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Stationary isothermic surfaces and a characterization of the spherical cylinder *

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1 Introduction

This is based on the author's recent work with R. Magnanini [MS4].

Let Ω be a domain in $\mathbb{R}^N$ with $N \geq 3$, and let $u = u(x,t)$ be the unique bounded solution of the following problem for the heat equation:

$$\partial_t u = \Delta u \quad \text{in } \Omega \times (0, +\infty), \tag{1.1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}. \tag{1.3}$$

The problem we consider is to characterize the boundary $\partial\Omega$ such that the solution u has a stationary isothermic surface, say Γ. A hypersurface Γ in Ω is said to be a *stationary isothermic surface* of u if at each time t the solution u remains constant on Γ (a constant depending on t). Examples we easily notice are isoparametric hypersurfaces. Namely, Γ and $\partial\Omega$ are either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. This complete classification of isoparametric hypersurfaces was given by Levi-Civita [LC] and Segre [Seg].

Almost complete characterizations of the sphere have already been obtained by [MS1, MS3] with the help of Aleksandrov's sphere theorem [Alek], and some

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characterizations of the hyperplane have been given by [MS3]. In this note, we give a characterization of the spherical cylinder in $\mathbb{R}^3$. See Theorem 3.1 in Section 3.

2 A preliminary lemma

Before proceeding to the spherical cylinder in $\mathbb{R}^3$, we consider general domains in $\mathbb{R}^N$ with unbounded boundaries. Hereafter in this note, we assume the following: Ω satisfies the uniform exterior sphere condition; Γ is a stationary isothermic surface of u; there exists a domain D in $\mathbb{R}^N$ with $\overline{D} \subset \Omega$ such that Γ equals a connected component of ∂D; $\mathrm{dist}(\Gamma, \partial\Omega) = \mathrm{dist}(\overline{D}, \partial\Omega)$; D satisfies the interior cone condition with respect to Γ.

We recall that Ω satisfies the *uniform exterior sphere condition* if there exists a number $r_0 > 0$ such that for every $\xi \in \partial\Omega$ there exists a ball $B_{r_0}(y)$ satisfying $\overline{B_{r_0}(y)} \cap \overline{\Omega} = \{\xi\}$, where $B_{r_0}(y)$ denotes an open ball centered at $y \in \mathbb{R}^N$ and with radius $r_0 > 0$. Also, D satisfies the *interior cone condition* with respect to Γ if for every $x \in \Gamma$ there exists a finite right spherical cone V_x with vertex x such that $V_x \subset \overline{D}$ and $\overline{V_x} \cap \partial D = \{x\}$.

Let $d = d(x)$ be the distance function defined by

$$d(x) = \mathrm{dist}(x, \partial\Omega), \quad x \in \Omega. \tag{2.1}$$

We start with a lemma.

Lemma 2.1 *The following assertions hold:*

(1) *There exists a number $R > 0$ such that $d(x) = R$ for every $x \in \Gamma$;*

(2) *Γ is a real analytic hypersurface;*

(3) *there exists a connected component γ of $\partial\Omega$ such that γ is also a real analytic hypersurface, and the mapping: $\gamma \ni \xi \mapsto x(\xi) \equiv \xi + R\nu(\xi) \in \Gamma$ is a diffeomorphism, where $\nu(\xi)$ denotes the inward unit normal vector to $\partial\Omega$ at $\xi \in \gamma$, that is, γ and Γ are parallel hypersurfaces with distance R;*

(4) *the following inequality holds:*

$$-\frac{1}{r_0} \le \kappa_j(\xi) < \frac{1}{R} \ (j = 1, \cdots, N-1) \ \textit{for every } \xi \in \partial\Omega, \tag{2.2}$$

where $\kappa_1(\xi), \cdots, \kappa_{N-1}(\xi)$ *denote the principal curvatures of* $\partial\Omega$ *at* $\xi \in \gamma$ *with respect to the inward unit normal vector to* $\partial\Omega$*, and where* $r_0 > 0$ *is the radius of the uniform exterior sphere condition for* Ω*;*

(5) *there exists a number* $c > 0$ *satisfying*

$$\prod_{j=1}^{N-1}\left(\frac{1}{R} - \kappa_j(\xi)\right) = c \quad \textit{for every } \xi \in \gamma. \tag{2.3}$$

Proof. Since Γ is stationary isothermic, (1) follows from a result of Varadhan [Va]:

$$-\frac{1}{\sqrt{s}}\log W(x,s) \to d(x) \quad \text{as } s \to \infty,$$

where $W = W(x,s)$ $(x \in \Omega,\ s > 0)$ is defined by

$$W(x,s) = s\int_0^\infty u(x,t)e^{-st}dt. \tag{2.4}$$

The inequality $-\frac{1}{r_0} \le \kappa_j(\xi)$ in (2.2) follows from the uniform exterior sphere condition for Ω. See Lemma 2.2 of [MS3] together with Lemma 3.1 of [MS1] for the remainder.

Finally, we illustrate an outline of the proof of (5), since it is helpful in understanding Theorem 3.2 in Section 3, which is the key to our characterization of the spherical cylinder. We use a balance law stated as follows: Let G be a domain in $\mathbb{R}^N$. For $x_0 \in G$, a solution $v = v(x,t)$ of the heat equation in $G \times (0,+\infty)$ is such that $v(x_0,t) = 0$ for every $t > 0$ if and only if

$$\int_{\partial B_r(x_0)} v(x,t)\, dS_x = 0, \text{ for every } r \in [0,\ \mathrm{dist}(x_0,\partial G)) \text{ and every } t > 0. \tag{2.5}$$

See [MS1] for a proof of this balance law. Let $P, Q \in \gamma$ be two distinct points, and let $p, q \in \Gamma$ be the points such that

$$\overline{B_R(p)} \cap \partial\Omega = \{P\} \quad \text{and} \quad \overline{B_R(q)} \cap \partial\Omega = \{Q\}.$$

Consider the function $v = v(x,t)$ defined by

$$v(x,t) = u(x+p,t) - u(x+q,t) \quad \text{for} \quad (x,t) \in B_R(0) \times (0,+\infty).$$

Since v satisfies the heat equation and $v(0,t) = 0$ for every $t > 0$, it follows from (2.5) that

$$\int_{B_R(p)} u(x,t)\,dx = \int_{B_R(q)} u(x,t)\,dx \quad \text{for every} \quad t > 0.$$

Therefore we obtain

$$\int_{B_R(p)} W(x,s)\,dx = \int_{B_R(q)} W(x,s)\,dx \quad \text{for every} \quad s > 0. \tag{2.6}$$

By using upper and lower barriers for W near γ in [MS3]:

$$W_\varepsilon^{\pm}(x,s) = \exp\left\{-\sqrt{s(1 \mp \varepsilon)}\; d(x)\right\}, \quad 0 < \varepsilon < 1, \tag{2.7}$$

and by integrating on the level surfaces of d with the co-area formula as in [MS2], we have that as $s \to \infty$

$$\int_{B_R(p)} W(x,s)\,dx = C_N \left\{\prod_{j=1}^{N-1}\left(\frac{1}{R} - \kappa_j(P)\right)\right\}^{-\frac{1}{2}} s^{-\frac{N+1}{4}} + o\left(s^{-\frac{N+1}{4}}\right) \tag{2.8}$$

for some positive constant C_N depending only on N. This together with (2.6) implies (2.3). □

3 A class of domains whose boundaries are unbounded surfaces of revolution and our main theorem

In this section we consider a class of domains Ω where each boundary $\partial\Omega$ is an unbounded surface of revolution in $\mathbb{R}^3$. Precisely, let $r = r(x_1)$ ($x_1 \in \mathbb{R}$) be a continuous positive function on $\mathbb{R}$, and consider a domain Ω defined by

$$\Omega = \{\, x = (x_1, x_2, x_3) \in \mathbb{R}^3 \;:\; x_2^2 + x_3^2 < \{r(x_1)\}^2 \,\}. \tag{3.1}$$

Then, in view of Lemma 2.1, since $\partial\Omega$ is connected, $\gamma = \partial\Omega$ and

$$\Gamma = \{\, x \in \Omega \;:\; d(x) = R \,\}.$$

Moreover, r is real analytic and equation (2.3) is written as

$$\left(\frac{1}{R}+\frac{\ddot{r}}{(1+(\dot{r})^2)^{\frac{3}{2}}}\right)\left(\frac{1}{R}-\frac{1}{(1+(\dot{r})^2)^{\frac{1}{2}}r}\right)=c \quad \text{on } \mathbb{R}, \tag{3.2}$$

where $\dot{r}=\frac{d}{dx_1}r$ and $\ddot{r}=\frac{d^2}{dx_1^2}r$, and ODE analysis yields that only the following three possibilities (I), (II), and (III) occur (see [MS4] for the details):

(I) r is constant on $\mathbb{R}$, that is, $\partial\Omega$ is a spherical cylinder;

(II) $c=\frac{1}{R^2}$, $\ddot{r}>0$ on $\mathbb{R}$, r has only one minimum point, say $x_1=a$, satisfying $r(2a-x_1)=r(x_1)$ for all $x_1\in\mathbb{R}$, the Gaussian curvature $K=K(x)(=\kappa_1(x)\kappa_2(x))$ of $\partial\Omega$ is negative at $x=(a,x_2,x_3)$, $\lim_{|x_1|\to\infty} r(x_1)=\infty$, and $\lim_{|x_1|\to\infty}K(x)=0$. Moreover, $\partial\Omega$ is parallel to a catenoid, namely, the surface $\{\ x\in\Omega\ :\ d(x)=\frac{R}{2}\ \}$ is a catenoid;

(III) $c<\frac{1}{R^2}$ and r is periodic on $\mathbb{R}$. Precisely, there exist $a\in\mathbb{R}$ and $\ell>0$ satisfying

$$r(x_1+2\ell)=r(x_1)\ \text{ for all } x\in\mathbb{R},\ \dot{r}(a+k\ell)=0\ \text{ for all } k\in\mathbb{Z},$$

$$\dot{r}(x_1)<0\ \text{ if } x_1\in(a-\ell,a),\ \text{ and } \dot{r}(x_1)>0\ \text{ if } x_1\in(a,a+\ell).$$

Moreover, the Gaussian curvature $K=K(x)$ of $\partial\Omega$ is negative at $x=(a,x_2,x_3)$ and positive at $x=(a+\ell,x_2,x_3)$.

We are in a position to state our main theorem:

Theorem 3.1 *In cases* (II) *and* (III), Γ *can not be a stationary isothermic surface of* u.

We will call several points $P\in\partial\Omega$ in cases (II) and (III) "*symmetric*" as follows: In case (II), any point $(a,x_2,x_3)\in\partial\Omega$ and any ideal point at infinity as $|x_1|\to\infty$ are called *symmetric*, since $\partial\Omega$ is symmetric with respect to the hyperplane $x_1=a$ and $\partial\Omega$ becomes flatter and flatter as $|x_1|\to\infty$. In case (III), any point $(a+k\ell,x_2,x_3)\in\partial\Omega$ ($k\in\mathbb{Z}$) is called *symmetric*, since $\partial\Omega$ is symmetric with respect to the hyperplane $x_1=a+k\ell$ for any $k\in\mathbb{Z}$.

For two positive constants c and R in Lemma 2.1, we set

$$T = cR^2 \quad \text{and hence in cases (II) and (III)} \quad 0 < T \leq 1. \tag{3.3}$$

Then it follows from (2.3) that

$$(1 - R\kappa_1)(1 - R\kappa_2) = T. \tag{3.4}$$

Our main theorem is drawn from the following asymptotic formula for the integral $\int_{B_R(p)} W(x,s)\, dx$ which is more precise than formula (2.8).

Theorem 3.2 *For each symmetric point $P \in \partial\Omega$ in cases* (II) *and* (III)*, let $p \in \Gamma$ be a unique point with $\overline{B_R(p)} \cap \partial\Omega = \{P\}$. Then, as $s \to \infty$,*

$$\int_{B_R(p)} W(x,s)\, dx = \frac{2\pi R}{T^{\frac{1}{2}}} s^{-1} + \Psi(K(P)) s^{-\frac{3}{2}} + O(s^{-2}), \tag{3.5}$$

where $\Psi = \Psi(K)$ is a quadratic function of K satisfying

$$\Psi'(K) < 0 \quad \text{if } K \leq \frac{1}{R^2}. \tag{3.6}$$

More precisely, Ψ is given by

$$\Psi(K) = C(R,T) + \frac{\pi R^2}{12 T^{\frac{5}{2}}} \left[3R^2(21 - T)K^2 - (134 + 6T^2)K \right], \tag{3.7}$$

where $C(R,T)$ is a constant depending only on R and T.

Remark. In order to apply Theorem 3.2 to any ideal point at infinity in case (II), we use a kind of blowup argument to reduce the problem to the case where $\partial\Omega$ and Γ are parallel planes with distance R. See [MS4] for the details.

4 Outline of the proof of Theorem 3.2

Throughout this section, see [MS4] for the details. For $\rho > 0$, we set

$$\Gamma_\rho = \{ \, x \in \Omega \; : \; d(x) = \rho \, \}. \tag{4.1}$$

Let us begin with a purely geometrical lemma.

Lemma 4.1 *For each symmetric point $P \in \partial\Omega$ in cases* (II) *and* (III)*, let $p \in \Gamma$ be a unique point with $\overline{B_R(p)} \cap \partial\Omega = \{P\}$. Then, as $\rho \to 0$,*

$$|\Gamma_\rho \cap B_R(p)| = \frac{2\pi R}{T^{\frac{1}{2}}}\rho + \psi(K(P))\rho^2 + O(\rho^3), \tag{4.2}$$

where $|\cdot|$ indicates the 2-dimensional Hausdorff measure of sets and $\psi = \psi(K)$ is a quadratic function of K given by

$$\psi(K) = C_1(R,T) + \frac{\pi R^2}{24T^{\frac{5}{2}}}\left[3R^2(21-T)K^2 - (134+30T^2)K\right], \tag{4.3}$$

where $C_1(R,T)$ is a constant depending only on R and T.

We write

$$\mathcal{N} = \left\{ x \in \Omega \ : \ d(x) \leq \frac{R}{2} \right\}. \tag{4.4}$$

By Lemma 2.1, for each point $x \in \mathcal{N}$, there exits a unique point $z(x) \in \partial\Omega$ with $\overline{B_{d(x)}(x)} \cap \partial\Omega = \{z(x)\}$. Then we set

$$x(t) = z(x) + t\nu(z(x)) \ \text{ for } 0 \leq t \leq d(x), \tag{4.5}$$

where $\nu(z(x))$ denotes the unit inward normal vector to $\partial\Omega$ at $z(x) \in \partial\Omega$.

Let us introduce upper and lower barriers $U^\pm$ for W in $\mathcal{N}$ which are more precise than $W_\varepsilon^\pm$ given by (2.7). By setting

$$\begin{aligned} A_0(x) &= \left\{\prod_{j=1}^{2}(1-\kappa_j(z(x))d(x))\right\}^{-\frac{1}{2}} \quad \text{and} \\ A^\pm(x) &= \int_0^{d(x)} \left(\frac{1}{2}\Delta A_0 \pm 1\right)(x(t)) \exp\left\{-\frac{1}{2}\int_t^{d(x)} \Delta d(x(s))ds\right\} dt, \end{aligned}$$

we define $U^\pm = U^\pm(x,s)$ $(x \in \mathcal{N},\ s > 0)$ by

$$U^\pm(x,s) = \exp\{-\sqrt{s}\, d(x)\}\left(A_0(x) + \frac{1}{\sqrt{s}}A^\pm(x)\right) \pm \exp\left\{-\frac{R}{4}\sqrt{s}\right\}. \tag{4.6}$$

Then we have

Lemma 4.2 *There exists $s_0 > 0$ such that, for any $s \geq s_0$ and for any $x \in \mathcal{N}$,*

$$U^-(x,s) \leq W(x,s) \leq U^+(x,s). \tag{4.7}$$

Another simple lemma is the following.

Lemma 4.3 *For each symmetric point $P \in \partial\Omega$ in cases* (II) *and* (III), *let $p \in \Gamma$ be a unique point with $\overline{B_R(p)} \cap \partial\Omega = \{P\}$. Then, as $\rho \to 0$,*

$$\kappa_j(z(x)) = \kappa_j(P) + O(\rho) \quad \textit{for } x \in \Gamma_\rho \cap B_R(p) \textit{ and for } j = 1, 2. \tag{4.8}$$

This lemma follows from the fact that $\frac{\partial \kappa_j}{\partial x_1}(P) = 0$.

By the co-area formula, we have

$$\int_{B_R(p)} W(x,s)\ dx = \int_0^{\frac{R}{2}} \left(\int_{\Gamma_\rho \cap B_R(p)} W(x,s)\ dS_x \right) d\rho + \int_{B_R(p)\setminus\mathcal{N}} W(x,s)\ dx.$$

Since the second term of the right-hand side decays exponentially as $s \to \infty$, it suffices to consider the first term. We estimate the first term with the help of Lemmas 4.1, 4.2, and 4.3, and hence we can prove Theorem 3.2.

Lemma 4.1 is purely geometrical, but we need hard computations to prove it (see [MS4] for the details). Here, we introduce one simple and useful lemma.

Lemma 4.4 *For each symmetric point $P \in \partial\Omega$ in cases* (II) *and* (III), *let $p \in \Gamma$ be a unique point with $\overline{B_R(p)} \cap \partial\Omega = \{P\}$. Then, for $0 < \rho \le \frac{R}{2}$,*

$$\int_0^\rho |\Gamma_t \cap B_R(p)|\ dt = \int_0^\rho |\Omega_\rho \cap \partial B_{R-\rho+t}(p)|\ dt, \tag{4.9}$$

where

$$\Omega_\rho = \{\ x \in \Omega\ :\ d(x) < \rho\ \}.$$

With the help of this lemma, we compute the right-hand side of (4.9) and then we proceed to Lemma 4.1. Since each set $\Omega_\rho \cap \partial B_{R-\rho+t}(p)$ is a subset of the sphere $\partial B_{R-\rho+t}(p)$, it is easier to compute the integrand of the right-hand side of (4.9).

5 Concluding remarks

When Ω is outside an unbounded surface of revolution, that is, when Ω is defined by

$$\Omega = \{\ x = (x_1, x_2, x_3) \in \mathbb{R}^3\ :\ x_2^2 + x_3^2 > \{r(x_1)\}^2\ \}, \tag{5.1}$$

equation (2.3) is written as

$$\left(\frac{1}{R}-\frac{\ddot{r}}{(1+(\dot{r})^2)^{\frac{3}{2}}}\right)\left(\frac{1}{R}+\frac{1}{(1+(\dot{r})^2)^{\frac{1}{2}}r}\right)=c \quad \text{on } \mathbb{R}. \tag{5.2}$$

Similarly, ODE analysis yields cases (I), (II), and (III), where in (III) $c < \frac{1}{R^2}$ is replaced by $c > \frac{1}{R^2}$. We also have Theorem 3.2. However, in case (III), we have $T > 1$, and hence, in (3.7) of Theorem 3.2, we have possibilities: $3R^2(21 - T) < 0$, $K > \frac{1}{R^2}$. We might need a little bit more consideration to draw Theorem 3.1 from Theorem 3.2.

In higher dimensional case, $N \geq 4$, we can consider for instance a domain Ω in $\mathbb{R}^N$ defined by

$$\Omega = \{\ x = (x_1, \cdots, x_N) \in \mathbb{R}^N\ :\ x_2^2 + \cdots + x_N^2 < \{r(x_1)\}^2\ \}.$$

In this case, equation (2.3) is written as

$$\left(\frac{1}{R}+\frac{\ddot{r}}{(1+(\dot{r})^2)^{\frac{3}{2}}}\right)\left(\frac{1}{R}-\frac{1}{(1+(\dot{r})^2)^{\frac{1}{2}}r}\right)^{N-2}=c \quad \text{on } \mathbb{R}. \tag{5.3}$$

Similarly, ODE analysis yields cases (I), (II), and (III), where R^2 is replaced by R^{N-1}. However, it seems harder to prove a theorem replacing Theorem 3.2.

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