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Asymptotic behavior of least energy solutions for a biharmonic problem with nearly critical growth

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1 Introduction

In this note, we concern the asymptotic behavior of blowing-up solutions to the fourth order semilinear problem

$$(P_{\varepsilon,K})\begin{cases}\Delta^2 u = c_0 K(x) u^{p_\varepsilon} & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=\Delta u = 0 & \text{on } \partial\Omega\end{cases}$$

as $\varepsilon \to +0$. Here, Ω is a smooth bounded domain in $\mathbb{R}^N (N\geq 5)$, $c_0 = (N-4)(N-2)N(N+2)$, $\varepsilon>0$ is a small positive parameter, $p_\varepsilon = p-\varepsilon$, $p=(N+4)/(N-4)$ is the critical Sobolev exponent from the view point of the Sobolev embedding $H^2\cap H_0^1(\Omega)\hookrightarrow L^{p+1}(\Omega)$, and $K\in C^2(\overline{\Omega})$ is a given positive function.

When $K\equiv 1$, Chou and Geng [1] obtained a result corresponding to the one of Han [5] on a strictly convex domain Ω for solutions u_ε minimizing the Sobolev quotient:

$$\frac{\|u_\varepsilon\|^2_{H^2\cap H_0^1(\Omega)}}{\|u_\varepsilon\|^2_{L^{p_\varepsilon+1}(\Omega)}} \to S \quad \text{as } \varepsilon\to 0.$$

Here

$$\|u_\varepsilon\|_{H^2\cap H_0^1(\Omega)} = \left(\int_\Omega |\Delta u|^2 dx\right)^{1/2}$$

is the norm of the Hilbert space $H^2 \cap H_0^1(\Omega)$, and $S = \inf\{\int_\Omega |\Delta u|^2 dx | u \in H^2 \cap H_0^1(\Omega), \|u\|_{L^{p+1}(\Omega)} = 1\}$ is the best Sobolev constant of the embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. In particular, they proved that the blow up point of solutions minimizing the Sobolev quotient is a critical point of the Robin function associated with the Green function under the Navier boundary condition.

Also when $K \not\equiv 1$, there always exists a function $\overline{u}_\varepsilon$ satisfying

$$\frac{\int_\Omega |\Delta \overline{u}_\varepsilon|^2 dx}{\left(\int_\Omega K(x)|\overline{u}_\varepsilon|^{p_\varepsilon+1} dx\right)^{2/(p_\varepsilon+1)}} = \inf_{u\in H^2\cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega K(x)|u|^{p_\varepsilon+1} dx\right)^{2/(p_\varepsilon+1)}}.$$

We may assume $\overline{u}_\varepsilon > 0$ by solving the equation $-\Delta v = |\Delta \overline{u}_\varepsilon|, v \in H^2 \cap H_0^1(\Omega)$; see [9]. Thus, an appropriate constant multiple of $\overline{u}_\varepsilon$ is a solution of $(P_{\varepsilon,K})$, which we call a *least energy solution* to $(P_{\varepsilon,K})$. In the following, we will treat only least energy solutions to $(P_{\varepsilon,K})$.

For non constant K, least energy solutions $\{u_\varepsilon\}$ are known to blow up at one point x_0, which is a maximum point of K in $\overline{\Omega}$:

$$\|u_\varepsilon\|_{L^\infty(\Omega)} = u_\varepsilon(x_\varepsilon) \to \infty \quad \text{and } x_\varepsilon \to x_0 \in K^{-1}(\max_{\overline{\Omega}} K). \tag{1.1}$$

In what follows, we assume the function K satisfies

Assumption (K) $K \in C^2(\overline{\Omega})$, $0 < K(x) \le 1$, K attains $\max_{\overline{\Omega}} K$ at the unique interior point $x_0 \in \Omega$ with $K(x_0) = 1$, and x_0 is a nondegenerate critical point of K.

In the sequel, let $G = G(x,y)$ denote the Green function of Δ^2 under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = \Delta G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $\Gamma(x,y)$ be the fundamental solution of Δ^2:

$$\Gamma(x,y) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N}|x-y|^{4-N}, & N \ge 5, \\ \frac{1}{\sigma_4}\log|x-y|^{-1}, & N = 4, \end{cases}$$

here σ_N is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$. Finally, let $R(x) = \lim_{y\to x}[\Gamma(x,y) - G(x,y)]$ denote the Robin function of Δ^2 with the Navier boundary condition. By the maximum principle, we see $R > 0$ on Ω and $R(x) \to +\infty$ as x tends to the boundary of Ω.

Main result of this note reads as follows.

Theorem 1 *Let $\Omega \subset \mathbb{R}^N, N \geq 5$ be a smooth bounded domain. Let u_ε be a least energy solution to $(P_{\varepsilon,K})$ for $\varepsilon > 0$ and let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty(\Omega)}$. Assume* **(K)**. *Then after passing to a subsequence, we have*

$$(1)\begin{cases} |x_\varepsilon - x_0| = O(\|u_\varepsilon\|_{L^\infty(\Omega)}^{-2}) & N = 5, \\ |x_\varepsilon - x_0| = o(\|u_\varepsilon\|_{L^\infty(\Omega)}^{-\frac{2}{N-4}}) & N \geq 6, \end{cases}$$

$$(2)\|u_\varepsilon\|_{L^\infty(\Omega)}^{\varepsilon} \to 1 \quad as\ \varepsilon \to 0,$$

$$(3)\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon(x) \to 2(N-4)(N-2)\sigma_N G(x,x_0) \quad as\ \varepsilon \to 0,\ (x \neq x_0)$$

$$(4)\begin{cases} \lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^{2} = \frac{2^{15}}{21}\pi R(x_0) & N = 5, \\ \lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^{2} = -\frac{1}{4}\Delta K(x_0) + 480\pi^3 R(x_0) & N = 6, \\ \lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{4}{N-4}} = -\frac{2}{(N-2)(N-4)}\Delta K(x_0) & N \geq 7. \end{cases}$$

Thus, the above theorem corresponds to the one proved by Hebey [6] for the second order Laplacian case problem. Our starting point of proof is to establish a key pointwise estimate for u_ε; see Lemma 2 below. To do this, we rely on the blow up analysis with the Navier boundary condition performed by Geng [4]. Although Geng assumed the strict convexity of the domain and $K \equiv 1$ in [4], his blow up analysis works well if the solution sequence considered is known a priori to blow up at the unique interior point of Ω. Note that in our case, the boundary blow up cannot occur since we know $x_\varepsilon \to x_0 \in \Omega$, the unique maximum point of K, for least energy solutions. Therefore, we confirm that the blow up point x_0 is indeed an *isolated simple* blow up point in the sense of [4], without any restriction of the domain dimension. The needed pointwise estimate can be derived from this fact. For local blow up analysis (without any boundary condition) for any solution sequence of subcritical biharmonic equations with nearly critical growth, see the works of Djadli, Malchiodi and Ahmedou [2] and Felli [3]. See also the original work of YanYan Li [10] for the Laplacian case.

In Theorem 1, we observe that the asymptotics depend sensitively on the dimension of the domain: The geometric effect (the Robin function $R(x_0)$) is dominant in the lowest dimension $N = 5$, the effect of the coefficient function ($\Delta K(x_0)$) is dominant when $N \geq 7$, and they are mixed when $N = 6$. This phenomenon was also observed in the second order Laplacian case by Hebey [6].

2 Proof of Theorem 1

In this section, we will show the sketch of proof of Theorem 1. We will treat the case $N \geq 6$ only for the sake of simplicity. Detailed arguments including the case $N = 5$ can be found in the forthcoming paper [8].

First, we recall the Pohozaev type identity for a biharmonic equation with the Navier boundary condition. Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of the following equation

$$\begin{cases} \Delta^2 u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is in $C^1(\overline{\Omega} \times \mathbb{R})$. Denote $F(x,u) = \int_0^u f(x,s)ds$ for any $x \in \overline{\Omega}$. Then we have an identity:

$$\begin{aligned} &\int_\Omega NF(x,u) - (\frac{N-4}{2})uf(x,u) + (x-y)\cdot\nabla_x F(x,u)dx \\ &= \int_{\partial\Omega} ((x-y)\cdot\nabla u)\frac{\partial v}{\partial\nu}ds_x \end{aligned} \tag{2.1}$$

for any $y \in \mathbb{R}^N$, where $v = -\Delta u$ and $\nu = \nu(x)$ is an outer unit normal at $x \in \partial\Omega$.

For a least energy solution u_ε of $(P_{\varepsilon,K})$, the identity (2.1) becomes

$$\begin{aligned} &\frac{c_0(N-4)^2}{2(2N-\varepsilon(N-4))}\varepsilon\int_\Omega K(x)u_\varepsilon^{p_\varepsilon+1}dx \\ &+\frac{c_0(N-4)}{2N-\varepsilon(N-4)}\int_\Omega (x-y)\cdot\nabla K(x)u_\varepsilon^{p_\varepsilon+1}dx = \int_{\partial\Omega}((x-y)\cdot\nabla u_\varepsilon)\frac{\partial v_\varepsilon}{\partial\nu}ds_x \end{aligned} \tag{2.2}$$

where $v_\varepsilon = -\Delta u_\varepsilon$. Also by differentiating (2.2) with respect to y_i, we have

$$\frac{c_0(N-4)}{2N-\varepsilon(N-4)}\int_\Omega \frac{\partial K}{\partial x_i}(x)u_\varepsilon^{p_\varepsilon+1}dx = \int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial\nu}\frac{\partial v_\varepsilon}{\partial\nu}\nu_i ds_x \tag{2.3}$$

for all $i = 1, \cdots, N$. Note that $u_\varepsilon, v_\varepsilon > 0$ in Ω and $\nabla u_\varepsilon = -|\nabla u_\varepsilon|\nu, \nabla v_\varepsilon = -|\nabla v_\varepsilon|\nu$ on $\partial\Omega$.

Next, define the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon \tag{2.4}$$

where $\Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} (\Omega - x_\varepsilon)$, and in the following, we abbreviate $\|\cdot\| = \|\cdot\|_{L^\infty(\Omega)}$. It holds that $0 < \tilde{u}_\varepsilon \leq 1, \tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 \tilde{K}_\varepsilon(y) \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\tilde{K}_\varepsilon(y) = K\left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right)$. By (1.1) and (**K**), we know

$$\|u_\varepsilon\| \longrightarrow \infty, \quad x_\varepsilon \longrightarrow x_0 \in \Omega \quad \text{as } \varepsilon \longrightarrow 0,$$

thus $\Omega_\varepsilon \longrightarrow \mathbb{R}^N$ and $\tilde{K}_\varepsilon \longrightarrow K(x_0) = 1$ compact uniformly on $\mathbb{R}^N$ as $\varepsilon \longrightarrow 0$. By standard elliptic estimates and the uniqueness of the limit, we have

$$\tilde{u}_\varepsilon \longrightarrow U \quad \text{compact uniformly in } \mathbb{R}^N \tag{2.5}$$

as $\varepsilon \longrightarrow 0$, where

$$U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-4}{2}}$$

is the unique solution of

$$\begin{cases} \Delta^2 U = c_0 U^p \quad \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \ U(0) = 1, \\ \lim_{|y| \to \infty} U(y) = 0. \end{cases}$$

By (2.5), we easily see that there exists a constant $M \geq 1$ independent of ε such that for any ε sufficiently small, there holds

$$1 \leq \|u_\varepsilon\|^\varepsilon \leq M. \tag{2.6}$$

See [5]: Corollary 1, or [1]: Lemma 4.1.

Also we have the following crucial pointwise estimate for u_ε through the theory of isolated simple blow up points in [4]; see also [2] and [3].

Lemma 2 *There exists a constant $C > 0$ independent of ε such that for any $R_\varepsilon \to \infty$ with $r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}} \to 0$, the following estimates hold true:*

$$u_\varepsilon(x) \le C \frac{\|u_\varepsilon\|}{\left(1 + \|u_\varepsilon\|^{\frac{4}{N-4}} |x - x_\varepsilon|^2\right)^{\frac{N-4}{2}}}, \quad \text{for } |x - x_\varepsilon| \le r_\varepsilon, \tag{2.7}$$

$$u_\varepsilon(x) \le \frac{C}{\|u_\varepsilon\|} \frac{1}{|x - x_\varepsilon|^{N-4}}, \quad \text{for } \{|x - x_\varepsilon| > r_\varepsilon\} \cap \Omega. \tag{2.8}$$

Proof. As stated in Introduction, we appeal to the blow up analysis in [4] to prove Lemma. We will see that the interior blow up point x_0 is indeed an isolated simple blow up one. We refer [4] for the definition of isolated, and isolated simple blow up points. See also the original work by YanYan Li [10] for the Laplacian case problem.

First, by a standard argument originally due to R. Schoen (for example, [7] Lemma 3.1), we know that any interior blow up point is an isolated one; see [4] Proposition 2.1. Note that though the convexity of the domain is assumed in [4], the assumption is used only to assure that any blow up point is in the interior of the domain Ω. Also since u_ε makes one point blow up in our case, we do not need an argument using the Pohozaev identity to deal with multiple blow up points and their interactions. Therefore, the coefficient function K does not have any effect on the validity of the proofs in [4]. Thus, by Proposition 2.2 in [4], we have the estimate

$$\left\| \tilde{u}_\varepsilon(\cdot) - (1 + |y|^2)^{-\frac{N-4}{2}} \right\|_{C^4(B_{R_\varepsilon(0)})} \le \delta_\varepsilon$$

for any $R_\varepsilon \to \infty$ with $R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}} \to 0$ and $\delta_\varepsilon \to 0$. By taking $\delta_\varepsilon \le (1 + R_\varepsilon^2)^{-\frac{N-4}{2}}$, (2.7) holds when $|x - x_\varepsilon| \le r_\varepsilon = R_\varepsilon \|u_\varepsilon\|^{-\frac{p_\varepsilon - 1}{4}}$.

Next, Proposition 4.1 in [4] is valid for least energy solutions of $(P_{\varepsilon,K})$ for any $N \ge 5$, when K is a positive function satisfying (**K**). Thus we have that any interior isolated blow up point is an isolated simple one by Proposition 4.1 in [4], and by Proposition 3.2 in [4], we have the estimate

$$u_\varepsilon(x) \le \frac{C}{\|u_\varepsilon\|} \frac{1}{|x - x_\varepsilon|^{N-4}} \tag{2.9}$$

for any $r_\varepsilon \le |x - x_\varepsilon| \le \rho$, where C and ρ are positive constants independent of ε. From this, we check that the estimate

$$u_\varepsilon(x) \le \frac{C}{\|u_\varepsilon\|} \frac{1}{\rho^{N-4}} \quad \text{for } \{|x - x_\varepsilon| > \rho\} \cap \Omega \tag{2.10}$$

holds true. Indeed, from (2.9) we have

$$u_\varepsilon(x) \le \frac{C}{\|u_\varepsilon\|}\frac{1}{\rho^{N-4}} \quad \text{for } |x-x_\varepsilon| = \rho. \tag{2.11}$$

If there exists a point $x' \in \{|x-x_\varepsilon| > \rho\} \cap \Omega$ such that $u_\varepsilon(x') > \frac{C}{\|u_\varepsilon\|}\frac{1}{\rho^{N-4}}$, we would have a maximum point in the region $\{|x-x_\varepsilon| > \rho\} \cap \Omega$. But this and (2.11) would contradict the fact that x_0 is an isolated simple blow up point. Finally, (2.8) follows easily from (2.9), (2.10) and the boundedness of the domain. □

In terms of $\tilde{u}_\varepsilon$ in (2.4), the above lemma reads

$$\tilde{u}_\varepsilon(y) \le \begin{cases} CU(y) & \text{for } |y| \le R_\varepsilon, \\ C\frac{1}{|y|^{N-4}} & \text{for } \{|y| > R_\varepsilon\} \cap \Omega_\varepsilon, \end{cases} \tag{2.12}$$

where $R_\varepsilon \to \infty$ is any sequence as in Lemma 2.

From Lemma 2, we also obtain the following:

Lemma 3 *There exists a constant $C > 0$ independent of ε such that*

$$\int_{\partial\Omega} |\nabla u_\varepsilon||\nabla v_\varepsilon| ds \le C\|u_\varepsilon\|^{-2}$$

holds true.

Proof. This is done by using Lemma 2 and the fact: *Let u solve*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $\omega' \subset\subset \omega$ be a neighborhood of $\partial\Omega$. Then

$$\|u\|_{W^{1,q}(\Omega)} + \|u\|_{C^{1,\alpha}(\omega')} \le C\left(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}\right) \tag{2.13}$$

holds for $q < \frac{N}{N-1}, \alpha \in (0,1)$. See [5] Lemma 2; there the left hand side of the claimed estimate is $\|u\|_{W^{1,q}(\Omega)} + \|\nabla u\|_{C^{0,\alpha}(\omega')}$, however the estimate (2.13) is indeed proved in the proof.

We apply (2.13) to

$$\begin{cases} -\Delta u_\varepsilon = v_\varepsilon & \text{in } \Omega, \\ -\Delta v_\varepsilon = c_0 K(x) u_\varepsilon^{p_\varepsilon} & \text{in } \Omega, \\ u_\varepsilon = v_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

As a consequence, it appears that we need to estimate $\|c_0K(x)u_\varepsilon^{p_\varepsilon}\|_{L^1(\Omega)}$ and $\|c_0K(x)u_\varepsilon^{p_\varepsilon}\|_{L^\infty(\omega)}$ to control both $\|\nabla u_\varepsilon\|_{L^\infty(\partial\Omega)}$ and $\|\nabla v_\varepsilon\|_{L^\infty(\partial\Omega)}$.

By (2.6) and the fact $0 < K(x) \le 1$, we have

$$\begin{aligned}\int_\Omega c_0K(x)u_\varepsilon^{p_\varepsilon}dx &\le C\int_\Omega u_\varepsilon^{p_\varepsilon}dx = C\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{4})N}\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon}(y)dy\\ &= C\|u_\varepsilon\|^{p_\varepsilon-(\frac{p_\varepsilon-1}{4})N}\left(\int_{\mathbb{R}^N}U^p(y)dy+o(1)\right)\\ &\le C\|u_\varepsilon\|^{-1+\varepsilon(\frac{N-4}{4})} \le C\|u_\varepsilon\|^{-1}\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small. Here we have used (2.5), (2.12) and the Lebesgue convergence theorem.

On the other hand, since we may take a neighborhood of $\partial\Omega$ small such that $x_0 \notin \omega$, we see by Lemma 2

$$\begin{aligned}c_0K(x)u_\varepsilon^{p_\varepsilon}(x) &\le C\frac{\|u_\varepsilon\|^{-p_\varepsilon}}{|x-x_0|^{(N-4)p_\varepsilon}}\\ &\le C\|u_\varepsilon\|^{-p_\varepsilon} \le C\|u_\varepsilon\|^{-1}\end{aligned}$$

for any $x \in \omega$, if $\varepsilon > 0$ small such that $1 < p_\varepsilon$. Thus we have $\|c_0K(x)u_\varepsilon^{p_\varepsilon}\|_{L^\infty(\omega)} \le C\|u_\varepsilon\|^{-1}$. These estimates with (2.13) leads to

$$\|\nabla u_\varepsilon\|_{L^\infty(\partial\Omega)} \le C\|u_\varepsilon\|^{-1} \text{ and } \|\nabla v_\varepsilon\|_{L^\infty(\partial\Omega)} \le C\|u_\varepsilon\|^{-1},$$

from which we obtain Lemma 3. □

Now, we will prove the estimates

$$|x_\varepsilon - x_0| = o(\|u_\varepsilon\|^{-\frac{2}{N-4}}), \quad N \ge 6 \tag{2.14}$$

under the assumption (**K**).

Indeed, by Taylor expansion, we have

$$K(x) = 1 + \frac{1}{2}\sum_{i,j=1}^N b_{ij}(x_i - x_i^0)(x_j - x_j^0) + O(|x-x_0|^3), \tag{2.15}$$

and

$$\frac{\partial K}{\partial x_i}(x) = \sum_{j=1}^N b_{ij}(x_j - x_j^0) + O(|x-x_0|^2) \tag{2.16}$$

for all $i = 1, \cdots, N$, where we set $b_{ij} = \frac{\partial^2 K}{\partial x_i \partial x_j}(x_0)$. Inserting (2.16) into (2.3), we have

$$\frac{c_0(N-4)}{2N-\varepsilon(N-4)} \int_\Omega \sum_{j=1}^N b_{ij}(x_j - x_j^0) u_\varepsilon^{p_\varepsilon+1} dx + \int_\Omega O(|x-x_0|^2) u_\varepsilon^{p_\varepsilon+1} dx$$
$$= \int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial \nu} \frac{\partial v_\varepsilon}{\partial \nu} \nu_i ds_x \qquad (2.17)$$

for $i = 1, \cdots, N$. The right hand side of (2.17) is $O(\|u_\varepsilon\|^{-2})$ by Lemma 3. Now, by the change of variables (2.4), we have

$$\int_\Omega O(|x-x_0|^2) u_\varepsilon^{p_\varepsilon+1} dx = \|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} O\left(\left|\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon - x_0\right|^2\right) \tilde{u}_\varepsilon^{p_\varepsilon+1} dy.$$

Splitting the integral as

$$\int_{\Omega_\varepsilon} O\left(\left|\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon - x_0\right|^2\right) \tilde{u}_\varepsilon^{p_\varepsilon+1} dy$$
$$= \int_{\{y\in\Omega_\varepsilon : |y| \le \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} |x_\varepsilon - x_0|\}} (\cdots) dy + \int_{\{y\in\Omega_\varepsilon : |y| > \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} |x_\varepsilon - x_0|\}} (\cdots) dy$$
$$=: I_1 + I_2,$$

and estimating

$$I_1 = O(|x_\varepsilon - x_0|^2),$$
$$I_2 \le C\|u_\varepsilon\|^{-(\frac{p_\varepsilon-1}{2})} \int_{\Omega_\varepsilon} |y|^2 \tilde{u}_\varepsilon^{p_\varepsilon+1} dy$$
$$= C\|u_\varepsilon\|^{-\frac{4}{N-4}} \left(\int_{\mathbb{R}^N} |y|^2 U^{p+1} dy + o(1)\right) = O(\|u_\varepsilon\|^{-\frac{4}{N-4}}),$$
$$\|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} = \|u_\varepsilon\|^{(\frac{N-4}{4})\varepsilon} = O(1)$$

by (2.6), (2.5), (2.12) and the Lebesgue convergence theorem, we have

$$\int_\Omega O(|x-x_0|^2) u_\varepsilon^{p_\varepsilon+1} dx = O(|x_\varepsilon - x_0|^2) + O(\|u_\varepsilon\|^{-\frac{4}{N-4}}). \qquad (2.18)$$

The same argument leads to

$$\int_\Omega O(|x-x_0|^3)u_\varepsilon^{p_\varepsilon+1}dx = O(|x_\varepsilon - x_0|^3) + O(\|u_\varepsilon\|^{-\frac{6}{N-4}}). \tag{2.19}$$

Now,

$$\begin{aligned}
&\sum_{j=1}^N b_{ij}\int_\Omega (x_j - x_j^0)u_\varepsilon^{p_\varepsilon+1}dx \\
&= \sum_{j=1}^N b_{ij}\int_{\Omega_\varepsilon}\left(\frac{y_j}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + (x_\varepsilon)_j - x_j^0\right)(\|u_\varepsilon\|\tilde{u}_\varepsilon)^{p_\varepsilon+1}(y)\|u_\varepsilon\|^{-(\frac{p_\varepsilon-1}{4})N}dy \\
&= \sum_{j=1}^N b_{ij}\|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N-(\frac{p_\varepsilon-1}{4})}\int_{\Omega_\varepsilon} y_j\tilde{u}_\varepsilon^{p_\varepsilon+1}dy \\
&+ \sum_{j=1}^N b_{ij}\|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N}((x_\varepsilon)_j - x_j^0)\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon+1}dy \\
&=: J_1 + J_2.
\end{aligned} \tag{2.20}$$

By (2.5), (2.12) and the Lebesgue convergence theorem, we see

$$\int_{\Omega_\varepsilon} y_j\tilde{u}_\varepsilon^{p_\varepsilon+1}dy = \int_{\mathbb{R}^N} y_jU^{p+1}(y)dy + o(1) = o(1)$$

for any $j = 1, \cdots, N$. Therefore, J_1 in (2.20) is

$$J_1 = C\|u_\varepsilon\|^{-(\frac{2}{N-4})+(\frac{N-3}{4})\varepsilon} \times o(1) = o(\|u_\varepsilon\|^{-\frac{2}{N-4}})$$

by (2.6). Similarly, we have

$$J_2 = \|u_\varepsilon\|^{\varepsilon(\frac{N-4}{4})}\sum_{j=1}^N b_{ij}((x_\varepsilon)_j - x_j^0)\int_{\Omega_\varepsilon}\tilde{u}_\varepsilon^{p_\varepsilon+1}dy = O(1)\times\sum_{j=1}^N b_{ij}((x_\varepsilon)_j - x_j^0).$$

Returning to (2.17) with these, we get that

$$\sum_{j=1}^N b_{ij}((x_\varepsilon)_j - x_j^0) + O(|x_\varepsilon - x_0|^2) = O(\|u_\varepsilon\|^{-2}) + o(\|u_\varepsilon\|^{-\frac{2}{N-4}}). \tag{2.21}$$

By our assumption that x_0 is a nondegenerate critical point of K, the matrix $(b_{ij})_{1\le i,j\le N} = (\frac{\partial^2 K}{\partial x_i \partial x_j}(x_0))$ is invertible. Hence from (2.21), we have (2.14).

Next we prove Theorem 1 (2):

$$\|u_\varepsilon\|^\varepsilon \longrightarrow 1, \quad \text{as } \varepsilon \longrightarrow 0 \tag{2.22}$$

by using (2.14).

In fact, inserting (2.15) and (2.16) into (2.2), we have

$$\begin{aligned}
&\frac{c_0(N-4)^2}{2(2N-\varepsilon(N-4))}\varepsilon \int_\Omega u_\varepsilon^{p_\varepsilon+1} dx \\
&+ \left\{ \frac{c_0(N-4)}{2N-\varepsilon(N-4)} + \frac{c_0(N-4)^2}{2(2N-\varepsilon(N-4))}\frac{\varepsilon}{2} \right\} \int_\Omega \sum_{i,j=1}^N b_{ij}(x_i - x_i^0)(x_j - x_j^0) u_\varepsilon^{p_\varepsilon+1} dx \\
&+ \int_\Omega O(|x-x_0|^3) u_\varepsilon^{p_\varepsilon+1} dx = \int_{\partial\Omega} |\nabla u_\varepsilon||\nabla v_\varepsilon|((x-x_0)\cdot\nu) ds_x
\end{aligned} \tag{2.23}$$

when $N \ge 6$. Hence by (2.23), (2.18), (2.19) and Lemma 3, we have

$$O(1)\times\varepsilon + O(|x_\varepsilon - x_0|^2) + O(\|u_\varepsilon\|^{-\frac{4}{N-4}}) = O(\|u_\varepsilon\|^{-2}).$$

This in turn implies

$$\varepsilon \le C\|u_\varepsilon\|^{-\frac{4}{N-4}}, \quad \text{when } N \ge 6 \tag{2.24}$$

for some constant $C > 0$, here we have used (2.14). By the mean value theorem, it holds

$$\big|\|u_\varepsilon\|^\varepsilon - 1\big| = \big|\|u_\varepsilon\|^{t\varepsilon}\varepsilon \log \|u_\varepsilon\|\big|$$

for some $t \in (0,1)$. Therefore by (2.6) and (2.24), it holds

$$\big|\|u_\varepsilon\|^\varepsilon - 1\big| = O(\|u_\varepsilon\|^{-\frac{4}{N-4}} \log \|u_\varepsilon\|), \qquad N \ge 6.$$

Thus we obtain (2.22).

Once (2.22) is established, we can check that the following lemma along the line of [1]: Proposition 5.1, or [5]: Proposition 1.

Lemma 4 *We have*

$$\Delta^2(\|u_\varepsilon\| u_\varepsilon) \rightharpoonup c_0 \frac{2\sigma_N}{N(N+2)}\delta_{x_0}$$

in the sense of Radon measures of $\overline{\Omega}$, *and*

$$\|u_\varepsilon\|u_\varepsilon \to c_0 \frac{2\sigma_N}{N(N+2)} G(\cdot, x_0) \quad \text{in } C^{3,\alpha}(\omega),$$
$$\|u_\varepsilon\|v_\varepsilon \to c_0 \frac{2\sigma_N}{N(N+2)} (-\Delta G)(\cdot, x_0) \quad \text{in } C^{1,\alpha}(\omega)$$

for some $\alpha \in (0,1)$, *where* ω *is any open neighborhood of* $\partial\Omega$, *not containing* x_0.

Finally, we will prove Theorem 1 (4) when $N \geq 6$.

By (2.5), (2.12), (2.14), (2.22) and the Lebesgue convergence theorem, we have the followings:

$$\int_\Omega u_\varepsilon^{p_\varepsilon+1} dx \to \int_{\mathbb{R}^N} U^{p+1} dy = \frac{\sigma_N}{2} \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}, \tag{2.25}$$

$$\begin{aligned}
&\sum_{i,j=1}^N b_{ij} \int_\Omega (x_i - x_i^0)(x_j - x_j^0) u_\varepsilon^{p_\varepsilon+1} dx \\
&= \sum_{i,j=1}^N b_{ij} \|u_\varepsilon\|^{\varepsilon(\frac{N-4}{4})} \times \\
&\int_{\Omega_\varepsilon} \left(\frac{y_i y_j}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{2}}} + \frac{y_i((x_\varepsilon)_j - x_j^0) + y_j((x_\varepsilon)_i - x_i^0)}{\|u_\varepsilon\|^{\frac{p_\varepsilon - 1}{4}}} + ((x_\varepsilon)_i - x_i^0)((x_\varepsilon)_j - x_j^0) \right) \tilde{u}_\varepsilon^{p_\varepsilon+1} dy \\
&= \sum_{i,j=1}^N b_{ij} \|u_\varepsilon\|^{-\frac{4}{N-4} + \varepsilon(\frac{N-2}{4})} \int_{\Omega_\varepsilon} y_i y_j \tilde{u}_\varepsilon^{p_\varepsilon+1} dy + o\left(\|u_\varepsilon\|^{-\frac{4}{N-4}}\right)
\end{aligned} \tag{2.26}$$

if $N \geq 6$, and

$$\begin{aligned}
\int_{\Omega_\varepsilon} y_i y_j \tilde{u}_\varepsilon^{p_\varepsilon+1} dy &\to \int_{\mathbb{R}^N} y_i y_j U^{p+1} dy = \frac{1}{N} \int_{\mathbb{R}^N} |y|^2 U^{p+1} dy \delta_{ij} \\
&= \frac{\sigma_N}{N} \int_0^\infty \frac{r^{N+1}}{(1+r^2)^N} dr \delta_{ij} = \frac{\sigma_N}{2(N-2)} \frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)} \delta_{ij},
\end{aligned} \tag{2.27}$$

where δ_{ij} is Kronecker's delta.

Furthermore, Lemma 4 leads to

$$\|u_\varepsilon\|^{\frac{4}{N-4}} \int_{\partial\Omega} |\nabla u_\varepsilon| |\nabla v_\varepsilon| ((x - x_0) \cdot \nu) ds_x \to 0, \quad \text{if } N \geq 7, \tag{2.28}$$

and

$$
\begin{aligned}
&\|u_\varepsilon\|^{\frac{4}{N-4}} \int_{\partial\Omega} |\nabla u_\varepsilon||\nabla v_\varepsilon|((x-x_0)\cdot\nu)ds_x \\
&\to \left(\frac{2c_0\sigma_N}{N(N+2)}\right)^2 \int_{\partial\Omega} |\nabla G||\nabla\Delta G|((x-x_0)\cdot\nu)ds_x \\
&= \left(\frac{2c_0\sigma_N}{N(N+2)}\right)^2 (N-4)R(x_0) = 2^9\sigma_6^2 R(x_0), \quad \text{if } N=6, \qquad (2.29)
\end{aligned}
$$

where we have used a formula in [1] Lemma 3.1:

$$
\int_{\partial\Omega} |\nabla G||\nabla\Delta G|((x-x_0)\cdot\nu)ds_x = (N-4)R(x_0)
$$

for any $x_0 \in \Omega$.

Thus, multiplying (2.23) by $\|u_\varepsilon\|^{\frac{4}{N-4}}$ when $N \geq 6$, using (2.25), (2.26), (2.27), and (2.28) or (2.29), we obtain

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|^{\frac{4}{N-4}} &= \frac{-\frac{(N-4)c_0}{2N}\frac{\sigma_N}{2(N-2)}\frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}\sum_{i=1}^N b_{ii}}{\frac{c_0(N-4)^2}{4N}\frac{\sigma_N}{2}\frac{\Gamma(\frac{N}{2})^2}{\Gamma(N)}} \\
&= -\frac{2}{(N-2)(N-4)}\Delta K(x_0), \quad \text{if } N \geq 7,
\end{aligned}
$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|^{\frac{4}{N-4}} &= \frac{2^9\pi^6 R(x_0) - \frac{4}{15}\pi^3\Delta K(x_0)}{\frac{16}{15}\pi^3} \\
&= 480\pi^3 R(x_0) - \frac{1}{4}\Delta K(x_0), \quad \text{if } N=6.
\end{aligned}
$$

Recall $\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$, in particular, $\sigma_6 = \pi^3$.
This proves Theorem 1 when $N \geq 6$. □

References

[1] K-S. Chou, and D. Geng: *Asymptotics of positive solutions for a biharmonic equation involving critical exponent,* Diff. Int. Eq. **13** (2000) 921-940.

[2] Z. Djadli, A. Malchiodi and M. O. Ahmedou: *Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **1** (2002) 387-434.

[3] V. Felli: *Existence of conformal metrics on $\mathbb{S}^n$ with prescribed fourth-order invariant,* Adv, Diff. Eq. **7** no.1, (2002) 47-76.

[4] D. Geng: *On blow-up of positive solutions for a biharmonic equation involving nearly critical exponent,* Comm. Partial Differential Equations. **24** no.11-12, (1999) 2333-2370.

[5] Z. C. Han: *Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent,* Ann. Inst. Henri Poincaré. **8** (1991) 159-174.

[6] E. Hebey: *Asymptotic behavior of positive solutions of quasilinear elliptic equations with critical Sobolev growth,* Diff. Int. Eq. **13** (2000) 1073-1080.

[7] R. Schoen, and D. Zhang: *Prescribed scalar curvature on the n-sphere,* Calc. Vari. PDE. **4** (1996) 1-25.

[8] F. Takahashi: *Asymptotic behavior of least energy solutions for a biharmonic problem with nearly critical growth,* to appear in Asymptotic Anal.

[9] R.C.A.M. van der Vorst: *Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$,* Diff. Int. Eq. **6** (1993) 259-276.

[10] Y. Y. Li: *Prescribing scalar curvature on $\mathbb{S}^n$ and related topics, Part I,* J. Differential Equations **120** (1995) 319-410.