 + Az}{1 + Bz}$ is analytic and univalent in $\mathbb{U}$ and $p(z)$ maps the open unit disk $\mathbb{U}$ onto the open disk given by
\[
\left| p(z) - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2}.
\]
Thus, it is clear that
\[
(1.9) \quad \Re(p(z)) > \frac{\Re(1 - AB) - |A - B|}{1 - |B|^2} \geq 0 \quad (z \in \mathbb{U}).
\]
Also, for some complex numbers $A$ and $B$ which satisfy the following condition
\[
(ii) \quad |B| = 1, A \neq B, |A| \leq 1, \text{ and } 1 - A\overline{B} > 0,
\]
the function $p(z) = \frac{1 + Az}{1 + Bz}$ is analytic and univalent in $\mathbb{U}$ and the domain $p(U)$ is the right half-plane satisfying
\[
(1.10) \quad \Re(p(z)) > \frac{1 - |A|^2}{2(1 - AB)} \geq 0.
\]
Hence, we see that the generalized Janowski function maps the open unit disk $\mathbb{U}$ onto some domain which is on the right half-plane.

We define the following differential operator due to Sălăgean [8].
For a function $f(z)$ and $j = 1, 2, 3, \cdots$,
\[
(1.11) \quad D^j f(z) = f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
\(D^1 f(z) = Df(z) = zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,\)

\(D^j f(z) = D(D^{j-1} f(z)) = z + \sum_{n=2}^{\infty} n^j a_n z^n.\)

Also, we meditate the following integral operator

\(D^{-1} f(z) = \int_{0}^{z} \frac{f(\zeta)}{\zeta} d\zeta = z + \sum_{n=2}^{\infty} n^{-1} a_n z^n,\)

\(D^{-j} f(z) = D^{-1}(D^{-(j-1)} f(z)) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n\)

for any negative integers.

Then, for \(f(z) \in \mathcal{A}\) given by (1.1), we know that

\(f^{-j} f(z) = z + \sum_{n=2}^{\infty} n^{-j} a_n z^n\)

Using the above operator \(D^j f(z)\), we consider the subclass \(S_j^k(\alpha)\) of \(\mathcal{A}\) as follows:

\[S_j^k(\alpha) = \left\{ f(z) \in \mathcal{A} : \text{Re} \left( \frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}.\]

**Remark 1.1** Noting

\[\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)}, \quad \frac{D^2 f(z)}{D^1 f(z)} = \frac{zf''(z)}{f'(z)} = 1 + \frac{zf''(z)}{f'(z)},\]

we see that

\[S_0^1(\alpha) \equiv S^*(\alpha), \quad S_1^1(\alpha) \equiv K(\alpha) \quad (0 \leq \alpha < 1).\]

**Remark 1.2** For some \(\alpha \ (0 \leq \alpha < 1)\), we find

\[\frac{D^k f(z)}{D^j f(z)} < \frac{1 + (1 - 2\alpha)z}{1 - z} \iff \text{Re} \left( \frac{D^k f(z)}{D^j f(z)} \right) > \alpha \quad (z \in \mathbb{U}).\]

In our investigation here, we need the following lemma concerning the differential subordination given by Miller and Mocanu [5] (see also [6, p. 132]).

**Lemma 1.3** Let the function \(q(z)\) be analytic and univalent in \(\mathbb{U}\). Also let \(\phi(\omega)\) and \(\psi(\omega)\) be analytic in a domain \(C\) containing \(q(\mathbb{U})\), with

\[\psi(\omega) \neq 0 \quad (\omega \in q(\mathbb{U}) \subset C).\]
Set
\[ Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z), \]
and suppose that
\begin{enumerate}[(i)]  
  \item \(Q(z)\) is starlike and univalent in \(\mathbb{U}\); and
  \item \(\Re\left( \frac{zh'(z)}{Q(z)} \right) = \Re\left( \frac{\phi'(q(z))}{\psi(q(z))} + \frac{zQ'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}). \)
\end{enumerate}

If \(p(z)\) is analytic in \(\mathbb{U}\), with
\[ p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset \mathbb{C}, \]
and
\[ \phi(p(z)) +zp'(z)\psi(p(z)) < \phi(q(z)) + zq'(z)\psi(q(z)) =: h(z) \quad (z \in \mathbb{U}), \]
then
\[ p(z) < q(z) \quad (z \in \mathbb{U}) \]
and \(q(z)\) is the best dominant of this subordination.

By making use of lemma 1.3, Kuroki, Owa and Srivastava [2] have investigated some subordination criteria for the generalized Janowski functions and deduced the following lemma.

**Lemma 1.4** Let the function \(f(z) \in \mathcal{A}\) be so chosen that \(\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U})\).

Also, let \(\alpha (\alpha \neq 0), \beta (-1 \leq \beta \leq 1)\), and some complex parameters \(A\) and \(B\) which satisfy one of following conditions
\begin{enumerate}[(i)]  
  \item \(|B| < 1, A \neq B, \text{ and } \Re(1 - AB) \geq |A - B| \text{ be so that} \)
    \[ \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(\Re(1 - AB) - |A - B|)}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0, \]
  \item \(|B| = 1, A \neq B, |A| \leq 1, \text{ and } 1 - AB > 0 \text{ be so that} \)
    \[ \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - AB)} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0. \]
\end{enumerate}

If
\[ (1.17) \quad \left( \frac{zf''(z)}{f(z)} \right)^{\beta} \left( 1 + \alpha\frac{zf''(z)}{f'(z)} \right) < h(z) \quad (z \in \mathbb{U}), \]
where
\[ h(z) = \left( \frac{1 + Az}{1 + Bz} \right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\}, \]
then
\[ \frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \]
2 Subordinations for the class defined by Sălăgean operator

Applying Sălăgean operator for $f(z) \in \mathcal{A}$, we deduced the following subordination criterion for the generalized Janowski function.

**Theorem 2.1** Let the function $f(z) \in \mathcal{A}$ be so chosen that $\frac{D^j f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. Also, let $\alpha$ ($\alpha \neq 0$), $\beta$ $(-1 \leq \beta \leq 1)$, and some complex parameters $A$ and $B$ which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\text{Re}(1 - AB) \geq |A - B|$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta) \{\text{Re}(1 - AB) - |A - B|\}}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - AB > 0$ be so that

$$\frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - AB)} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If

$$\left(\frac{D^k f(z)}{D^j f(z)}\right)^\beta \left\{ (1 - \alpha) + \alpha \left( \frac{D^k f(z)}{D^j f(z)} + \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right) \right\} \prec h(z),$$

where

$$h(z) = \left(\frac{1 + Az}{1 + Bz}\right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\},$$

then

$$\frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

**Proof.** If we define the function $p(z)$ by

$$p(z) = \frac{D^k f(z)}{D^j f(z)} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Further, since

$$zp'(z) = \left(\frac{D^k f(z)}{D^j f(z)}\right) \left( \frac{D^{k+1} f(z)}{D^k f(z)} - \frac{D^{j+1} f(z)}{D^j f(z)} \right),$$

the condition (2.1) can be written as follows:

$$\{p(z)\}^\beta \left\{ (1 - \alpha) + \alpha p(z) \right\} + \alpha z p'(z) \{p(z)\}^{\beta-1} \prec h(z) \quad (z \in \mathbb{U}).$$
We also set
\[ q(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \]
and
\[ \phi(\omega) = \omega^\beta (1 - \alpha + \alpha \omega), \quad \text{and} \quad \psi(\omega) = \alpha \omega^{\beta - 1} \]
for \( \omega \in q(\mathbb{U}) \). Then, it is clear that the function \( q(z) \) is analytic and univalent in \( \mathbb{U} \) and have a positive real part in \( \mathbb{U} \) for the conditions (i) and (ii).
Therefore, \( \phi \) and \( \psi \) are analytic in a domain \( C \) containing \( q(\mathbb{U}) \), with
\[ \psi(\omega) = \alpha \omega^{\beta - 1} \neq 0 \quad (\omega \in q(\mathbb{U}) \subset C). \]
Also, for the function \( Q(z) \) given by
\[ Q(z) = zq'(z)\psi(q(z)) = \frac{\alpha(A - B)z(1 + Az)^{\beta - 1}}{(1 + Bz)^{\beta + 1}}, \]
we obtain
\[ (2.2) \quad \frac{zQ'(z)}{Q(z)} = \frac{1 - \beta}{1 + Az} + \frac{1 + \beta}{1 + Bz} - 1. \]
Furthermore, we have
\[ h(z) = \phi(q(z)) + Q(z) \]
\[ = \left( \frac{1 + Az}{1 + Bz} \right)^\beta \left( 1 - \alpha + \alpha \frac{1 + Az}{1 + Bz} \right) + \frac{\alpha(A - B)z(1 + Az)^{\beta - 1}}{(1 + Bz)^{\beta + 1}} \]
and
\[ (2.3) \quad \frac{zh'(z)}{Q(z)} = \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta)q(z) + \frac{zQ'(z)}{Q(z)}. \]
Hence,
(i) For the complex numbers \( A \) and \( B \) such that
\[ |B| < 1, \ A \neq B, \ \text{and} \ \text{Re}(1 - A\overline{B}) \geq |A - B|, \]
it follows from (2.2) and (2.3) that
\[ \text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) > \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0, \]
and
\[ \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \frac{\beta(1 - \alpha)}{\alpha} + (1 + \beta)\left\{ \text{Re}(1 - A\overline{B}) - |A - B| \right\} \]
\[ + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0 \quad (z \in \mathbb{U}). \]
(ii) For the complex numbers $A$ and $B$ such that
\[ |B| = 1, \ |A| \leq 1, \ A \neq B, \text{ and } 1 - AB > 0, \]
from (2.2) and (2.3), we get
\[ \text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) > \frac{1 - \beta}{1 + |A|} + \frac{1}{2} (1 + \beta) - 1 = \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0, \]
and
\[ \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - AB)} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0 \quad (z \in \mathbb{U}). \]
Since all conditions of Lemma 1.3 are satisfied, we conclude that
\[ \frac{D^k f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}), \]
which completes the proof of Theorem 2.1.

Letting $k = j + 1$ in Theorem 2.1, we obtain

**Corollary 2.2** Let the function $f(z) \in A$ be so chosen that $\frac{D^j f(z)}{z} \neq 0$ $(z \in \mathbb{U})$. Also, let $\alpha$ ($\alpha \neq 0$), $\beta$ ($-1 \leq \beta \leq 1$), and some complex parameters $A$ and $B$ which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\text{Re}(1 - AB) \geq |A - B|$ be so that
\[ \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{1 - |B|^2} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0, \]
(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - AB > 0$ be so that
\[ \frac{\beta(1 - \alpha)}{\alpha} + \frac{(1 + \beta)(1 - |A|^2)}{2(1 - AB)} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0. \]

If
\[ (2.2) \quad \left( \frac{D^{j+1} f(z)}{D^j f(z)} \right)^{\beta} \left( 1 - \alpha + \alpha \frac{D^{j+2} f(z)}{D^{j+1} f(z)} \right) \prec h(z), \]
where
\[ h(z) = \left( \frac{1 + Az}{1 + Bz} \right)^{\beta-1} \left\{ (1 - \alpha) \frac{1 + Az}{1 + Bz} + \frac{\alpha(1 + Az)^2 + \alpha(A - B)z}{(1 + Bz)^2} \right\}, \]
then
\[ \frac{D^{j+1} f(z)}{D^j f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \]

**Remark 2.3** Setting $j = 0$ in Corollary 2.2, we obtain Lemma 1.4 proven by Kuroki, Owa and Srivastava [2].
Also, if we assume that $\alpha = 1$, $\beta = A = 0$, and $B = \frac{1-\mu}{1+\mu} e^{i\theta}$ ($0 \leq \mu < 1, 0 \leq \theta < 2\pi$), Corollary 2.2 becomes the following corollary.

**Corollary 2.4** If $f(z) \in A \left( \frac{D^j f(z)}{z} \neq 0 \text{ in } U \right)$ satisfies

$$\frac{D^{j+2} f(z)}{D^{j+1} f(z)} < \frac{1+\mu - (1-\mu) e^{i\theta} z}{1+\mu + (1-\mu) e^{i\theta} z} \quad (z \in U; 0 \leq \theta < 2\pi)$$

for some $\mu$ ($0 \leq \mu < 1$), then

$$\frac{D^{j+1} f(z)}{D^j f(z)} < \frac{1+\mu}{1+\mu + (1-\mu) e^{i\theta} z} \quad (z \in U).$$

From the above corollary, we have

$$\text{Re} \left( \frac{D^{j+2} f(z)}{D^{j+1} f(z)} \right) > \mu \iff \text{Re} \left( \frac{D^{j+1} f(z)}{D^j f(z)} \right) > \frac{1+\mu}{2} \quad (z \in U; 0 \leq \mu < 1).$$

Thus, we see that

$$f(z) \in S_{j+1}^{j+2}(\mu) \iff f(z) \in S_j^{j+1} \left( \frac{1+\mu}{2} \right) \iff f(z) \in S_{j-1}^{j} \left( \frac{3+\mu}{4} \right)$$

$$\iff \ldots \iff f(z) \in S_{0}^{1} \left( \frac{2^j - 1 + \mu}{2^j} \right)$$

$$\iff f(z) \in S_{0}^{1} \left( \frac{2^{j+1} - 1 + \mu}{2^{j+1}} \right) \quad (z \in U; 0 \leq \mu < 1).$$

In particular, we find

$$f(z) \in S_{j+1}^{j+2}(\mu) \iff f(z) \in \mathcal{K} \left( \frac{2^j - 1 + \mu}{2^j} \right)$$

$$\iff f(z) \in \mathcal{K}^* \left( \frac{2^{j+1} - 1 + \mu}{2^{j+1}} \right) \quad (z \in U; 0 \leq \mu < 1).$$

And, taking $j = 0$ and $\mu = 0$, we find the fact that every convex function is starlike of order $\frac{1}{2}$. This fact is well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

### 3 Subordination criteria for other analytic functions

In this section, by making use of Lemma 1.3, we consider some subordination criteria concerning analytic function $\frac{D^j f(z)}{z}$ for $f(z) \in A$. 
**Theorem 3.1** Let $\alpha (\alpha \neq 0)$, $\beta (-1 \leq \beta \leq 1)$, and some complex parameters $A$ and $B$ which satisfy one of following conditions

(i) $|B| < 1$, $A \neq B$, and $\text{Re}(1 - A\overline{B}) \geq |A - B|$ be so that

$$\frac{\beta}{\alpha} + \frac{1 - \beta}{1 + |A|} + \frac{1 + \beta}{1 + |B|} - 1 \geq 0,$$

(ii) $|B| = 1$, $A \neq B$, $|A| \leq 1$, and $1 - A\overline{B} > 0$ be so that

$$\frac{\beta}{\alpha} + \frac{(1 - \beta)(1 - |A|)}{2(1 + |A|)} \geq 0.$$

If $f(z) \in A$ satisfies

$$(3.1) \quad \left( \frac{D^j f(z)}{z} \right)^\beta \left( 1 - \alpha + \alpha \frac{D^{j+1} f(z)}{D^j f(z)} \right) \prec \left( \frac{1 + Az}{1 + Bz} \right)^\beta + \frac{\alpha(A - B)z(1 + Az)^{\beta - 1}}{(1 + Bz)^{\beta + 1}},$$

then

$$\frac{D^j f(z)}{z} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

**Proof.** If we define the function $p(z)$ by

$$p(z) = \frac{D^j f(z)}{z} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and the condition (3.1) can be written as follows:

$$\{p(z)\}^\beta + \alpha p'(z)\{p(z)\}^{\beta - 1} \prec h(z) \quad (z \in \mathbb{U}).$$

We also set

$$q(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

and

$$\phi(\omega) = \omega^\beta, \quad \psi(\omega) = \alpha \omega^{\beta - 1}$$

for $\omega \in q(\mathbb{U})$. Then, the function $q(z)$ is analytic and univalent in $\mathbb{U}$ and satisfies

$$\text{Re}(q(z)) > 0 \quad (z \in \mathbb{U})$$

for the condition (i) and (ii).

Thus, the functions $\phi$ and $\psi$ satisfy the conditions required by Lemma 1.3.

Further, for the functions $Q(z)$ and $h(z)$ given by

$$Q(z) = zq'(z)\psi(q(z)) \quad \text{and} \quad h(z) = \phi(q(z)) + Q(z),$$

we have

$$\frac{zQ'(z)}{Q(z)} = \frac{1 - \beta}{1 + Az} + \frac{1 + \beta}{1 + Bz} - 1 \quad \text{and} \quad \frac{zh'(z)}{Q(z)} = \frac{\beta}{\alpha} + \frac{zQ'(z)}{Q(z)}.$$
Then, similarly to proof of Theorem 2.1, we see that
\[
\text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) > 0 \quad \text{and} \quad \text{Re} \left( \frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U})
\]
for the conditions (i) and (ii).

Thus, by applying Lemma 1.3, we conclude that \( p(z) \prec q(z) \) \( (z \in \mathbb{U}) \).
The proof of the theorem is completed.

In Theorem 3.1, taking \( \alpha = 1, \beta = A = 0, \) and \( B = \frac{1-\nu}{\nu} e^{i\theta} \left( \frac{1}{2} \leq \nu < 1, 0 \leq \theta < 2\pi \right) \),
we obtain the following corollary.

**Corollary 3.2** If \( f(z) \in A \) satisfies
\[
\frac{D^{j+1}f(z)}{D^j f(z)} < \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)
\]
for some \( \nu \left( \frac{1}{2} \leq \nu < 1 \right) \), then
\[
\frac{D^j f(z)}{z} < \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).
\]

Also, making \( \alpha = \beta = 1, A = 0, \) and \( B = \frac{1-\nu}{\nu} e^{i\theta} \left( \frac{1}{2} \leq \nu < 1, 0 \leq \theta < 2\pi \right) \) in
Theorem 3.1, we get

**Corollary 3.3** If \( f(z) \in A \) satisfies
\[
\frac{D^{j+1}f(z)}{D^j f(z)} < \left( \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \right)^2 \quad (z \in \mathbb{U}; 0 \leq \theta < 2\pi)
\]
for some \( \nu \left( \frac{1}{2} \leq \nu < 1 \right) \), then
\[
\frac{D^j f(z)}{z} < \frac{\nu}{\nu + (1-\nu)e^{i\theta}z} \quad (z \in \mathbb{U}).
\]

The above corollaries derive each of the facts that
\[
\text{Re} \left( \frac{D^{j+1}f(z)}{D^j f(z)} \right) > \nu \quad \Rightarrow \quad \text{Re} \left( \frac{D^j f(z)}{z} \right) > \nu \quad (z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1),
\]
and
\[
\text{Re} \sqrt{\frac{D^{j+1}f(z)}{z}} > \nu \quad \Rightarrow \quad \text{Re} \left( \frac{D^j f(z)}{z} \right) > \nu \quad (z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1).
\]
In particular, for $j = 0$, we see that
\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \nu \quad \Rightarrow \quad \text{Re} \left( \frac{f(z)}{z} \right) > \nu \quad \left( z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right), \]
and
\[ \text{Re} \sqrt{f'(z)} > \nu \quad \Rightarrow \quad \text{Re} \left( \frac{f(z)}{z} \right) > \nu \quad \left( z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right). \]
Here, taking $\nu = \frac{1}{2}$, we find some results well-known as the Marx-Strohhäcker theorem in Univalent Function Theory (cf. [4], [9]).

Also, letting $j = 1$ in Corollary 3.2, we get the following fact:
\[ \text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \nu \quad \Rightarrow \quad \text{Re} (f'(z)) > \nu \quad \left( z \in \mathbb{U}; \frac{1}{2} \leq \nu < 1 \right). \]

References


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