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Global branches of solutions to a semilinear elliptic Neumann problem

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ABSTRACT. Let $D \subset \mathbb{R}^2$ be a disk, and let $f \in C^3$. We assume that there is $a \in \mathbb{R}$ such that $f(a) = 0$ and $f'(a) > 0$. In this article, we are concerned with non-radially symmetric solutions to the Neumann problem

$$\Delta u + \lambda f(u) = 0 \text{ in } D, \qquad \partial_\nu u = 0 \text{ on } \partial D.$$

We announce some results on branches of non-radially symmetric solutions emanating from the second and third eigenvalues, respectively. The proofs are given in [M08a, M08b].

1. INTRODUCTION

In this article, we announce the main results of [M08a, M08b]. Let $D \subset \mathbb{R}^2$ be a disk centered at the origin with radius 1, and let $f \in C^3$. Throughout the present article, we assume that

(A0) there is $a \in \mathbb{R}$ such that $f(a) = 0$ and $f'(a) > 0$.

We are concerned with non-radially symmetric solutions to the Neumann problem in a domain $\Omega \subset \mathbb{R}^N$

$$(\mathrm{BP}_\Omega) \qquad \Delta u + \lambda f(u) = 0 \text{ in } \Omega, \qquad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where $\lambda > 0$. We can assume without loss of generality that $f'(a) = 1$.

The following are examples of f:

(A1) There are $a_-, a_+ \in \mathbb{R}$ such that $a_- < a < a_+$, $f(a_-) = f(a_+) = 0$, $f < 0$ in (a_-, a), and $f > 0$ in (a, a_+),

(A2) $f(u) = (-u + u^p)/(p-1)$ $(p > 1)$, and $a = 1$.

The problem (BP_Ω) with (A2) is equivalent to the problem

$$(1.1) \qquad \varepsilon^2 \Delta u - u + u^p = 0 \text{ in } \Omega, \qquad \partial_\nu u = 0 \text{ on } \partial\Omega,$$

where $\varepsilon = \sqrt{(p-1)/\lambda}$.

Since, for any $\lambda > 0$, $u \equiv a$ is a solution of (BP_Ω), we call $u \equiv a$ *a trivial solution* (or *a trivial branch*).

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Let X be a functional space to which the solution u of (BP$_\Omega$) belongs. We call $(\lambda^*, a) \in \mathbb{R} \times X$ a bifurcation point if for any neighborhood $\mathcal{U} \subset \mathbb{R} \times X$ of (λ^*, a) there is a non-trivial solution (λ, u) in $\mathcal{U}$.

We mainly consider (BP$_D$). When u is a non-radially symmetric solution of (BP$_D$), then the rotation of u is also a solution. Hence the continuum of non-radially symmetric solutions is rather a sheet than a branch. We fix the phase in the statements of Theorems 2.1 and 2.2 below.

This article consists of four sections. In Section 2, we state the main results of [M08a] (Theorems 2.1, 2.2, 2.3, and 2.4). In Section 3, we state the main results of [M08b] (Theorems 3.1, 3.2, and 3.3).

2. Main results of [M08a]

We need some more notation to state the results. We define D_n by

$$D_n := \begin{cases} \{(r,\theta);\ 0<r<1,\ 0<\theta<\pi/n\} & \text{if } n \in \{1,2,3,\ldots\}, \\ D & \text{if } n=0. \end{cases} \tag{2.1}$$

Here (r,θ) is the polar coordinate of $\mathbb{R}^2$. Let $\mu_j^{(n)}$ $(j \geq 0)$ be the eigenvalues of the Neumann Laplacian on D_n with counting multiplicities. Let $\Gamma_1 := \{(\cos\theta, \sin\theta);\ 0 < \theta < \pi\}$, $\Gamma_2 := \{(x,0);\ -1 < x < 1\}$, $O = (0,0)$, $P := (1,0)$, and $Q := (-1,0)$.

The first result is the existence of global branches of non-radially symmetric solutions.

Theorem 2.1 ([M08a, Theorem 3.1]). *There is an unbounded continuum of (BP$_D$), $\tilde{\mathcal{C}}_1$, emanating from $(\mu_1^{(0)}, a)$ and consisting of non-radially symmetric solutions such that, for any $(\lambda, u) \in \tilde{\mathcal{C}}_1$, u is symmetric with respect to $\{y=0\}$,*

$$-u_\theta > 0 \text{ in } D_1 \cup \Gamma_1, \qquad \text{and} \qquad u_x > 0 \text{ in } \overline{D_1} \backslash \{P, Q\}. \tag{2.2}$$

Hence P and Q are the maximum and minimum points of u in $\overline{D}$, respectively.

Theorem 2.2 ([M08a, Theorem 4.1]). *There is an unbounded continuum of (BP$_D$), $\tilde{\mathcal{C}}_2$, emanating from $(\mu_2^{(0)}, a)$ and consisting of non-radially symmetric solutions such that if $(\lambda, u) \in \tilde{\mathcal{C}}_2$, then u is symmetric with respect to $\{x=0\}$ and $\{y=0\}$,*

$$u_\theta > 0 \ \text{ in } \ R_{\pi/2}D_2 \cup R_{3\pi/2}D_2, \ \text{ and } \ u_\theta < 0 \ \text{ in } \ D_2 \cup R_\pi D_2,$$

where R_θ is the counterclockwise rotation with center O and angle θ.

The second result is the local uniqueness of the branch emanating from the second eigenvalue.

Theorem 2.3 ([M08a, Theorem 3.5]). *Let $\mathcal{C}$ be a continuum consisting of non-trivial solutions to (BP$_D$) and emanating from $(\mu_1^{(0)}, a)$. Then there is a neighborhood $\mathcal{U}_0 \subset \mathbb{R} \times X$ of $(\mu_1^{(0)}, a)$ such that if $(\lambda, u) \in \mathcal{C} \cap \mathcal{U}_0$, then u is symmetric with respect to a line containing the origin. Moreover if $f'''(a) \neq 0$, then $\mathcal{C}$ is unique up to rotation near $(\mu_1^{(0)}, a)$. Specifically, there is a neighborhood $\mathcal{U}_1 \subset \mathbb{R} \times X$ of $(\mu_1^{(0)}, a)$ such that if (λ_0, u), $(\lambda_0, v) \in \mathcal{C} \cap \mathcal{U}_1$, then $u = R_\theta v$ for some $\theta \in [0, 2\pi)$.*

The third result is the direction of the global branches. Specifically, the branches do not blow up if (A1) holds.

Theorem 2.4 ([M08a, Theorem 3.6]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and let $\{\mu_j(\Omega)\}_{j\geq 0}$ denote the set of the eigenvalues of the Neumann Laplacian on Ω. Suppose that (A1) holds. If (BP_Ω) has an unbounded continuum of non-trivial solutions, $\mathcal{C}$, emanating from $(\mu_n(\Omega), a)$ $(n \geq 1)$, then $\mathcal{C}$ is unbounded in the positive direction of λ. Hence branches of (BP_D) obtained in Theorems 2.1 and 2.2 are unbounded in the positive direction of λ.*

When (A2) holds, there is a priori bound and the branches do not blow up.

In proofs of Theorems 2.1, 2.2, and 2.3, we analyze the zero level sets of u_x, u_y, and u_θ in detail. The main tool is the theory of Carleman-Hartman-Wintner [C33, HW53]. The zero level sets are corresponding to the zero number in a one-dimensional case. Using this technique, we can exclude the case where the branch meets another eigenvalue, in the Rabinowitz alternative [R71]. We obtain a global branch.

3. Main results of [M08b]

We continue to study (BP_D). We assume the following conditions on f:

(f0) $$f \text{ is of class } C^3,$$

(f1) $$f(-t) = -f(t) \quad \text{for} \quad t \in \mathbb{R},$$

(f2) $$f'(t) < \frac{f(t)}{t} \quad \text{for} \quad t > 0,$$

(f3) $$f'(0) > 0 \quad \text{and} \quad f'''(0) < 0.$$

Let $\mathcal{C}$ be the branch obtained by Theorem 2.1, which emanates from the second eigenvalue. The zero is an eigenvalue of the linearized eigenvalue problem which comes from the rotation invariance. Thus we cannot directly apply the implicit function theorem. However, when the zero eigenvalue comes only from the rotation invariance, we can show that $\mathcal{C}$ does not have a secondary bifurcation point.

Theorem 3.1 ([M08b, Theorem C]). *Assume that (f0)–(f3) hold. Then $\mathcal{C}$ is the unique maximal continuum consisting of non-trivial solutions to (BP_Ω) and emanating from $(\mu_1^{(0)}, 0)$. Hence, $\mathcal{C}$ is homeomorphic to $\mathbb{R} \times S^1 (\simeq \mathbb{R}^2 \backslash \{(0,0)\})$ and the closure of $\mathcal{C}$ is homeomorphic to $\mathbb{R}^2$.*

3.1. **The first abstract result.** Theorem 3.1 is proven in a rather abstract setting. Let X be a Banach space, and let $I_{c,\varepsilon} := (c-\varepsilon, c+\varepsilon) \subset \mathbb{R}$ $(c \in \mathbb{R}, \varepsilon > 0)$. Let G be a continuous group acting on X, and let σ_θ be an element of G parameterized by $\theta \in I_{0,\varepsilon}$ such that $\sigma_0 = \mathrm{id}$ ($(\sigma(I_{0,\varepsilon}), \sigma^{-1})$ is a local chart of G including id). Hereafter, we locally identify an element of G with a real number.

We consider the mapping $F : \mathbb{R} \times X \to X$ such that

(F0) $$\sigma_\theta F(\lambda, u) = F(\lambda, \sigma_\theta u) \quad \text{for all} \quad \theta \in I_{0,\varepsilon}.$$

We say that $\bar{u}$ is a trivial solution of $F(\lambda, u) = 0$ if $\bar{u}$ satisfies $F(\lambda, \bar{u}) = 0$ and if $\sigma_\theta \bar{u} = \bar{u}$ for all $\theta \in I_{0,\varepsilon}$.

First, we assume the existence of a branch consisting of non-trivial solutions that can be described as a graph of λ near λ^*. Specifically, we assume that

(F1) there exists a one-parameter family $\tilde{u}(\lambda)$ $(\lambda \in I_{\lambda^*,\delta})$ consisting of non-trivial solutions such that $F(\lambda, \tilde{u}(\lambda)) = 0$ for all $\lambda \in I_{\lambda^*,\delta}$.

If $\tilde{u}(\lambda)$ is a non-trivial solution, then $\sigma_\theta \tilde{u}(\lambda)$ is also a non-trivial solution, because $F(\lambda, \sigma_\theta \tilde{u}(\lambda)) = \sigma_\theta F(\lambda, \tilde{u}(\lambda)) = 0$. Hence $\sigma_\theta \tilde{u}(\lambda)$ is a two-parameter family of non-trivial solutions. By $u^*(\lambda, \theta)$ we define $u^*(\lambda, \theta) := \sigma_\theta \tilde{u}(\lambda)$ $(\lambda \in I_{\lambda^*,\varepsilon}, \theta \in I_{0,\delta})$.

Second, we assume that

(F2) $u^*(\lambda, \theta)$ is of class C^1 with respect to (λ, θ) near $(\lambda^*, 0)$.

We define $Y_{1,\lambda} := \mathrm{Ran} F_u(\lambda, u^*(\lambda, 0))$, $Z_{1,\lambda} := \ker F_u(\lambda, u^*(\lambda, 0))$. The third assumption is the essential one for Theorem 3.2 below.

(F3) The zero is a simple eigenvalue of $F_u(\lambda^*, u^*(\lambda^*, 0))$,

$$Z_{1,\lambda^*} = \mathrm{span}\, \langle u^*_\theta(\lambda^*, 0) \rangle, \text{ and } Y_{1,\lambda^*} \oplus Z_{1,\lambda^*} = X.$$

Here we say that the zero is a simple eigenvalue of $F_u(\lambda, u^*(\lambda, 0))$ if

$$\dim \bigcup_{n \geq 1} \ker (F_u(\lambda, u^*(\lambda, 0)))^n = 1.$$

The first abstract theorem is

Theorem 3.2 ([M08b, Theorem A]). *Let $\{(\lambda, u^*(\lambda, \theta))\}_{\lambda \in I_{\lambda^*,\varepsilon}, \theta \in I_{0,\delta}}$ be a two-parameter family of solutions to $F(\lambda, u) = 0$ defined above. Suppose that (F0), (F1), (F2), and (F3) hold. Then $(\lambda^*, u^*(\lambda^*, 0))$ is not a secondary bifurcation point. Specifically, there is a neighborhood $\mathcal{U} \subset \mathbb{R} \times X$ of $(\lambda^*, u^*(\lambda^*, 0))$ such that there is no solution in $\mathcal{U}$ except $(\lambda, u^*(\lambda, \theta))$.*

Roughly speaking, when the zero eigenvalue comes only from the G-invariance, then the secondary bifurcation does not occur.

This theorem is applicable not only for the rotation invariance but also for the translation invariance. We give an example. Let us consider

$$u_{xx} - \lambda u + u^p = 0 \quad \text{in } \mathbb{R}.$$

This equation has a two-parameter family of one-peak solutions $u(\lambda, \theta)$ corresponding to a heteroclinic orbit. This solution can be written explicitly

$$u^*(x; \lambda, \theta) := \left(\frac{p+1}{2} \lambda \right)^{\frac{1}{p-1}} \left(\cosh \left(\frac{p-1}{2} \sqrt{\lambda} (x - \theta) \right) \right)^{-\frac{2}{p-1}} \quad (\lambda \in \mathbb{R}_+, \theta \in \mathbb{R}).$$

The linearization has a zero eigenvalue. However, the Sturm-Liouville theory tells us that the zero eigenvalue is simple. Therefore the zero eigenvalue comes only from the translation invariance, and $u(\lambda, \theta)$ does not have a secondary bifurcation point.

3.2. **The second abstract result.** We consider the case where the zero eigenvalue is not simple. A turning point is a typical example. We state three assumptions (F4), (F5), and (F6).

First, we assume that

(F4) there is a continuum $(\lambda(s), \hat{u}(s))$ $(s \in I_{0,\delta})$ consisting of non-trivial solutions to $F(\lambda, u) = 0$.

We define $\lambda^* := \lambda(0)$. Since $\sigma_\theta \hat{u}(s)$ is a two-parameter family of non-trivial solutions, we define $u^{**}(s, \theta) := \sigma_\theta \hat{u}(s)$ $(s \in I_{0,\delta}, \theta \in I_{0,\varepsilon})$.

Second, we assume that

(F5) $\lambda(s)$ is of class C^1 with respect to s near 0, $\lambda_s(0) = 0$, and $u^{**}(s, \theta)$ is of class C^1 with respect to (s, θ) near $(0, 0)$.

We define $Y_{2,s} := \mathrm{Ran} F_u(\lambda(s), u^{**}(s, \theta))$, $Z_{2,s} := \ker F_u(\lambda(s), u^{**}(s, \theta))$.

The third assumption is the essential one for Theorem 3.3 below.

(F6) Zero is an eigenvalue of $F_u(\lambda^*, u^{**}(0, 0))$,

$$Z_{2,0} = \mathrm{span}\,\langle u_s^{**}(0,0), u_\theta^{**}(0,0)\rangle\,,\ \dim Z_{2,0} = 2,\ Y_{2,0} \oplus Z_{2,0} = X,\ \text{and}$$

$$\mathrm{proj}_{\mathrm{span}\langle u_s^{**}(0,0)\rangle} F_\lambda(\lambda^*, u^{**}(0,0)) \neq 0.$$

Since $\dim Z_{2,0} = 2$, $u_s^{**}(0, 0)$ is not parallel to $u_\theta^{**}(0, 0)$.

The second abstract theorem is

Theorem 3.3 ([M08b, Theorem B]). *Let $\{(\lambda(s), u^{**}(s, \theta))\}_{s \in I_{0,\delta}, \theta \in I_{0,\varepsilon}}$ be a two-parameter family of solutions to $F(\lambda, u) = 0$ defined above. Suppose that (F0), (F4), (F5), and (F6) hold. Then $(\lambda^*, u^{**}(0, 0))$ is not a secondary bifurcation point. Specifically, there is a neighborhood $\mathcal{U} \subset \mathbb{R} \times X$ of $(\lambda^*, u^{**}(0, 0))$ such that there is no solution of $F(\lambda, u) = 0$ in $\mathcal{U}$ except $(\lambda(s), u^{**}(s, \theta))$.*

When the zero eigenvalue comes only from a turning point and the G-invariance, then the secondary bifurcation does not occur.

We give an application of Theorem 3.3. By $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ we define

$$F(\lambda, (x, y)) := (h(\lambda, r)x, h(\lambda, r)y), \tag{3.1}$$

where $h(\lambda, r) := r^4 - 2r^2 - 1 + \lambda$ and $r = \sqrt{x^2 + y^2}$. In this subsection we consider the equation

$$F(\lambda, (x, y)) = (0, 0). \tag{3.2}$$

Since, for each $\lambda \in \mathbb{R}$, $(x, y) = (0, 0)$ is a solution, we call this solution the trivial solution. Let σ_θ be a rotation operator on $\mathbb{R}^2$, i.e., $\sigma_\theta(x, y) := (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. Since

$$\begin{aligned} F(\lambda, \sigma_\theta(x, y)) &= (h(\lambda, r)(x \cos\theta - y \sin\theta), h(\lambda, r)(x \sin\theta + y \cos\theta)) \\ &= \sigma_\theta(h(\lambda, r)x, h(\lambda, r)y) = \sigma_\theta F(\lambda, (x, y)), \end{aligned} \tag{3.3}$$

(F0) is satisfied.

The solution of $h(\lambda, r) = 0$ is a solution of (3.2). Hence (3.2) has a one-parameter family of non-trivial solutions

$$(\lambda, (x, y)) = (-s^4 + 2s^2 + 1, (s, 0)) \quad (s > 0). \tag{3.4}$$

Let $u^{**}(s, \theta) := \sigma_\theta(s, 0)(= (s\cos\theta, s\sin\theta))$. Because of (3.3),

$$(\lambda, (x, y)) = (-s^4 + 2s^2 + 1, u^{**}(s, \theta)) \quad (s > 0, \theta \in S^1) \tag{3.5}$$

is also a solution of (3.2).

Since $\{(\lambda, (x, y));\ \lambda = -r^4+2r^2+1,\ r \neq 0\}$ is a continuum of non-trivial solutions, this continuum has a turning point $(2, (1, 0))$ in the $(\lambda, (x, y))-$space. We will check (F4)–(F6) and apply Theorem 3.3 to $(2, (1, 0))$. Since $u^{**}(s, \theta)$ is a two-parameter family of non-trivial solutions, (F4) holds. It is clear that $u^{**}(s, \theta)$ is of class C^1 in (s, θ). Since $\lambda_s(1) = 0$, (F5) holds. The linearization of (3.2) at the turning point is

$$\begin{aligned} &\partial_{(x,y)} F(\lambda, (x, y))|_{(\lambda,(x,y))=(2,u^{**}(1,0))} \\ &= \begin{pmatrix} 4(r^2 - 1)x^2 + h(r) & 4(r^2 - 1)xy \\ 4(r^2 - 1)xy & 4(r^2 - 1)y^2 + h(r) \end{pmatrix}\Bigg|_{(\lambda,(x,y))=(2,u^{**}(1,0))} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

On the other hand, we have $u_s^{**}(s, \theta)|_{(s,\theta)=(1,0)} = (1, 0)$, $u_\theta^{**}(s, \theta)|_{(s,\theta)=(1,0)} = (0, 1)$, and $F_\lambda(\lambda, (x, y))|_{(\lambda,(x,y))=(2,u^{**}(1,0))} = (1, 0)$. Using these relations, we see that $Y :=$ Ran $\partial_{(x,y)} F(2, u^{**}(1, 0)) = 0$, $Z :=$ ker $\partial_{(x,y)} F(2, u^{**}(1, 0)) =$ span $\langle (1, 0), (0, 1) \rangle$, $Z =$ span $\langle u_s^{**}(1, 0), u_\theta^{**}(1, 0) \rangle$, $\dim Z = 2$, $Y \oplus Z = \mathbb{R}^2$, and $\mathrm{proj}_{\mathrm{span}\langle u_s^{**}(1,0)\rangle} F_\lambda(2, u^{**}(1, 0)) \neq 0$. Hence (F6) is satisfied. Applying Theorem 3.3, we see that the turning point $(2, (1, 0))$ is not a secondary bifurcation point. Because of the rotation equivalence (3.3), $(2, u^{**}(1, \theta))$ $(\theta \in S^1)$ is not a secondary bifurcation point as well.

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