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On Convergence of the Simplicial Branch-and-Bound Algorithm

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Abstract

The simplicial algorithm is a kind of branch-and-bound method for computing a globally optimal solution of a convex maximization problem. Its convergence under the ω-subdivision branching strategy was an open problem for years until Locatelli and Raber proved it in 2000 [11]. In this paper, we modify the linear programming relaxation and give a different and simpler proof of the convergence.

Key words: Global optimization, convex maximization, branch-and-bound, simplicial algorithm, subdivision strategy.

1 Introduction

The branch-and-bound is a popular approach to intractable problems such as combinatorial optimization and integer programming problems. It can also be used for finding a globally optimal solution of multiextremal nonlinear optimization problems. A typical example of such a class of problems is a convex maximization problem of maximizing a convex function on a polyhedral set. In order to solve this problem globally, Tuy proposed in 1964 the first two rigorous algorithms [13], one of which is the conical branch-and-bound algorithm. It subdivides a simplicial cone including the feasible set into a number of simplicial cones, and computes an upper bound on the objective function over each cone by solving a linear programming relaxation. In 69, Falk and Soland assumed the objective function to be separable into univariate functions and devised the rectangular branch-and-bound algorithm [2], which is similar to combinatorial branch-and-bound algorithms for integer programs and subdivides the feasible set recursively

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into hyperrectangles. In 76, Horst developed the simplicial branch-and-bound algorithm, which requires no separability assumption and subdivides the feasible set into simplices. Since then, lots of branch-and-bound algorithms have been proposed to find a globally optimal solution efficiently, but each of them is usually a variant on one of the three pioneering algorithms.

A key step common to the three algorithms is computation of an upper bound on the objective function over each fundamental set, i.e., cone, rectangle or simplex, by solving a linear programming relaxation. It is intuitively rational to exploit its optimal solution to subdivide the fundamental set. Although this so-called ω-subdivision is known to be efficient empirically [14], the convergence of the algorithms with ω-subdivision was an open problem for years, except for the rectangular algorithm. However, in 98-99, Jaumard, Meyer [6] and Locatelli [10] completed the proof in different ways for the convergence of the conical algorithm under the ω-subdivision strategy. Locatelli and Raber [11, 12] extended their idea and proved the convergence of the simplicial algorithm with ω-subdivision in 2000.

Jaumard and Meyer's proof is based on the concept of nondegenerate subdivision process [5], though they did not mention it in [6]. Unfortunately, no one has yet succeeded in proving the convergence of the simplicial algorithm using this concept. In this paper, we try to prove it along those lines, by making a little modifications to the linear programming relaxation. In Section 2, we define the convex maximization problem formally, and illustrate how the usual simplicial branch-and-bound algorithm works on it. In Section 3, we state the condition for nondegenerate subdivision process, which ensures the convergence of the simplicial algorithm with ω-subdivision. We then show that the condition is satisfied if we slightly modify the linear programming relaxation. In Section 4, we further show that the ω-subdivision algorithm converges even under a certain generalization of the original ω-subdivision strategy. Lastly, we discuss some future issues.

2 Convex minimization and the simplicial algorithm

The convex maximization problem dealt with in this paper is as follows

$$\left|\begin{array}{ll} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{A}\mathbf{x} \le \mathbf{b}, \end{array}\right. \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and f is a real-valued convex function defined on some open set including the feasible set

$$D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}.$$

We impose the following on the problem (1) throughout the paper.

Asumption 2.1. *The feasible set D is bounded and contains the origin $\mathbf{0} \in \mathbb{R}^n$ as an interior point.*

This assumption implies that all components of $\mathbf{b}$ are positive.

As is well known, problem (1) has multiple locally optimal solutions which are not globally optimal. To locate a globally optimal solution, a number of algorithms have been developed so far (see e.g., [4, 5, 14]). Among them, as stated in Section 1, we are concerned here with the simplicial branch-and-bound algorithm, originally proposed by Horst [3] in 1976. Our main interest is in its convergence property, which has been poorly understood for over thirty years.

First of all, let us briefly illustrate how the simplicial algorithm works on our target problem (1).

WORKINGS OF THE SIMPLICIAL ALGORITHM

The basic procedures characterizing the simplicial branch-and-bound algorithm are naturally branching and bounding.

In preprocessing, the feasible set D is enclosed in an n-simplex Δ^1, which is given as $\operatorname{conv}\{\mathbf{u}_1^1, \ldots, \mathbf{u}_{n+1}^1\}$, a convex hull of $n+1$ affinely independent vectors $\mathbf{u}_j^1$s. The branching procedure subdivides Δ^1 into a set of subsimplices Δ^k, $k \in \mathcal{K}$, as follows

$$\Delta^1 = \bigcup_{k \in \mathcal{K}} \Delta^k, \quad \operatorname{int}\Delta^k \bigcap \operatorname{int}\Delta^\ell = \emptyset \text{ if } k, \ell \in \mathcal{K} \text{ and } k \neq \ell. \tag{2}$$

where $\mathcal{K}$ is an (infinite) index set, and $\operatorname{int} \cdot$ represents the set of interior points. The bounding procedure sifts through those subsimplices Δ^ks. Namely, for each $k \in \mathcal{K}$, if Δ^k shares no points with D, then Δ^k is discarded from consideration. Otherwise, the following subproblem is considered

$$\left| \begin{array}{ll} \text{maximize} & f(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in D \cap \Delta^k. \end{array} \right. \tag{3}$$

Since (3) is essentially the same as (1), it cannot be solved directly. Instead, replacing f by its concave envelope g^k, a minimal concave function overestimating f on Δ^k, the bounding procedure solves a relaxation of (3)

$$(\mathrm{P}^k) \left| \begin{array}{ll} \text{maximize} & g^k(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in D \cap \Delta^k. \end{array} \right.$$

In our case, where f is convex, g^k is an affine function which agrees with f at the vertices $\mathbf{u}_j^k$, $j = 1, \ldots, n+1$, of Δ^k. Therefore, (P^k) is a linear program and can be solved using any one of the simplex algorithms or the interior point algorithms. Let $\overline{\mathbf{x}}^k$ denote an optimal solution of (P^k). Then we have

$$g^k(\overline{\mathbf{x}}^k) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in D \cap \Delta^k. \tag{4}$$

If $g^k(\overline{\mathbf{x}}^k) \leq f(\mathbf{x}^*)$ holds for some feasible solution $\mathbf{x}^*$ of (1) found in the course of executing the algorithm, we can conclude that Δ^k contains no solution better than $\mathbf{x}^*$. The bounding procedure then discards Δ^k from further consideration. Otherwise, the branching procedure again subdivides Δ^k into smaller subsimplices. If $f(\overline{\mathbf{x}}^k) > f(\mathbf{x}^*)$, the incumbent $\mathbf{x}^*$ is updated with $\overline{\mathbf{x}}^k$ because it is a feasible solution of (1).

Simplicial subdivision strategies

The convergence of the simplicial algorithm depends largely on how to subdivide $\Delta^k = \mathrm{conv}\{\mathbf{u}_1^k, \ldots, \mathbf{u}_{n+1}^k\}$ for each $k \in \mathcal{K}$. The simplest subdivision strategy is bisection, which divides the longest edge, say $\mathbf{u}_s^k$–$\mathbf{u}_t^k$, at the midpoint $\boldsymbol{\beta}^k = (\mathbf{u}_s^k + \mathbf{u}_t^k)/2$. The resulting subsimplices of Δ^k are given as

$$\Delta^{k_j} = \mathrm{conv}\{\mathbf{u}_1^k, \ldots, \mathbf{u}_{j-1}^k, \boldsymbol{\beta}^k, \mathbf{u}_{j+1}^k, \ldots, \mathbf{u}_{n+1}^k\}, \quad j = s, t.$$

Each of these subsimplices is referred to as a *child* of Δ^k. If the branching procedure is recursively applied to Δ^k, an infinite sequence of simplices can be generated as follows

$$\Delta^k = \Delta^{k_1} \supset \cdots \supset \Delta^{k_q} \supset \Delta^{k_{q+1}} \supset \cdots,$$

where $\Delta^{k_{q+1}}$ is a child of Δ^{k_q}. Under the bisection strategy, the sequence shrinks to a single point. Since $\overline{\mathbf{x}}^{k_q} \in \Delta^{k_q}$ for $q = 1, 2, \ldots$, we have $g^{k_q}(\overline{\mathbf{x}}^{k_q}) - f(\overline{\mathbf{x}}^{k_q}) \to 0$ as $q \to 0$. This exhaustiveness guarantees that the incumbent $\mathbf{x}^*$ converges to a globally optimal solution of (1). Unfortunately, however, exhaustive subdivision strategies are still unknown except for bisection.

Although not exhaustive, an often-used alternative is ω-subdivision. This strategy exploits the optimal solution $\overline{\mathbf{x}}^k$ of (P^k) and subdivides Δ^k radially from $\boldsymbol{\omega}^k = \overline{\mathbf{x}}^k$ into up to $k+1$ subsimplices. Let J^k be an index set such that $j \in J^k$ if $\boldsymbol{\omega}^k$ is affinely independent of $\mathbf{u}_1^k, \ldots, \mathbf{u}_{j-1}^k, \mathbf{u}_{j+1}^k, \ldots, \mathbf{u}_{n+1}^k$. Then the children of Δ^k are

$$\Delta^{k_j} = \mathrm{conv}\{\mathbf{u}_1^k, \ldots, \mathbf{u}_{j-1}^k, \boldsymbol{\omega}^k, \mathbf{u}_{j+1}^k, \ldots, \mathbf{u}_{n+1}^k\}, \quad j \in J^k.$$

The ω-subdivision strategy has been said to be more efficient than bisection empirically. The theoretical convergence, however, was an open problem for years until Locatelli and Raber proved it in 2000, in some combinatorial manner [11]. In the succeeding sections, we will give another proof, which is different from and easier than theirs.

3 Nondegeneracy of a modified ω-subdivision

For each $k \in \mathcal{K}$, suppose that the concave envelope g^k of f on Δ^k is given as

$$g^k(\mathbf{x}) = \mathbf{c}^k\mathbf{x} + r^k, \tag{5}$$

where $(\mathbf{c}^k)^\top \in \mathbb{R}^n$ and $r^k \in \mathbb{R}$. Actually, even if the subdivision strategy is not exhaustive, the simplicial algorithm is known to be convergent if $\mathbf{c}^k$s possess a certain property. Select an arbitrary sequence of nested simplices generated in the simplicial algorithm and renumber the indices as follows

$$\Delta^1 \supset \cdots \supset \Delta^k \supset \Delta^{k+1} \supset \cdots, \tag{6}$$

where Δ^{k+1} is a child of Δ^k.

Nondegenerate subdivision process

Definition 3.1. [5] The sequence (6) is said to be *nondegenerate* if there exists a subsequence $\mathcal{K}' \subset \{1, 2, \ldots\}$ and a constant L such that

$$\|\mathbf{c}^k\| \leq L, \quad \forall k \in \mathcal{K}'. \tag{7}$$

Also, the subdivision process is *nondegenerate* if every sequence of nested simplices is nondegenerate.

Proposition 3.1. [5] *Suppose that (6) is generated by ω-subdivision, i.e., Δ^{k+1} is yielded by subdividing Δ^k radially from $\boldsymbol{\omega}^k$ for $k = 1, 2, \ldots$. If (6) is nondegenerate, then*

$$\liminf_{k \to \infty} |g^k(\boldsymbol{\omega}^k) - f(\boldsymbol{\omega}^k)| = 0. \tag{8}$$

When (6) satisfies the condition (8), the sequence is said to be *normal*. It is known [4, 5, 14] that the simplicial algorithm is convergent if every sequence of nested simplices is normal. In other words, to prove the convergence of the subdivision process, we need only to show the existence of a constant L in (7) for all sequences. For this purpose, we will make a little modifications to the linear programming relaxation (P^k).

RELAXATION OF THE RELAXATION (P^k)

Introducing an auxiliary variable $\tau \geq 0$, let us relax the feasible set D into

$$D(\tau) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq (1+\tau)\mathbf{b}\}.$$

Definition 3.2. For a positive constant σ, a vector $\mathbf{x}$ is referred to as a σ-feasible solution of (1) if $\mathbf{x} \in D(\sigma)$.

Let $\sigma > 0$ be a given tolerance, and select a number M to satisfy

$$M > F/\sigma, \tag{9}$$

where

$$F = \max\{f(\mathbf{u}_j^1) \mid j = 1, \ldots, n+1\}.$$

Also, define a function h^k of variables $\mathbf{x}$ and τ as follows

$$h^k(\mathbf{x}, \tau) = g^k(\mathbf{x}) - M\tau.$$

Instead of (P^k), consider

$$(\mathrm{Q}^k) \left| \begin{array}{ll} \text{maximize} & h^k(\mathbf{x}, \tau) \\ \text{subject to} & \mathbf{x} \in D(\tau) \cap \Delta^k, \quad \tau \geq 0. \end{array} \right.$$

It is easy to see that (Q^k) is equivalent to a linear program

$$(\mathrm{PQ}) \left| \begin{array}{ll} \text{maximize} & \mathbf{g}^{\mathsf{T}}\boldsymbol{\lambda} - M\tau \\ \text{subject to} & \mathbf{AU}\boldsymbol{\lambda} - \mathbf{b}\tau \leq \mathbf{b}, \quad \mathbf{e}^{\mathsf{T}}\boldsymbol{\lambda} = 1, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \tau \geq 0, \end{array} \right.$$

where $\mathbf{e} \in \mathbb{R}^{n+1}$ is the all-ones vector and

$$\mathbf{g} = [f(\mathbf{u}_1^k), \ldots, f(\mathbf{u}_{n+1}^k)]^{\mathsf{T}}, \quad \mathbf{U} = [\mathbf{u}_1^k, \ldots, \mathbf{u}_{n+1}^k].$$

The dual problem of (PQ) is written as

$$(\mathrm{DQ}) \left| \begin{array}{ll} \text{minimize} & \mathbf{b}^{\mathsf{T}}\boldsymbol{\mu} + \nu \\ \text{subject to} & \mathbf{U}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\boldsymbol{\mu} + \mathbf{e}\nu \geq \mathbf{g}, \quad \mathbf{b}^{\mathsf{T}}\boldsymbol{\mu} \leq M, \quad \boldsymbol{\mu} \geq \mathbf{0}. \end{array} \right.$$

Note that both (PQ) and (DQ) have optimal solutions because the objective function value of (PQ) is bounded from above by $\max\{f(\mathbf{u}_j^k) \mid j = 1, \ldots, n+1\}$. Let $(\widetilde{\boldsymbol{\lambda}}, \widetilde{\tau})$ and $(\widetilde{\boldsymbol{\mu}}, \widetilde{\nu})$ denote their respective optimal solutions. Then $(\widetilde{\mathbf{x}}^k, \widetilde{\tau}^k) = (\mathbf{U}\widetilde{\boldsymbol{\lambda}}, \widetilde{\tau})$ is obviously an optimal solution of (Q^k).

Proposition 3.2. *The optimal value of* (Q^k) *is an upper bound on the optimal value of* (P^k), *i.e.,*

$$h^k(\widetilde{\mathbf{x}}^k, \widetilde{\tau}^k) \geq g(\overline{\mathbf{x}}).$$

This proposition, together with (4), implies that (Q^k) can serve as a substitute for (P^k) in the bounding procedure. We can also prove the following:

Proposition 3.3. *If* $h^k(\widetilde{\mathbf{x}}^k, \widetilde{\tau}^k) < 0$, *then* $D \cap \Delta^k = \emptyset$. *Otherwise,* $\widetilde{\mathbf{x}}^k$ *is a* σ*-feasible solution of (1).*

If $D \cap \Delta^k = \emptyset$, the bounding procedure discards Δ^k from consideration. We may therefore assume $\widetilde{\mathbf{x}}^k \in D(\sigma)$ for every k. Let

$$\Delta_+^k = \mathrm{conv}\{\mathbf{u}_j^k \mid j \in J_+\}, \quad J_+ = \{j \mid \widetilde{\lambda}_j > 0\}.$$

Then $\widetilde{\mathbf{x}}^k$ belongs to Δ_+^k, and besides we obtain the following from the complementary slackness conditions on $(\widetilde{\boldsymbol{\lambda}}, \widetilde{\tau})$ and $(\widetilde{\boldsymbol{\mu}}, \widetilde{\nu})$ (see e.g., [1]).

Proposition 3.4. *For any* $\mathbf{x} \in \Delta^k$, *it holds that*

$$g^k(\mathbf{x}) = \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{A}\mathbf{x} + \widetilde{\nu}. \tag{10}$$

This result suggests that we may choose $\mathbf{c}^k$ and r^k in (5) as follows

$$\mathbf{c}^k = \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{A}, \quad r^k = \widetilde{\nu}. \tag{11}$$

Existence of the bounding constant

Now, we are ready to show the existence of L in (7) for the sequence (6).

Using $\mathbf{c}^k$ in (11), define a halfspace

$$H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{c}^k \mathbf{x} \leq \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{b}\}.$$

The distance $\delta(\mathbf{0}, \partial H)$ from the origin $\mathbf{0} \in \mathbb{R}^n$ to the boundary hyperplane ∂H is given by $\widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{b} / \|\mathbf{c}^k\|$. For any $\mathbf{x} \in D = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$, however,

$$\mathbf{c}^k \mathbf{x} = \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{A}\mathbf{x} \leq \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{b}. \tag{12}$$

Recall Assumption 2.1 states that $\mathbf{0}$ is an interior point of D. Since (12) implies that D is a subset of H, the distance $\delta(\mathbf{0}, \partial H)$ must be bounded from below by $\delta(\mathbf{0}, \partial D)$, the distance from $\mathbf{0}$ to the boundary of D. Thus, by noting the constraints of (DQ), we have

$$\|\mathbf{c}^k\| = \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{b} / \delta(\mathbf{0}, \partial H) \leq \widetilde{\boldsymbol{\mu}}^{\mathsf{T}} \mathbf{b} / \delta(\mathbf{0}, \partial D) \leq M / \delta(\mathbf{0}, \partial D),$$

where $M/\delta(\mathbf{0}, \partial D)$ is a constant for each instance of (1).

Proposition 3.5. *If $\mathbf{c}^k$ is chosen as in (11), then there exists a constant L such that*

$$\|\mathbf{c}^k\| \leq L, \quad k = 1, 2, \ldots.$$

4 Convergence of the subdivision processes

Let us reexamine the results in the preceding section. To make the subdivision process nondegenerate, we may simply choose $\widetilde{\mathbf{x}}^k$ from the optimal solution of (Q^k) as the center $\boldsymbol{\omega}^k$ for subdividing Δ^k, as we set $\boldsymbol{\omega}^k = \overline{\mathbf{x}}^k$ in the usual ω-subdivision algorithm, because $\widetilde{\mathbf{x}}^k$ is a point of Δ_+^k. However, it is noteworthy in Proposition 3.4 that (10) holds not only for $\mathbf{x} = \widetilde{\mathbf{x}}^k$, but for any $\mathbf{x} \in \Delta_+^k$. Moreover, we should remark in Proposition 3.5 that $\|\mathbf{c}^k\|$ is bounded by a constant for every k. On the basis of these observations, we have the following proposition somewhat stronger than Proposition 3.1.

Proposition 4.1. *If Δ^{k+1} is yielded by subdividing Δ^k radially from $\boldsymbol{\omega}^k \in \Delta_+^k$ for $k = 1, 2, \ldots,$ then*

$$\lim_{k \to \infty} |g^k(\boldsymbol{\omega}^k) - f(\boldsymbol{\omega}^k)| = 0. \tag{13}$$

Proof. Let us denote the hypograph of g^k by

$$G^k = \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \leq g^k(\mathbf{x})\},$$

which is an $n+1$-dimensional halfspace because g^k is an affine function. Also let

$$\boldsymbol{\xi}^k = (\boldsymbol{\omega}^k, g^k(\boldsymbol{\omega}^k)), \quad \boldsymbol{\eta}^k = (\boldsymbol{\omega}^k, f(\boldsymbol{\omega}^k)).$$

The sequence $\{\boldsymbol{\xi}^k \mid k = 1, 2, \ldots\}$ is bounded, and by the convexity of f it satisfies

$$\boldsymbol{\xi}^k \notin G^{k+1}, \quad \boldsymbol{\xi}^k \in \textstyle\bigcap_{j=1}^k G^j, \quad k = 1, 2, \ldots.$$

From the *bounded convergence principle* (see e.g., [5]) we see, therefore, that $\delta(\boldsymbol{\xi}^k, G^{k+1}) \to 0$, and hence $\delta(\boldsymbol{\xi}^k, \partial G^{k+1}) \to 0$ as $k \to \infty$. Note that

$$\delta(\boldsymbol{\xi}^k, \partial G^{k+1}) = \frac{|(\mathbf{c}^{k+1})^\top \boldsymbol{\omega}^k + r^{k+1} - g^k(\boldsymbol{\omega}^k)|}{(1 + \|\mathbf{c}^{k+1}\|^2)^{1/2}},$$

because $\partial G^{k+1} = \{(\mathbf{x}, y) \in \mathbb{R}^n \times \mathbb{R} \mid y = (\mathbf{c}^{k+1})^\top \mathbf{x} + r^{k+1}\}$. We also have

$$\delta(\boldsymbol{\xi}^k, \partial G^{k+1}) = \frac{|f(\boldsymbol{\omega}^k) - g^k(\boldsymbol{\omega}^k)|}{(1 + \|\mathbf{c}^{k+1}\|^2)^{1/2}},$$

because $\boldsymbol{\eta}^k \in \partial G^{k+1}$ and hence $f(\boldsymbol{\omega}^k) = (\mathbf{c}^{k+1})^{\mathrm{T}}\boldsymbol{\omega}^k + r^{k+1}$. By noting Proposition 3.5 we have

$$|f(\boldsymbol{\omega}^k) - g^k(\boldsymbol{\omega}^k)| \leq (1 + L^2)^{1/2}\delta(\boldsymbol{\xi}^k, \partial G^{k+1}) \to 0$$

as $k \to \infty$. □

The convergence result for the usual ω-subdivision process, where $\boldsymbol{\omega}^k = \widetilde{\mathbf{x}}^k$ for $k = 1, 2, \ldots$, is just a corollary of this proposition.

Corollary 4.2. *If Δ^{k+1} is yielded by subdividing Δ^k radially from $\boldsymbol{\omega}^k = \widetilde{\mathbf{x}}^k$ for $k = 1, 2, \ldots$, then*

$$\lim_{k \to \infty} |g^k(\boldsymbol{\omega}^k) - f(\boldsymbol{\omega}^k)| = 0.$$

Proposition 4.1, however, is not sufficient to ensure the convergence of the algorithm to an optimal solution or an approximately optimal solution of the target problem (1). For this purpose, we need to restrict the selection of $\boldsymbol{\omega}^k$ from Δ^k_+ for each k. If we select $\boldsymbol{\omega}^k = \widetilde{\mathbf{x}}^k$ for each k, as in Corollary 4.2, and update the incumbent $\mathbf{x}^*$ with $\widetilde{\mathbf{x}}^k$ appropriately, we can prove that $\mathbf{x}^*$ converges to an *optimal σ-feasible solution* $\mathbf{x}^\sigma$, i.e.,

$$\mathbf{x}^\sigma \in D(\sigma) \quad \text{and} \quad \mathrm{f}(\mathbf{x}^\sigma) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in D.$$

Although it is a rather satisfactory result from a theoretical viewpoint, this subdivision strategy inherits a serious weak point from the original ω-subdivision strategy, i,e., Δ^k is subdivided into up to $n + 1$ subsimplices for each k. We have to solve $n + 1$ linear programs, in the worst case, to update the upper bound on the values of f over $D \cap \Delta^k$. From a practical viewpoint, this is a major drawback of ω-subdivision compared to bisection, in particular when we terminate the algorithm prematurely and use the incumbent $\mathbf{x}^*$ as a heuristic solution [8, 9]. One way to improve it is to divide some edge of Δ^k_+. Since this strategy equivalent to subdividing Δ^k radially from $\boldsymbol{\omega}^k$ on some edge of Δ^k_+, the resulting subdivision process is convergent. To guarantee the convergence of the algorithm as well, we need to introduce additional devices, the detail of which will be reported elsewhere.

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