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Quasiconformal extension of univalent functions and Becker's theorem (Study on Non-Analytic and Univalent Functions and Applications)

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Abstract

This is a research for a subclass of univalent holomorphic functions on the unit disc normalized by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), which can be extended to \( k \)-quasiconformal mappings on the disc \( \{ |z| < R \} \) where \( R > 1 \). Such a subclass is denoted by \( S(k, R) \). In this note, the class \( S(k, R) \) is introduced through the observation of Becker's theorem which ensures a \( k \)-quasiconformal extendibility of univalent holomorphic functions on the disc to the Riemann sphere with L"owner chains.

1 Motivation

Let \( D = \{ z \mid |z| < 1 \} \) and

\[
\mathcal{S} = \{ f \mid f \text{ is holomorphic and univalent on } D, \ f(0) = f'(0) - 1 = 0 \},
\]

\[
\mathcal{S}(k) = \{ f \mid f \in \mathcal{S}, \ f \text{ can be extended to a } k\text{-quasiconformal mapping on } \overline{\mathbb{C}} \},
\]

\[
\mathcal{S}_0(k) = \{ f \mid f \in \mathcal{S}(k), \text{the extended mappings fix } \infty \},
\]

respectively, where \( k \in [0,1) \). The class \( \mathcal{S}(k) \) has been studied by numerous authors in connection with the theory of Teichmüller spaces. In those investigations, an interesting method for quasiconformal extension of univalent functions was obtained by Becker ([1], see also [5]) which relies on the L"owner chains described by the L"owner equation

\[
\frac{\partial f(z,t)}{\partial t} = zp(z,t) \frac{\partial f(z,t)}{\partial z} \tag{1}
\]

for \( z \in D \) and \( t \in [0, \infty) \). This equation determines an expanding flow. Here, the function \( f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n \) is holomorphic in \( |z| < 1 \) for each \( t \in [0, \infty) \), absolutely continuous in \( t \in [0, \infty) \) for each \( |z| < r_0 \) and satisfies the inequality

\[
|f(z,t)| \leq K_0 e^t \ (|z| < r_0, t \leq 0)
\]

for some positive constants \( K_0 \) and \( r_0 \). Also a function \( p(z,t) \) is measurable on \( D \times [0, \infty) \), holomorphic in \( |z| < 1 \), and satisfies \( \text{Re } p(z,t) > 0 \) and the partial differential equation (1) for
Theorem 1 ([1]). If $f(z, t)$ is a univalent solution to (1) with $p(z, t)$ satisfying the condition

$$\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1$$

then, for each $t \geq 0$, the function $f(z) = f(z, t)$ maps $D$ onto a Jordan domain bounded by a $k$-quasiconformal image of $\partial D$, and the map $\hat{f}(z)$ defined by

$$\hat{f}(re^{i\theta}) = \begin{cases} f(re^{i\theta}, 0) & r \leq 1 \\ f(e^{i\theta}, \log r) & r > 1 \end{cases}$$

is a $k$-quasiconformal extension of $f(z, 0)$ onto $\hat{C}$ with $\hat{f}(\infty) = \infty$ (thus $\hat{f}(z) \in S_0(k)$).

Observe that $p(D, t)$ must be contained in the disc $|z - (1 + k^2)/(1 - k^2)| \leq 2k/(1 - k^2)$ for all $t \in [0, \infty)$ so that we can apply Theorem 1 to the Löwner chains (Fig.1). This strong assumption can be weakened by restricting the range of the parameter $t$. In fact, the following is true;

![Figure 1](image-url)

Figure 1: $p(z, t)$ must be in this circle for all $z \in D$ and $t \in [0, \infty)$. 

Theorem 1 (1)). If $f(z, t)$ is a univalent solution to (1) with $p(z, t)$ satisfying the condition.
Corollary 2. If \( f(z,t) \) is a univalent solution to (1) and there exists \( t_0 > 0 \) such that for all \( t \in [0, t_0] \) \( p(z, t) \) satisfies the condition

\[
\left| \frac{p(z, t) - 1}{p(z, t) + 1} \right| \leq k < 1,
\]

then the map \( \hat{f}(z) \) is a \( k \)-quasiconformal extension of \( f(z, 0) \) defined on \( \{z \mid |z| < e^b\} \) with \( \hat{f} \neq \infty \).

Now we shall introduce the classes \( S(k, R) \) and \( S_0(k, R) \); namely

\[
S(k, R) = \{f \mid f \in S, f \text{ can be extended to a } k \text{-quasiconformal mapping } \hat{f} \text{ on } \{|z| < R\}\}
\]

and

\[
S_0(k, R) = \{f \mid f \in S(k, R), \text{ the extended mapping } \hat{f} \text{ doesn't take } \infty \text{ on } \{|z| < R\}\}
\]

respectively, where \( R > 1 \).

2 Properties of the class \( S(k, R) \)

The class \( S(k, R) \) was studied by some authors in another context. We shall give some known results for the classes \( S(k, R) \) and \( \Sigma(k, r) \), where \( \Sigma(k, r) \) is a family of univalent holomorphic functions on \( \{z \in \hat{C} - \bar{D} \} \) which can be extended to a \( k \)-quasiconformal mapping on \( \{|z| > r\}, r < 1 \).

McLeavey [8] (see also [9]) first considered the subclass of \( \Sigma \) with \( K(|z|) \)-quasiconformal extensions into the interior of \( D \) where \( K(|z|) \) is a piecewise continuous function of bounded variation on \([r, 1], 0 \leq r < 1 \). She obtained for this class the analogs of the classical Grunsky and Goluzin inequalities and sharp estimates for the coefficients \( b_0 \) and \( b_1 \) of \( \Sigma(k, r) \) and \( a_2 \) of \( S_0(k, R) \) with extremal function as follow;

**Theorem 3** ([8]). If \( g \in \Sigma(k, r) \) and \( g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \) for \( |z| > 1 \), then

\[
|b_1| \leq \frac{k + r^2}{1 + kr^2}.
\]
Equality occurs if and only if

\[
g(z) = \begin{cases} 
  z + b_0 + \left(\frac{k + r^2}{1 + kr^2}\right) \frac{e^{i\alpha}}{z} & |z| \geq 1 \\
  \left(\frac{1}{1 + kr^2}\right) \left( z + \frac{r^2 e^{i\alpha}}{z} + ke^{i\alpha \overline{z}} + \frac{kr^2}{z} \right) & r < |z| \leq 1.
\end{cases}
\]

Corollary 4 ([8]). Suppose \( f \in S(k, R) \) and \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) for \( z \in D \). Then

\[ |a_3 - a_2^2| \leq \frac{1 + kR^2}{k + R^2}. \]

If, in addition, extend mappings do not take \( \infty \) on \( \{|z| < R\} \), then

\[ |a_2| \leq 2 \frac{1 + kR}{k + R}. \quad (4) \]

Kühnau [6] also proved similar results of those through introducing the class \( \Sigma(Q_1, \cdots, Q_n) \) of \( K(|z|) \)-quasiconformal mapping of the plane which are conformal on \( \{z; |z| > 1\} \) with a development \( f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \) and which have piecewise bounded dilatation in \( D : K(|z|) \leq Q_i \) \((Q_i \geq 1)\) in \( R_i < |z| < R_{i-1} \) \((i = 1, \cdots, n)\), with \( R_0 = 1, R_n = 0 \). Schober [9] mentioned above results in his book, Chap.14. He also gave some more results, for instance, generalized Gronwall's area theorem for \( \Sigma(k, r) \);

**Theorem 5 ([9]).** If \( g \in \Sigma(k, r) \) and \( g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n} \) for \( |z| > 1 \), then

\[ \sum_{m=1}^{\infty} m |b_m|^2 \leq \left( \frac{k + r^2}{1 + kr^2} \right)^2. \quad (5) \]

Under the more general case, Lehto [7] showed a majorant principle for a holomorphic functional as follow;

Let \( A \) be a domain in \( \overline{C} \) which is bounded by a quasicircle, \( B \) be a domain whose closure \( \overline{B} \subset A \), and \( F_B, 0 \leq k < 1 \), be a family of functions which are \( k \)-quasiconformal on \( A \) and conformal on \( \overline{B} \). Denote by \( F_1 \) the family of all conformal mappings on \( B \).

We introduce four different normalizations to cover a large number of cases appearing in applications. Let \( z_1, z_2, z_3 \) be distinct points of \( B \) and \( \alpha_1, \alpha_2, \alpha_3, \beta \) be complex numbers, the
\(\alpha\)'s are different from each other and \(\beta \neq 0\). The families \(\mathcal{F}_k\) and \(\mathcal{F}_1\) are called normalized if all the functions \(f\) of \(A\) contained in \(\mathcal{F}_k\) or \(\mathcal{F}_1\) have one of the following conditions:

1. \(f(z_i) = \alpha_i, \ i = 1, 2, 3\),
2. \(f(z_i) = \alpha_i, \ i = 1, 2\), and \(f(z) \neq \infty \) in \(A\),
3. \(f(z_1) = \alpha_1, f'(z_1) = \beta\) and \(f(z) \neq \infty\) in \(A\),
4. If \(\infty \in B\), then \(f(z) - z \rightarrow 0\) as \(z \rightarrow \infty\).

We shall suppose here \(\mathcal{F}_k\) and \(\mathcal{F}_1\) are normalized. Remark that normalized \(\mathcal{F}_k\) and \(\mathcal{F}_1\) are closed normal families.

Let \(\Psi\) be a holomorphic functional defined on the family \(F_k\) or \(F_1\), i.e. \(\Psi(f) = \omega(f(z_0), f'(z_1), \cdots, f^{(n)}(z_n))\), where \(\omega\) is a complex-valued holomorphic function of the variables \(f^{(0)}(z_i), i = 1, 2, \cdots\), each \(f^{(0)}(z_i)\) being the value at fixed point \(z_i \in B\).

Set
\[
M(k) = \sup_{f \in \mathcal{F}_k} |\Psi(f)|, \quad 0 \leq k \leq 1.
\]

Since \(\mathcal{F}_k\) is a closed normal family, there exists an extremal function maximizing \(|\Psi(f)|\) in \(\mathcal{F}_k\).

**Theorem 6 ([7]).** For a holomorphic functional in \(\mathcal{F}_k\),
\[
M(k) \leq M(1) \frac{k + M(1)}{1 + k M(1)}.
\]

This result contains some coefficient estimates as corollaries; for the class \(\Sigma(k, r)\)
\[
\max_{\Sigma(k, r)} |b_n| \leq \frac{k + r^{n+1}}{1 + kr^{n+1}} \max_{\Sigma} |b_n|, \quad n = 1, 2, \cdots,
\]
which imply
\[
|b_1| \leq \frac{k + r^2}{1 + kr^2} \quad \text{and} \quad |b_2| \leq \frac{2}{3} \frac{k + r^3}{1 + kr^3}.
\]
The Grunsky type inequalities for \(\Sigma(k, r)\) also follow easily from the general inequality (6).

For \(f \in \Sigma\), let \(A_{mn}, m, n = 1, 2, \cdots\), be the numbers determined by
\[
\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} A_{mn} z^{-m} \zeta^{-n}.
\]
Then for any complex numbers $x_1, x_2, \ldots, x_N$,

$$\left| \sum_{m,n}^{N} A_{mn} x_m x_n \right| \leq \frac{k + r^2}{1 + kr^2} \sum_{n}^{N} \frac{|x_n|^2}{n} \tag{7}$$

and

$$\sum_{n}^{N} n \left| \sum_{m}^{N} A_{m} x_m \right|^2 \leq \left( \frac{k + r^2}{1 + kr^2} \right)^2 \sum_{n=1}^{N} \frac{|x_2|^2}{n}. \tag{8}$$

Remark that (7) and (8) is not sharp (see [8]).

Deiernann treats several similar problems of those in [2] and [3] with the method of extremal length. Recently, Krushkal gives a short mention for those reseaches in his survey [4], Chap.6.3.

3 Main Results

Now the more applications of Theorem 6 are given to $S_0(k, R)$ and $\Sigma(k, r)$ (again remark that these results are not sharp because (7) and (8) is not sharp);

Theorem 7.

$$\sup_{S_0(k, R)} |a_n| \leq n \frac{1 + kR^{n-1}}{k + R^{n-1}}.$$

Proof. Let us take $F_k = S_0(k, R)$, then $F_1$ is the well-known class $S$. Choose $\Psi(f) = a_n$. Then $M(1) = n$, and $M(0) = n/R^{n-1}$ because $Rf(z/R) \in S$ for arbitrary $f \in F_0$. Hence the inequality (6) follow the theorem. 

Theorem 8 (Generalized Goluzin inequality). If $g \in \Sigma(k, r)$ and $z_\nu \in \overline{C} - D$, $\gamma_\nu \in C (\nu = 1, 2, \ldots, n)$, $n = 1, 2, \ldots$, then

$$\left| \sum_{\mu} \sum_{\nu} \gamma_\mu \gamma_\nu \log \frac{g(z_\mu) - g(z_\nu)}{z_\mu - z_\nu} \right| \leq \frac{k + r^2}{1 + kr^2} \sum_{\mu} \sum_{\nu} \gamma_\mu \overline{\gamma_\nu} \log \frac{1}{1 - (z_\mu \overline{z_\nu})^{-1}}. \tag{9}$$

Proof. We shall apply the inequality (7) with $x_m = \sum_{\nu=1}^{N} \gamma_\nu z_\nu^{-m}$, $m = 1, 2, \cdots$. In fact, we
\[
\sum_{\mu} \sum_{v} \gamma_{\mu} \gamma_{v} \log \frac{g(z_{\mu}) - g(z_{v})}{z_{\mu} - z_{v}} = -\sum_{m,n} \sum_{\mu,v} A_{mn} \gamma_{\mu} \gamma_{v} z_{\mu}^{-m} z_{v}^{-n}
\]

\[
= -\sum_{m,n} A_{mn} x_{m} x_{n}.
\]

Hence (7) shows that the left-hand side of (9) is

\[
\leq \frac{1 + kr^{2}}{k + r^{2}} \sum_{n} \frac{1}{n} |x_{n}|^{2} = \frac{1 + kr^{2}}{k + r^{2}} \sum_{n} \frac{1}{n} \sum_{\mu,v} \gamma_{\mu} \overline{\gamma_{v}} z_{\mu}^{-k} z_{v}^{-k}
\]

\[
= \frac{1 + kr^{2}}{k + r^{2}} \sum_{\mu,v} \gamma_{\mu} \overline{\gamma_{v}} \log \frac{1}{1 - (z_{\mu} \overline{z_{v}})^{-1}}.
\]

This completes the proof of the theorem. \(\square\)

**Theorem 9.** For \(f \in S_{0}(k,R)\) and \(z \in D\),

\[
\left| \log \frac{zf'(z)}{f(z)} \right| \leq \frac{1 + kR}{k + R} \log \frac{1 + |z|}{1 - |z|}.
\]

**Proof.** In (9) let \(n = 2, \gamma_{1} = 1, \gamma_{2} = -1\), then

\[
\left| \log \frac{g'(z)g'(-\zeta)(z - \zeta)^{2}}{(g(z) - g(-\zeta))^{2}} \right| \leq \frac{k + r^{2}}{1 + kr^{2}} \log \frac{|z^{2} - 1|^{2}}{(|z|^{2} - 1)(|\zeta|^{2} - 1)} \quad (z, \zeta \in \hat{C} - D). \quad (10)
\]

We want to apply (9) to the function \(f \in S_{0}(k,R)\). If we put

\[
g(\zeta) = 1/\sqrt{f(\zeta^{-2})},
\]

then \(g \in \Sigma(k, 1/\sqrt{R})\). Since \(g\) is odd function, it follows from (10) with \(z = -\zeta\) that

\[
\left| 2 \log \frac{\xi g'(\xi)}{g(\xi)} \right| \leq \frac{k + (1/R)}{1 + k(1/R)} 2 \log \frac{|\xi|^{2} + 1}{|\xi|^{2} - 1} \quad (|\xi| > 1).
\]

If we choose \(z = \zeta^{-2}\) and use (11) we obtain the desired inequality. \(\square\)

**Corollary 10.** For \(f \in S_{0}(k,R)\) and \(z \in D\),

\[
\left( \frac{1 - |z|}{1 + |z|} \right)^{(1+kR)/(k+R)} \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \left( \frac{1 + |z|}{1 - |z|} \right)^{(1+kR)/(k+R)}.
\]
References


