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Coefficient conditions for certain classes concerning starlike functions of complex order

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Abstract

For functions $f(z)$ which are starlike of complex order $b$ ($b \neq 0$) in the open unit disk $U$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (a_0 = 0, \ a_1 = 1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in $U$. If $p(z) \in \mathcal{P}$ satisfies $\text{Re} \ p(z) > 0 (z \in U)$, then we say that $p(z)$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha$ ($0 \leq \alpha < 1$), then $f(z)$ is said to be starlike of order $\alpha$ in $U$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order $\alpha$ in $U$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ in $U$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha$ ($0 \leq \alpha < 1$).

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As usual, in the present investigation, we write
\[ S^* \equiv S^*(0) \quad \text{and} \quad \mathcal{K} \equiv \mathcal{K}(0). \]

Classes \( S^*(\alpha) \) and \( \mathcal{K}(\alpha) \) were introduced by Robertson [5].

Next, a function \( f(z) \in \mathcal{A} \) is called \( \lambda \)-spiral like of order \( \alpha \) in \( \mathbb{U} \) if and only if
\[
\Re \left[ e^{i\lambda} \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right] > 0 \quad (z \in \mathbb{U})
\]
for some real \( -\frac{\pi}{2} < \lambda < \frac{\pi}{2} \) and \( 0 \leq \alpha < 1 \). We denote this class by \( \mathcal{SP}(\lambda, \alpha) \).

Moreover, for some non-zero complex number \( b \), we consider the subclasses \( S_b^* \) and \( \mathcal{K}_b \) of \( \mathcal{A} \) as follows:
\[
S_b^* = \left\{ f(z) \in \mathcal{A} : \Re \left[ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right] > 0 \quad (b \neq 0; \ z \in \mathbb{U}) \right\}
\]
and
\[
\mathcal{K}_b = \left\{ f(z) \in \mathcal{A} : \Re \left[ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right] > 0 \quad (b \neq 0; \ z \in \mathbb{U}) \right\}.
\]

If a function \( f(z) \) belongs to the class \( S_b^* \) or \( \mathcal{K}_b \), we say that \( f(z) \) is starlike or convex of complex order \( b \) (\( b \neq 0 \)), respectively. In [3], Nasr and Aouf introduced the class \( S_b^* \).

Then, we can see that
\[
S_{1-\alpha}^* = S^*(\alpha), \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha) \quad \text{and} \quad S_{(1-\alpha)e^{-i\lambda}c\infty\lambda}^* = \mathcal{SP}(\lambda, \alpha).
\]

Example 1.1
\[
f(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n} (j+2(b-1))}{(n-1)!} z^n \in S_b^* \quad (b \neq 0)
\]
and
\[
f(z) = \begin{cases} \frac{1-(1-z)^{1-2b}}{1-2b} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n} (j+2(b-1))}{n!} z^n \in \mathcal{K}_b & (b \neq \frac{1}{2}) \\ \log \left( \frac{1}{1-z} \right) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n \in \mathcal{K}_1 = \mathcal{K} \left( \frac{1}{2} \right) \end{cases}
\]

We apply the following lemma to obtain our results.

Lemma 1.2 A function \( p(z) \in \mathcal{P} \) satisfies \( \Re p(z) > 0 \ (z \in \mathbb{U}) \) if and only if
\[
p(z) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})
\]
for all \( |x| = 1 \).
Then, by using Lemma 1.2, various conditions for starlike functions are studied. The following results are enumerated as the some examples.

**Lemma 1.3** A function $f(z) \in A$ is in $S^*(\alpha)$ if and only if

$$(1.3) \quad 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \quad (z \in U; \ |x| = 1)$$

where

$$A_n = \frac{n + 1 - 2\alpha + (n - 1)x}{2 - 2\alpha} a_n.$$

Silverman, Silvia, and Telage [6] have given

**Remark 1.4** The relation (1.3) of Lemma 1.3 is equivalent to

$$\frac{1}{z} \left( f(z) * \frac{z + x + 2\alpha - 1}{2 - 2\alpha} z^2 \right) \neq 0 \quad (z \in U, \ |x| = 1)$$

where $*$ means the convolution or Hadamard product of two functions.

Furthermore, letting $\alpha = 0$ in Lemma 1.3, Nezhmetdinov and Ponnusamy [4] have given the sufficient conditions for coefficients of $f(z)$ to be in the class $S^*$.

Hayami, Owa and Sirivastava [2] have shown the following results.

**Theorem 1.5** If $f(z) \in A$ satisfies the following condition

$$\sum_{n=2}^{\infty} \left\| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j+1-2\alpha)(-1)^{k-j} \binom{\beta k-j}{k-j} a_j \right\} \binom{\gamma n-k}{n-k} \right\| \leq 2(1-\alpha)$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in S^*(\alpha)$.

**Theorem 1.6** If $f(z) \in A$ satisfies the following condition

$$\sum_{n=2}^{\infty} \left\| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j+1-2\alpha)(-1)^{k-j} \binom{\beta k-j}{k-j} a_j \right\} \binom{\gamma n-k}{n-k} \right\| \leq 2(1-\alpha)$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{K}(\alpha)$.
Theorem 1.7 If \( f(z) \in A \) satisfies the following condition
\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left( \sum_{j=1}^{k} (j - \alpha + (1 - \alpha)e^{-2\lambda})(-1)^{k-j}\begin{pmatrix} \beta k-j \\ k-j \end{pmatrix}a_{j} \right) \right| \left( \begin{pmatrix} \gamma n-k \\ n-k \end{pmatrix} \right) + \sum_{k=1}^{\infty} \left| \sum_{j=1}^{k} (j-1)(-1)^{k-j}\begin{pmatrix} \beta k-j \\ k-j \end{pmatrix}a_{j} \right| \left( \begin{pmatrix} \gamma n-k \\ n-k \end{pmatrix} \right) \right| \leq 2(1-\alpha)\cos \lambda
\]
for some \( \alpha (0 \leq \alpha < 1) \), \( \lambda (-\frac{\pi}{2} < \lambda < \frac{\pi}{2}) \), \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R}_{l} \) then \( f(z) \in S\mathcal{P}(\lambda, \alpha) \).

2 Main results

Main result for starlike of complex order \( b \) is contained in

Theorem 2.1 If \( f(z) \in A \) satisfies the following condition
\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left( \sum_{j=1}^{k} (j - 1 + 2b)(-1)^{k-j}\begin{pmatrix} \beta k-j \\ k-j \end{pmatrix}a_{j} \right) \right| \left( \begin{pmatrix} \gamma n-k \\ n-k \end{pmatrix} \right) + \sum_{k=1}^{\infty} \left| \sum_{j=1}^{k} (j-1)(-1)^{k-j}\begin{pmatrix} \beta k-j \\ k-j \end{pmatrix}a_{j} \right| \left( \begin{pmatrix} \gamma n-k \\ n-k \end{pmatrix} \right) \right| \leq 2|b|
\]
for some \( b \in \mathbb{C} (b \neq 0) \), \( \beta \in \mathbb{R} \), and \( \gamma \in \mathbb{R} \), then \( f(z) \in S_{b}^{*} \).

Proof. Let us define the function \( p(z) \) by \( p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \) for \( f(z) \in A \). Applying Lemma 1.2, \( f(z) \in S_{b}^{*} \) if and only if
\[
p(z) = 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \neq \frac{x-1}{x+1} \quad (z \in \mathbb{U})
\]
for all \( |x| = 1 \).

Then, we need not consider Lemma 1.2 for \( z = 0 \), because it follows that
\[
p(0) = 1 \neq \frac{x-1}{x+1} \quad (|x| = 1).
\]

Hence, the relation (2.1) is equivalent to
\[
2bz + \sum_{n=2}^{\infty} \left\{ (n - 1 + 2b) + x(n - 1) \right\} n^2a_nz^n \neq 0.
\]

Dividing the both sides of (2.2) by \( 2bz \) \( (z \neq 0) \), we obtain that
\[
1 + \sum_{n=2}^{\infty} B_nz^{n-1} \neq 0
\]
where
\[
B_n = \frac{(n - 1 + 2b) + x(n - 1)}{2b} n^2a_n \quad (n \geq 2).
\]
Therefore, it is sufficient that we prove

\[
\left(1 + \sum_{n=2}^{\infty} B_n z^{n-1}\right) (1-z)^\beta (1+z)^\gamma = 1 + \sum_{n=2}^{\infty} \sum_{k=1}^{n} \{ \sum_{j=1}^{k} B_j (-1)^{k-j} \left(\begin{array}{l} \gamma k-j \\ k-j \end{array}\right) \left(\begin{array}{l} \delta n-k \\ n-k \end{array}\right) \} \right] z^{n-1} \neq 0
\]

where \( \beta, \gamma \in \mathbb{R} \) and \( B_1 = 1 \). Thus, if \( f(z) \) satisfies

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (j-1+2b)(-1)^{k-j} \left(\begin{array}{l} \beta k-j \\ k-j \end{array}\right) a_j \left(\begin{array}{l} \gamma n-k \\ n-k \end{array}\right) \right\} \right| + |x| \cdot \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j-1)(-1)^{k-j} \left(\begin{array}{l} \beta k-j \\ k-j \end{array}\right) a_j \left(\begin{array}{l} \gamma n-k \\ n-k \end{array}\right) \right\} \right| \leq 2|b|
\]

then \( f(z) \in S_b^* \). The proof of Theorem 2.1 is completed.

We next derive the coefficient condition for functions \( f(z) \) to be in the class \( \mathcal{K}_b \).

**Theorem 2.2** If \( f(z) \in \mathcal{A} \) satisfies the following condition

\[
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j-1+2b)(-1)^{k-j} \left(\begin{array}{l} \beta k-j \\ k-j \end{array}\right) a_j \right\} \right| + |x| \cdot \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} j(j-1)(-1)^{k-j} \left(\begin{array}{l} \beta k-j \\ k-j \end{array}\right) a_j \right\} \right| \leq 2|b|
\]

for some \( b \in \mathbb{C} (b \neq 0) \), \( \beta \in \mathbb{R} \), and \( \gamma \in \mathbb{R} \), then \( f(z) \in \mathcal{K}_b \).

**Proof.** Since \( zf'(z) \in S_b^* \) if and only if \( f(z) \in \mathcal{K}_b \) and since

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n,
\]

replacing \( a_j \) in Theorem 2.1 by \( ja_j \), we easily prove Theorem 2.2.

Putting \( \beta = \gamma = 0 \) in Theorem 2.1 and Theorem 2.2, we have

**Corollary 2.3** If \( f(z) \in \mathcal{A} \) satisfies the following inequality

\[
\sum_{n=2}^{\infty} \{|n-1+2b| + (n-1)|a_n| \leq 2|b|
\]

for some \( b \in \mathbb{C} (b \neq 0) \), then \( f(z) \in S_b^* \).
Corollary 2.4 If $f(z) \in A$ satisfies the following inequality

$$
\sum_{n=2}^{\infty} n \left\{ |n - 1 + 2b| + (n - 1) \right\} |a_n| \leq 2|b|
$$

for some $b \in \mathbb{C} \ (b \neq 0)$, then $f(z) \in \mathcal{K}_b$.

Finally, taking $b = 1 - \alpha$ in Theorem 2.1 and Theorem 2.2, or $b = (1 - \alpha) e^{-\lambda} \cos \lambda$ in Theorem 2.1, we arrive Theorem 1.5, Theorem 1.6 and Theorem 1.7.

References


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