数理解析研究所講究録
第 1626 巻 2009 年 1-6

Coefficient conditions for certain classes concerning starlike functions of complex order

Toshio Hayami and Shigeyoshi Owa

Abstract

For functions $f(z)$ which are starlike of complex order b ($b \neq 0$) in the open unit disk $\mathbb{U}$, some interesting sufficient conditions for coefficient inequalities of $f(z)$ are discussed.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n} \qquad (a_{0}=0,\ a_{1}=1)$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

(1.2) $$p(z)=1+\sum_{n=1}^{\infty}p_{n}z^{n}$$

which are analytic in $\mathbb{U}$. If $p(z)\in\mathcal{P}$ satisfies $\mathrm{Re}\ p(z)>0$ ($z\in\mathbb{U}$), then we say that $p(z)$ is the Carathéodory function (cf. [1]).

If $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some α ($0\leqq\alpha<1$), then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^{*}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Similary, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z)\in\mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some α ($0\leqq\alpha<1$).

2000 Mathematics Subject Classification: Primary 30C45.
Keywords and Phrases: Coefficient inequality, analytic function, univalent function, starlike function of complex order, λ-spiral like function.

As usual, in the present investigation, we write

$$\mathcal{S}^* \equiv \mathcal{S}^*(0) \quad \text{and} \quad \mathcal{K} \equiv \mathcal{K}(0).$$

Classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were introduced by Robertson [5].

Next, a function $f(z) \in \mathcal{A}$ is called λ-spiral like of order α in $\mathbb{U}$ if and only if

$$\operatorname{Re}\left[e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right] > 0 \qquad (z \in \mathbb{U})$$

for some real λ $(-\frac{\pi}{2} < \lambda < \frac{\pi}{2})$ and α $(0 \leqq \alpha < 1)$. We denote this class by $\mathcal{SP}(\lambda, \alpha)$.

Moreover, for some non-zero complex number b, we consider the subclasses $\mathcal{S}_b^*$ and $\mathcal{K}_b$ of $\mathcal{A}$ as follows:

$$\mathcal{S}_b^* = \left\{ f(z) \in \mathcal{A} \ : \ \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] > 0 \quad (b \neq 0;\ z \in \mathbb{U}) \right\}$$

and

$$\mathcal{K}_b = \left\{ f(z) \in \mathcal{A} \ : \ \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{zf''(z)}{f'(z)}\right)\right] > 0 \quad (b \neq 0;\ z \in \mathbb{U}) \right\}.$$

If a function $f(z)$ belongs to the class $\mathcal{S}_b^*$ or $\mathcal{K}_b$, we say that $f(z)$ is starlike or convex of complex order b $(b \neq 0)$, respectively. In [3], Nasr and Aouf introduced the class $\mathcal{S}_b^*$.

Then, we can see that

$$\mathcal{S}_{1-\alpha}^* = \mathcal{S}^*(\alpha), \quad \mathcal{K}_{1-\alpha} = \mathcal{K}(\alpha) \quad \text{and} \quad \mathcal{S}_{(1-\alpha)e^{-i\lambda}\cos\lambda}^* = \mathcal{SP}(\lambda, \alpha).$$

Example 1.1

$$f(z) = \frac{z}{(1-z)^{2b}} = z + \sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}\left(j + 2(b-1)\right)}{(n-1)!} z^n \in \mathcal{S}_b^* \qquad (b \neq 0)$$

and

$$f(z) = \begin{cases} \dfrac{1-(1-z)^{1-2b}}{1-2b} = z + \displaystyle\sum_{n=2}^{\infty} \frac{\prod_{j=2}^{n}\left(j + 2(b-1)\right)}{n!} z^n \in \mathcal{K}_b & \left(b \neq \dfrac{1}{2}\right) \\[2ex] \log\left(\dfrac{1}{1-z}\right) = z + \displaystyle\sum_{n=2}^{\infty} \frac{1}{n} z^n \in \mathcal{K}_{\frac{1}{2}} = \mathcal{K}\left(\tfrac{1}{2}\right). & \end{cases}$$

We apply the following lemma to obtain our results.

Lemma 1.2 *A function $p(z) \in \mathcal{P}$ satisfies* $\operatorname{Re} p(z) > 0$ $(z \in \mathbb{U})$ *if and only if*

$$p(z) \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$.

Then, by using Lemma 1.2, various conditions for starlike functions are studied. The following results are enumerated as the some examples.

Lemma 1.3 *A function $f(z) \in \mathcal{A}$ is in $\mathcal{S}^*(\alpha)$ if and only if*

$$1+\sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \qquad (z \in \mathbb{U};\ |x|=1) \tag{1.3}$$

where

$$A_n = \frac{n+1-2\alpha+(n-1)x}{2-2\alpha} a_n.$$

Silverman, Silvia, and Telage [6] have given

Remark 1.4 The relation (1.3) of Lemma 1.3 is equivalent to

$$\frac{1}{z}\left(f(z) * \frac{z+\dfrac{x+2\alpha-1}{2-2\alpha}z^2}{(1-z)^2} \right) \neq 0 \qquad (z \in \mathbb{U},\ |x|=1)$$

where $*$ means the convolution or Hadamard product of two functions.

Furthermore, letting $\alpha = 0$ in Lemma 1.3, Nezhmetdinov and Ponnusamy [4] have given the sufficient conditions for coefficients of $f(z)$ to be in the class $\mathcal{S}^*$.

Hayami, Owa and Sirivastava [2] have shown the following results.

Theorem 1.5 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j+1-2\alpha)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2(1-\alpha)$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{S}^*(\alpha)$.

Theorem 1.6 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j+1-2\alpha)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2(1-\alpha)$$

for some α $(0 \leqq \alpha < 1)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{K}(\alpha)$.

Theorem 1.7 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-\alpha+(1-\alpha)e^{-2i\lambda})(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{\infty}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2(1-\alpha)\cos\lambda$$

for some α $(0 \leqq \alpha < 1)$, λ $\left(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\right)$, $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{SP}(\lambda, \alpha)$.

2 Main results

Main result for starlike of complex order b is contained in

Theorem 2.1 *If $f(z) \in \mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1+2b)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right] \leqq 2|b|$$

for some $b \in \mathbb{C}$ $(b \neq 0)$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, then $f(z) \in \mathcal{S}_b^*$.

Proof. Let us define the function $p(z)$ by $p(z) = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)$ for $f(z) \in \mathcal{A}$. Applying Lemma 1.2, $f(z) \in \mathcal{S}_b^*$ if and only if

(2.1)
$$p(z) = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \neq \frac{x-1}{x+1} \qquad (z \in \mathbb{U})$$

for all $|x| = 1$.
Then, we need not consider Lemma 1.2 for $z = 0$, because it follows that

$$p(0) = 1 \neq \frac{x-1}{x+1} \qquad (|x| = 1).$$

Hence, the relation (2.1) is equivalent to

(2.2)
$$2bz + \sum_{n=2}^{\infty}\left\{(n-1+2b) + x(n-1)\right\}n^j a_n z^n \neq 0.$$

Dividing the both sides of (2.2) by $2bz$ $(z \neq 0)$, we obtain that

$$1 + \sum_{n=2}^{\infty} B_n z^{n-1} \neq 0$$

where

$$B_n = \frac{(n-1+2b) + x(n-1)}{2b} n^j a_n \quad (n \geqq 2).$$

Therefore, it is sufficient that we prove

$$\left(1+\sum_{n=2}^{\infty}B_n z^{n-1}\right)(1-z)^{\beta}(1+z)^{\gamma}=1+\sum_{n=2}^{\infty}\left[\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}B_j(-1)^{k-j}\binom{\gamma}{k-j}\right\}\binom{\delta}{n-k}\right]z^{n-1}\neq 0$$

where $\beta,\gamma\in\mathbb{R}$ and $B_1=1$. Thus, if $f(z)$ satisfies

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1+2b)(-1)^{k-j}\binom{\gamma}{k-j}a_j\right\}\binom{\delta}{n-k}\right|\right.$$
$$\left.+|x|\cdot\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}(j-1)(-1)^{k-j}\binom{\gamma}{k-j}a_j\right\}\binom{\delta}{n-k}\right|\right]\leqq 2|b|$$

then $f(z)\in\mathcal{S}_b^*$. The proof of Theorem 2.1 is completed. □

We next derive the coefficient condition for functions $f(z)$ to be in the class $\mathcal{K}_b$.

Theorem 2.2 *If $f(z)\in\mathcal{A}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j-1+2b)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left\{\sum_{j=1}^{k}j(j-1)(-1)^{k-j}\binom{\beta}{k-j}a_j\right\}\binom{\gamma}{n-k}\right|\right]\leqq 2|b|$$

for some $b\in\mathbb{C}$ $(b\neq 0)$, $\beta\in\mathbb{R}$, and $\gamma\in\mathbb{R}$, then $f(z)\in\mathcal{K}_b$.

Proof. Since $zf'(z)\in\mathcal{S}_b^*$ if and only if $f(z)\in\mathcal{K}_b$ and since

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad\text{and}\quad zf'(z)=z+\sum_{n=2}^{\infty}na_n z^n,$$

replacing a_j in Theorem 2.1 by ja_j, we easily prove Theorem 2.2. □

Putting $\beta=\gamma=0$ in Theorem 2.1 and Theorem 2.2, we have

Corollary 2.3 *If $f(z)\in\mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty}\Big\{|n-1+2b|+(n-1)\Big\}|a_n|\leqq 2|b|$$

for some $b\in\mathbb{C}$ $(b\neq 0)$, then $f(z)\in\mathcal{S}_b^*$.

Corollary 2.4 *If $f(z) \in \mathcal{A}$ satisfies the following inequality*

$$\sum_{n=2}^{\infty} n\Big\{|n-1+2b|+(n-1)\Big\}|a_n| \leqq 2|b|$$

for some $b \in \mathbb{C}$ $(b \neq 0)$, then $f(z) \in \mathcal{K}_b$.

Finally, taking $b = 1-\alpha$ in Theorem 2.1 and Theorem 2.2, or $b = (1-\alpha)e^{-i\lambda}\cos\lambda$ in Theorem 2.1, we arrive Theorem 1.5, Theorem 1.6 and Theorem 1.7.

References

[1] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] T. Hayami, S. Owa and H. M. Srivastava, *Coefficient inequalities for certain classes of analytic and univalent functions*, J. Ineq. Pure and Appl. Math., **8**(4) Article 95 (2007), 1-10.

[3] M. A. Nasr and M. K. Aouf, *Starlike functions of complex order*, J. Natural Sci. Math., Vol.25, No 1 (1985), 1-12.

[4] I. R. Nezhmetdinov and S. Ponnusamy, *New coefficient conditions for the starlikeness of analytic functions and their applications*, Houston J. Math. **31**, No. 2 (2005), 587-604.

[5] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. **37** (1936), 374-408.

[6] H. Silverman, E. M. Silvia, and D. Telage, *Convolution conditions for convexity, starlikeness and spiral-likeness*, Math. Z., **162** (1978), 125-130.

Toshio Hayami
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: ha_ya_to112@hotmail.com

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: owa@math.kindai.ac.jp