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ロバスト Nash 均衡問題に対する解の一意存在性について

The uniqueness of the robust Nash equilibrium in N-person noncooperative games

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1 Introduction

In this paper, we consider N-person noncooperative games with uncertain data. For them, distribution-free models based on the worst-case analysis attract much attention in recent years [1, 13]. In such models, each player makes a decision according to the idea of robust optimization [5, 6, 8]. Originally, robust optimization is a technique for handling optimization problems with uncertain parameters, in which those uncertain parameters are assumed to belong to so-called *uncertainty sets*, and then the objective function is minimized (or maximized) by taking into account the worst possible case. An equilibrium resulting from the robust optimization by each player is called a robust Nash equilibrium, and the problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem. Hayashi, Yamashita, and Fukushima [13] defined the concept of robust Nash equilibria for bimatrix games. Under the assumption that uncertain sets are expressed by means of the Euclidean or the Frobenius norm, they showed that each player's problem reduces to a second-order cone program (SOCP) [2] and the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) [11, 12]. In this paper, we extend the definition of robust Nash equilibria in [1] and [13] to N-person non-cooperative games with nonlinear cost functions. In particular, we show existence of robust Nash equilibria under the assumption that each player's cost function is *convex* with respect to his strategy, while [1] and [13] only considered the *linear* case. Moreover, we give some sufficient conditions for uniqueness of a robust Nash equilibrium. In order to solve certain classes of robust Nash equilibrium problems, we reformulate them to second-order cone complementarity problems.

Throughout the paper, we use the following notations. For a set X, $\mathcal{P}(X)$ denotes the set consisting of all the subsets of X. $\Re^n_+$ denotes the nonnegative orthant in $\Re^n$, that is, $\Re^n_+ := \{x \in \Re^n \mid x_i \geq 0\ (i = 1, \ldots, n)\}$. For a vector $x \in \Re^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \Re^{n\times m}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^n \sum_{j=1}^m (M_{ij})^2)^{1/2}$.

2 Robust Nash equilibrium

In this paper, we consider an N-person non-cooperative game in which each player tries to minimize his own cost. Let $x^i \in \Re^{m_i}$, $S_i \subseteq \Re^{m_i}$, and $f_i : \Re^{m_i} \times \Re^{m_{-i}} \to \Re$ be player i's strategy, strategy set, and cost function, respectively. Moreover, we denote

$$\mathcal{I} := \{1, \ldots, N\}, \quad \mathcal{I}_{-i} := \mathcal{I} \setminus \{i\}, \quad m := \sum_{j\in\mathcal{I}} m_j, \quad m_{-i} := \sum_{j\in\mathcal{I}_{-i}} m_j,$$
$$x := (x^j)_{j\in\mathcal{I}} \in \Re^m, \quad x^{-i} := (x^j)_{j\in\mathcal{I}_{-i}} \in \Re^{m_{-i}},$$
$$S := \prod_{j\in\mathcal{I}} S_j \subseteq \Re^m, \quad S_{-i} := \prod_{j\in\mathcal{I}_{-i}} S_j \subseteq \Re^{m_{-i}}.$$

When the complete information is assumed, each player i decides his own strategy by solving the following optimization problem with the opponents' strategy x^{-i} fixed:

$$\begin{aligned} \underset{x^i}{\text{minimize}} \quad & f_i(x^i, x^{-i}) \\ \text{subject to} \quad & x^i \in S_i. \end{aligned} \tag{2.1}$$

A tuple $(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N)$ satisfying $\bar{x}^i \in \operatorname{argmin}_{x^i \in S_i} f_i(x^i, \bar{x}^{-i})$ for each player $i = 1, \ldots, N$ is called a Nash equilibrium. In other words, if each player i chooses the strategy $\bar{x}^i$, then no player has an incentive to change his own strategy. The Nash equilibrium is well-defined only when each player can estimate his opponents' strategies and evaluate his own cost exactly. In the real situation, however, any information may contain uncertainty such as observation errors or estimation errors. Thus, in this paper, we focus on games with uncertainty.

To deal with such uncertainty, we introduce uncertainty sets U_i and $X_i(x^{-i})$, and assume the following statements for each player $i \in \mathcal{I}$:

(A) Player i's cost function involves a parameter $\hat{u}^i \in \Re^{\nu_i}$, i.e., it can be expressed as $f_i^{\hat{u}^i} : \Re^{m_i} \times \Re^{m_{-i}} \to \Re$. Although player i do not know the exact value of $\hat{u}^i$ itself, he can estimate that it belongs to a given nonempty set $U_i \subseteq \Re^{\nu_i}$.
(B) Although player i knows his opponents' strategies x^{-i}, his actual cost is evaluated with x^{-i} replaced by $\hat{x}^{-i} = x^{-i} + \delta x^{-i}$, where δx^{-i} is a certain error or noise. Player i cannot know the exact value of $\hat{x}^{-i}$. However, he can estimate that $\hat{x}^{-i}$ belongs to a certain nonempty set $X_i(x^{-i})$.

Then, each player is required to address the following family of problems involving uncertain parameters $\hat{u}^i$ and $\hat{x}^{-i}$:

$$\begin{aligned} \underset{x^i}{\text{minimize}} \quad & f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \\ \text{subject to} \quad & x^i \in S_i, \end{aligned} \tag{2.2}$$

where $\hat{u}^i \in U_i$ and $\hat{x}^{-i} \in X_i(x^{-i})$. We further assume that each player chooses his strategy according to the following criterion:

(C) Player i tries to minimize his worst cost under assumptions (A) and (B).

From assumption (C), each player considers the worst cost function $\tilde{f}_i : \Re^{m_i} \times \Re^{m_{-i}} \to (-\infty, +\infty]$ defined by

$$\tilde{f}_i(x^i, x^{-i}) := \sup\{f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \mid \hat{u}_i \in U_i, \hat{x}^{-i} \in X_i(x^{-i})\}, \tag{2.3}$$

and solves the following worst cost minimization problem:

$$\begin{aligned} \underset{x^i}{\text{minimize}} \quad & \tilde{f}_i(x^i, x^{-i}) \\ \text{subject to} \quad & x^i \in S_i. \end{aligned} \tag{2.4}$$

Note that (2.4) is regarded as a complete information game with cost functions $\tilde{f}_i$. Based on the above discussions, we define the robust Nash equilibrium.

Definition 2.1. *Let $\tilde{f}_i$ be defined by* (2.3) *for $i = 1, \ldots, N$. A tuple $(\bar{x}^i)_{i \in \mathcal{I}}$ is called a robust Nash equilibrium of game* (2.2)*, if $\bar{x}^i \in \operatorname{argmin}_{x^i \in S_i} \tilde{f}_i(x^i, \bar{x}^{-i})$ for all i, i.e., a Nash equilibrium of game* (2.4)*. The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.*

3 Existence of robust Nash equilibria

In this section, we give sufficient conditions for the existence of a robust Nash equilibria. Note that $X_i(x^{-i})$ given in (B) can be regarded as a set-valued mapping $X_i(\cdot)$ with variable x^{-i}.

In what follows, we suppose that $X_i(\cdot)$, U_i, f^{u^i} and S_i in (A) and (B) satisfy the following assumption.

Assumption 1. *For every $i \in \mathcal{I}$, the following statements hold.*

(a) *The function $G_i : \Re^{m_i} \times \Re^{m_{-i}} \times \Re^{\nu_i} \to \Re$ defined by $G_i(x^i, x^{-i}, u^i) := f_i^{u^i}(x^i, x^{-i})$ is continuous.*
(b) *The set-valued mapping $X_i : \Re^{m_{-i}} \to \mathcal{P}(\Re^{m_{-i}})$ is continuous, and $X_i(x^{-i})$ is nonempty and compact for any $x^{-i} \in S_{-i}$.*
(c) *The set $U_i \subseteq \Re^{\nu_i}$ is nonempty and compact.*
(d) *The set S_i is nonempty, compact and convex, and function $f_i^{u^i}(\cdot, x^{-i}) : \Re^{m_i} \to \Re$ is convex on S_i for any fixed x^{-i} and u^i.*

Under Assumption 1, the function $\tilde{f}_i(x^i, x^{-i})$ defined by (2.3) has the following properties:

- $\tilde{f}_i(x^i, x^{-i})$ is continuous and finite at any $(x^i, x^{-i}) \in S_i \times S_{-i}$.
- For any fixed $x^{-i} \in S_{-i}$, function $\tilde{f}_i(\cdot, x^{-i}) : \Re^{m_i} \to \Re$ is convex on S_i.

The continuity and finiteness of $\tilde{f}_i$ can be verified from [4, Theorem 1.4.16], while the convexity of $\tilde{f}_i(\cdot, x^{-i})$ follows from [7, Proposition 1.2.4(c)].

The following lemma is a well-known result for N-person non-cooperative games.

Lemma 3.1. [3, Theorem 9.1.1] *Suppose that, for every player $i \in \mathcal{I}$,* (i) *the strategy set S_i is nonempty, convex and compact,* (ii) *the cost function $f_i : \Re^{m_i} \times \Re^{m_{-i}} \to \Re$ is continuous, and* (iii) *$f_i(\cdot, x^{-i})$ is convex for any $x^{-i} \in S_{-i}$. Then, game* (2.1) *has at least one Nash equilibrium.*

By this lemma, we obtain the following theorem for the existence of a robust Nash equilibrium in game (2.2). For the proof of the following theorem, refer to [14].

Theorem 3.2. *Suppose that Assumption 1 holds. Then, game* (2.2) *has at least one robust Nash equilibrium.*

4 Uniqueness of the robust Nash equilibrium

In the previous section, we have studied sufficient conditions for existence of robust Nash equilibria. Under such conditions, there exist a number of robust Nash equilibria in general, and it is difficult to find them all. In this section, we therefore study conditions for uniqueness of a robust Nash equilibrium.

For complete information games, Rosen [15] gave some conditions for the uniqueness of a Nash equilibrium. Those conditions are essentially equivalent to the strict monotonicity of the vector-valued function involved in the equivalent variational inequality problem (VIP) [9]. Moreover, such a vector-valued function is defined by using the derivatives of all players' cost functions. However, since the worst cost function $\tilde{f}_i$ defined by (2.3) is in general nondifferentiable, the VIP reformulation approach cannot be applied directly. This fact prompts us to consider the generalized VIP (GVIP), which is defined by means of a set-valued mapping. Then, by using the uniqueness results for GVIP, we establish sufficient conditions for the uniqueness of a robust Nash equilibrium.

For a given set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ and a nonempty closed convex set Ω, GVIP($\mathcal{F}, \Omega$) is to find a vector $x \in \Omega$ such that

$$\text{GVIP}(\mathcal{F}, \Omega): \qquad \exists \xi \in \mathcal{F}(x), \quad \langle \xi, y - x \rangle \geq 0 \quad \forall y \in \Omega. \tag{4.1}$$

If the set-valued mapping $\mathcal{F}$ is given by $\mathcal{F}(x) = \{F(x)\}$ for a vector-valued function $F : \Re^n \to \Re^n$, then the GVIP reduces to the following VIP:

$$\text{VIP}(F, \Omega): \qquad \langle F(x), y - x \rangle \geq 0 \quad \forall y \in \Omega. \tag{4.2}$$

It is well known that if the function F is strictly monotone, then VIP (4.2) has at most one solution [9]. In fact, a similar result holds for GVIP [10]. Recall that the set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is said to be monotone (strictly monotone) on a nonempty convex set $\Omega \subseteq \Re^n$ if

$$\langle x - y, \xi - \eta \rangle \geq (>)\, 0$$

for all $x, y \in \Omega$ $(x \neq y)$ and $\xi \in \mathcal{F}(x)$, $\eta \in \mathcal{F}(y)$.

Proposition 4.1. *Suppose that the set-valued mapping $\mathcal{F} : \Re^n \to \mathcal{P}(\Re^n)$ is strictly monotone on Ω. Then, GVIP* (4.1) *has at most one solution.*

Next, we reformulate a robust Nash equilibrium problem as a GVIP. Specifically, the robust Nash equilibrium problem (2.4) is equivalent to GVIP($\tilde{\mathcal{F}}, \Omega$) with $\tilde{\mathcal{F}} : \Re^m \to \mathcal{P}(\Re^m)$ and Ω defined by

$$\tilde{\mathcal{F}}(x) := \left(\partial_i \tilde{f}_i(x^i, x^{-i})\right)_{i \in \mathcal{I}} \tag{4.3}$$

and $\Omega := S = S_1 \times \cdots \times S_N$, respectively. Here, $\partial_i \tilde{f}_i$ denotes the subdifferential of $\tilde{f}_i$ with respect to player i's strategy x^i.

If Assumption 1 holds, then there exists at least one robust Nash equilibrium from Theorem 3.2. Moreover, by Proposition 4.1, if the set-valued mapping $\tilde{\mathcal{F}}$ defined by (4.3) is strictly monotone, then game (2.2) has a unique robust Nash equilibrium.

Next, we give sufficient conditions for $\tilde{\mathcal{F}}$ to be strictly monotone. To this end, we introduce the following assumption:

Assumption 2. *For each $i \in \mathcal{I}$, the following conditions hold:*

(a) *The set $X_i(x^{-i})$ is given by $X_i(x^{-i}) = x^{-i} + D_i$ for a nonempty compact set $D_i \subseteq \Re^{m_{-i}}$.*
(b) *Function $f_i^{u^i}$ is expressed as $f_i^{u^i}(x^i, x^{-i}) := g_i^{u^i}(x^i) + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top A_{ij} x^j$ with a convex function $g_i^{u^i} : \Re^{m_i} \to \Re$ and matrices $A_{ij} \in \Re^{m_i \times m_j}$ $(j \in \mathcal{I}_{-i})$.*
(c) *Either of the following statements holds:*
    (c-i) *For any $u^i \in U_i$ and $i \in \mathcal{I}$, the function $g_i^{u^i}$ is strongly convex with modulus $\gamma > -\lambda_{\min}(\overline{A}_0)$, where $\lambda_{\min}(\overline{A}_0)$ denotes the minimum eigenvalue of $\overline{A}_0 := (A_0 + A_0^\top)/2$ with*

$$A_0 := \begin{bmatrix} 0 & A_{12} & \cdots & A_{1N} \\ A_{21} & 0 & & A_{2N} \\ \vdots & & \ddots & \vdots \\ A_{N1} & A_{N2} & \cdots & 0 \end{bmatrix}.$$

(c-ii) *U_i is a singleton, i.e., $U_i = \{u^i\}$, and the set-valued mapping $\mathcal{F} : \Re^m \to \mathcal{P}(\Re^m)$ defined by*

$$\mathcal{F}(x) := (\partial_i f_i^{u^i}(x^i, x^{-i}))_{i \in \mathcal{I}} \tag{4.4}$$

is strictly monotone.

Under the above assumption, we have the following lemma. For the proof of the lemma, refer to [14].

Lemma 4.2. *Suppose that Assumption 2 holds. Then, the set-valued mapping $\tilde{\mathcal{F}}$ defined by (4.3) is strictly monotone.*

By the above lemmas, we obtain the following theorem on the uniqueness of a robust Nash equilibrium. For the proof of the theorem, refer to [14].

Theorem 4.3. *Suppose that Assumptions 1 and 2 hold. Then, game (2.2) has a unique robust Nash equilibrium.*

5 SOCCP formulation of robust Nash equilibrium problem

In this section, we focus on the game in which each player takes a mixed strategy and minimizes a convex quadratic cost function with respect to his own strategy. We show that the robust Nash equilibrium problem then reduces to an SOCCP. We also discuss the existence and uniqueness properties by using the results obtained heretofore.

Here, we consider an SOCCP [11, 12] of the form

$$\mathcal{K} \ni M\zeta + q \perp N\zeta + r \in \mathcal{K}, \quad C\zeta = d \tag{5.1}$$

with variable $\zeta \in \Re^{l+\tau}$ and constants $M, N \in \Re^{l \times (l+\tau)}$, $q, r \in \Re^l, C \in \Re^{\tau \times (l+\tau)}$ and $d \in \Re^\tau$. SOCCP can be solved by some existing algorithms such as a smoothing and regularization method [12].

Throughout this section, the cost functions and the strategy sets are given as follows.

(i) Player i's cost function $f_i^{\hat{u}^i}$ is given by

$$f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) = \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + (x^i)^\top \Big(\sum_{j \in \mathcal{I}_{-i}} \hat{A}_{ij} \hat{x}^j + \hat{c}^i \Big), \tag{5.2}$$

where $\hat{A}_{ij} \in \Re^{m_i \times m_j}$ $(j \in \mathcal{I})$ and $\hat{c}^i \in \Re^{m_i}$ are given constants involving uncertainties.

(ii) Player i takes a mixed strategy, i.e.,

$$S_i = \{x^i \mid x^i \geq 0,\ e_{m_i}^\top x^i = 1\}, \tag{5.3}$$

where e_{m_i} denotes the vector $(1, 1, \ldots, 1)^\top \in \Re^{m_i}$.

We call $\hat{A}_{ij}$ and $\hat{c}^i$ a cost matrix and a cost vector, respectively. Note that these constants correspond to the cost function parameter $\hat{u}^i$, i.e.,

$$\hat{u}^i = \mathrm{vec}\,[\hat{A}_{i1} \cdots \hat{A}_{iN}\ \hat{c}^i] \in \Re^{m_i(m+1)} \tag{5.4}$$

where vec denotes the vectorization operator that creates an nm-dimensional vector $[(p_1^c)^\top \cdots (p_m^c)^\top]^\top$ from a matrix $P \in \Re^{n\times m}$ with column vectors $p_1^c, \dots, p_m^c$.

5.1 Uncertainty in the opponents' strategy

In this subsection, we consider the case where each player knows the cost matrices and vectors exactly but the opponents' strategies uncertainly. More specifically, we suppose the following assumption holds.

Assumption 3. *For each $i \in \mathcal{I}$, uncertainty sets $X_i(\cdot)$ and U_i $(i \in \mathcal{I})$ are given as follows.*

(a) $X_i(x^{-i}) = \prod_{j\in\mathcal{I}_{-i}} X_{ij}(x^j)$, *where* $X_{ij}(x^j) := \{x^j + \delta x^{ij} \mid \|\delta x^{ij}\| \le \rho_{ij},\ e_{m_j}^\top \delta x^{ij} = 0\}$ *with a given constant* $\rho_{ij} \ge 0$.

(b) U_i *is a singleton, i.e.,* $U_i := \{u^i\} = \{\mathrm{vec}[A_{i1}\cdots A_{iN}\ c^i]\}$. *Moreover,* A_{ii} *is symmetric and positive semidefinite.*

In Assumption 3(a), the condition $e_{m_j}^\top \delta x^{ij} = 0$ is provided so that $e_{m_j}^\top (x^j + \delta x^{ij}) = 1$ holds for $x^j \in S_j$. Under this assumption, the worst cost function $\tilde{f}_i$ can be expressed explicitly as follows:

$$\tilde{f}_i(x^i, x^{-i}) = \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j\in\mathcal{I}_{-i}} A_{ij} x^j + (c^i)^\top x^i + \sum_{j\in\mathcal{I}_{-i}} \rho_{ij} \|\tilde{A}_{ij}^\top x^i\|, \tag{5.5}$$

where $\tilde{A}_{ij} := A_{ij}(I_{m_j} - m_j^{-1} e_{m_j} e_{m_j}^\top)$.

5.1.1 Reformulation as SOCCP

We first show that the robust Nash equilibrium problem reduces to the SOCCP (5.1). By using the explicit expression (5.5) of $\tilde{f}_i$ and auxiliary variables $y_{ij} \in \Re$ $(j \in \mathcal{I}_{-i})$, player i's worst cost minimization problem (2.4) can be reformulated as the following SOCP:

$$\begin{array}{ll} \underset{x^i, y_{ij}}{\text{minimize}} & \frac{1}{2}(x^i)^\top A_{ii} x^i + (x^i)^\top \sum_{j\in\mathcal{I}_{-i}} A_{ij} x^j + (c^i)^\top x^i + \sum_{j\in\mathcal{I}_{-i}} \rho_{ij} y_{ij} \\ \text{subject to} & \|\tilde{A}_{ij}^\top x^i\| \le y_{ij} \ (j \in \mathcal{I}_{-i}), \quad x^i \ge 0, \quad e_{m_i}^\top x^i = 1. \end{array}$$

Moreover, the Karush-Kuhn-Tucker (KKT) conditions of this problem can be written as the following SOCCP:

$$\begin{gathered} \mathcal{K}^{m_j+1} \ni \begin{bmatrix} \mu_{ij} \\ \lambda^{ij} \end{bmatrix} \perp \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_{ij}^\top \end{bmatrix} \begin{bmatrix} y_{ij} \\ x^i \end{bmatrix} \in \mathcal{K}^{m_j+1} \ (j \in \mathcal{I}_{-i}) \\ \Re_+^{m_i} \ni x^i \perp A_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} \left(A_{ij} x^j - \tilde{A}_{ij} \lambda^{ij}\right) + c^i + e_{m_i} s_i \in \Re_+^{m_i}, \quad e_{m_i}^\top x^i = 1, \\ \mu_{ij} = \rho_{ij} \ (j \in \mathcal{I}_{-i}), \end{gathered}$$

where $\lambda^{ij} \in \Re^{m_j}$ and $s_i \in \Re$ are Lagrange multipliers, and $\mu_{ij} \in \Re$ are auxiliary variables. Noticing that the above KKT conditions hold for all players simultaneously, the robust Nash equilibrium problem can be reformulated as the SOCCP (5.1).

5.1.2 Existence and uniqueness of robust Nash equilibrium

Next, we study existence and uniqueness of the robust Nash equilibrium under Assumption 3. In the following analyses, we make use of the results from Theorems 3.2 and 4.3. For the proofs of the following theorems, refer to [14].

Theorem 5.1. *Suppose that the cost functions and the strategy sets are given by* (5.2) *and* (5.3), *respectively. Suppose further that Assumption* 3 *holds. Then, there exists at least one robust Nash equilibrium.*

Theorem 5.2. *Suppose that the cost functions and the strategy sets are given by* (5.2) *and* (5.3), *respectively. Suppose further that Assumption* 3 *holds. Then there exists a unique robust Nash equilibrium, provided that*

$$A := \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{N1} & \cdots & \cdots & A_{NN} \end{bmatrix} \succ 0. \tag{5.6}$$

5.2 Uncertainty in the cost matrices and vectors

In this subsection, we consider the case where each player can estimate the opponents' strategies exactly, but estimates his cost matrices and vectors uncertainly. We first make the following assumption.

Assumption 4. *For each $i \in \mathcal{I}$, uncertainty sets $X_i(\cdot)$ and U_i ($i \in \mathcal{I}$) are given as follows.*

(a) $X_i(x^{-i}) := \{x^{-i}\}$.

(b) $U_i := (\prod_{j \in \mathcal{I}} D_{A_{ij}}) \times D_{c^i}$ *with* $D_{A_{ij}} := \{A_{ij} + \delta A_{ij} \mid \|\delta A_{ij}\|_F \le \rho_{ij}\} \subseteq \Re^{m_i \times m_j}$ *and* $D_{c^i} := \{c^i + \delta c^i \mid \|\delta c^i\| \le \gamma_i\} \subseteq \Re^{m_i}$ *for some nonnegative scalars ρ_{ij} and γ_i. Moreover, $A_{ii} + \rho_{ii} I$ is symmetric and positive semidefinite.*

Under this assumption, the worst cost function $\tilde{f}_i$ in (2.4) can be rewritten as follows:

$$\tilde{f}_i(x^i, x^{-i}) = \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I)x^i + (c^i)^\top x^i + \sum_{j \in \mathcal{I}_{-i}} \left((x^i)^\top A_{ij} x^j + \rho_{ij} \|x^i\| \|x^j\|\right) + \gamma_i \|x^i\|. \tag{5.7}$$

5.2.1 Reformulation as SOCCP

We first reformulate the robust Nash equilibrium problem as SOCCP (5.1) under Assumption 4. By using (5.7) and an auxiliary variable $y_i \in \Re$, the minimization problem (2.4) can be rewritten as the following SOCP:

$$\begin{array}{ll} \underset{x^i, y_i}{\text{minimize}} & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I)x^i + (c^i)^\top x^i + \sum_{j \in \mathcal{I}_{-i}} \left((x^i)^\top A_{ij} x^j + \rho_{ij} \|x^j\| y_i\right) + \gamma_i y_i \\ \text{subject to} & \|x^i\| \le y_i, \quad x^i \ge 0, \quad e_{m_i}^\top x^i = 1, \end{array} \tag{5.8}$$

and its KKT conditions are given by

$$\mathcal{K}^{m_i+1} \ni \begin{bmatrix} y_i \\ x^i \end{bmatrix} \perp \begin{bmatrix} \sum_{j\in\mathcal{I}_{-i}} \rho_{ij}\|x^j\| + \gamma_i \\ (A_{ii} + \rho_{ii} I)x^i + \sum_{j\in\mathcal{I}_{-i}} A_{ij}x^j + e_{m_i}s_i - \lambda^i + c^i \end{bmatrix} \in \mathcal{K}^{m_i+1}$$
$$\Re_+^{m_i} \ni \lambda^i \perp x^i \in \Re_+^{m_i}, \quad e_{m_i}^\top x^i = 1, \tag{5.9}$$

where $\lambda^i \in \Re^{m_i}$ and $s_i \in \Re$ are Lagrange multipliers. It is not straightforward to reformulate the robust Nash equilibrium problem as SOCCP (5.1), since the KKT conditions (5.9) contains the nonlinear term $\|x^j\|$. However, by introducing auxiliary variables $z_j \in \Re$, $u^j \in \Re^{m_j}$, we can rewrite (5.9) as follows:

$$\mathcal{K}^{m_i+1} \ni \begin{bmatrix} y_i \\ x^i \end{bmatrix} \perp \begin{bmatrix} \sum_{j\in\mathcal{I}_{-i}} \rho_{ij}z_j + \gamma_i \\ (A_{ii} + \rho_{ii} I)x^i + \sum_{j\in\mathcal{I}_{-i}} A_{ij}x^j + e_{m_i}s_i - \lambda^i + c^i \end{bmatrix} \in \mathcal{K}^{m_i+1}, \quad e_{m_i}^\top x^i = 1,$$
$$\Re_+^{m_i} \ni \lambda^i \perp x^i \in \Re_+^{m_i}, \quad \mathcal{K}^{m_j+1} \ni \begin{bmatrix} z_j \\ x^j \end{bmatrix} \perp \begin{bmatrix} y_j \\ u^j \end{bmatrix} \in \mathcal{K}^{m_j+1} \ (j \in \mathcal{I}_{-i}). \tag{5.10}$$

So, we can reformulate the robust Nash equilibrium problem as SOCCP (5.1).

5.2.2 Existence and uniqueness of robust Nash equilibrium

Next, we study existence and uniqueness of the robust Nash equilibrium under Assumption 4. Unlike the analyses in Subsection 5.1.2, Assumption 4 cannot imply Assumption 1(d), 2(b) or 2(c). So, we do not use the results from Theorems 3.2 and 4.3. Instead of them, we exploit the concrete structure (5.7) of the worst cost function $\tilde{f}_i$. For the proof of the following theorem, refer to [14].

Theorem 5.3. *Suppose that the cost functions and the strategy sets are given by* (5.2) *and* (5.3), *respectively. Suppose further that Assumption* 4 *holds. Then, there exists at least one robust Nash equilibrium.*

We next give sufficient conditions for the uniqueness of a robust Nash equilibrium. To simplify the notations, we define the following vector and matrices:

$$A := (A_{ij})_{i\in\mathcal{I},\, j\in\mathcal{I}}, \quad P := (\rho_{ij})_{i\in\mathcal{I},\, j\in\mathcal{I}}$$
$$Q(x) := \mathrm{diag}\left[\left(\frac{1}{\|x^i\|}\sum_{j=1}^{N} \rho_{ij}\|x^j\|\right)\left(I - v^i (v^i)^\top\right)\right],$$
$$V(x) := \mathrm{diag}\left(v^1, \dots, v^N\right), \quad \text{where } v^i := x^i/\|x^i\|.$$

Then, we have the following lemma. For the proof of the lemma, refer to [14].

Lemma 5.4. *For each* $i \in \mathcal{I}$, *let* $\tilde{f}_i : \Re^{m_i} \to \Re$ *and* $S_i \subset \Re^m$ *be given by* (5.7) *and* (5.3), *respectively. Then, for any* $x \in S$, *the set-valued mapping* $\tilde{\mathcal{F}}$ *given by* (4.3) *satisfies* $\tilde{\mathcal{F}}(x) = \{\tilde{F}(x)\}$ *with* $\tilde{F}(x) := \left(\nabla_i \tilde{f}_i(x^i, x^{-i})\right)_{i\in\mathcal{I}}$. *Moreover, the following statements hold.*

(a) *Function* $\tilde{F}$ *is differentiable at any* $x \in S$ *with the Jacobian* $\nabla \tilde{F}(x)^\top = A + V(x)P\,V(x)^\top + Q(x)$. (b) $Q(x) \succeq 0$ *for any* $x \in S$. (c) *If* $P \succ 0$, *then* $V(x)PV(x)^\top + Q(x) \succ 0$ *for any* $x \in S$.

We now obtain the following theorem. For the proof of the theorem, refer to [14].

Theorem 5.5. *Suppose that the cost functions and the strategy sets are given by* (5.2) *and* (5.3), *respectively. Suppose further that Assumption* 4 *holds. Then, there exists a unique robust Nash equilibrium, if either* (i) $A \succ 0$ *and* $P \succeq 0$ *or* (ii) $A \succeq 0$ *and* $P \succ 0$ *holds.*

6 Numerical experiments

In this section, we solve some robust Nash equilibrium problems with various sizes of uncertainty sets, by using the SOCCP reformulation approaches discussed in the previous section. Then, we change the size of uncertain sets variously, and see the trajectory of the robust Nash equilibria. For solving the reformulated SOCCPs, we apply the Newton-type method combined with a smoothing regularization technique [12]. All programs are coded in MATLAB 7 and run on a computer with 3.06GHz CPU and 1GB memories.

We consider another game where the cost functions are defined by (5.2) with cost matrices and vectors:

$$A_{11} = \begin{bmatrix} 12.486 & 1.249 & 5.650 \\ 1.249 & 2.516 & 4.361 \\ 5.650 & 4.361 & 13.980 \end{bmatrix}, A_{12} = \begin{bmatrix} -5.095 & -7.403 & -4.152 \\ -1.459 & -8.215 & -2.511 \\ -6.228 & -3.783 & -5.306 \end{bmatrix}, A_{13} = \begin{bmatrix} -8.250 & -8.514 & -7.015 \\ -8.178 & -2.222 & -1.091 \\ -2.004 & -5.367 & -4.486 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -7.236 & -2.175 & -5.223 \\ -1.980 & -7.579 & -3.141 \\ -3.180 & -4.678 & -1.155 \end{bmatrix}, A_{22} = \begin{bmatrix} 2.064 & 3.041 & 3.228 \\ 3.041 & 6.563 & 2.341 \\ 3.228 & 2.341 & 14.720 \end{bmatrix}, A_{23} = \begin{bmatrix} -5.420 & -1.153 & -1.514 \\ -4.874 & -6.610 & -3.609 \\ -7.741 & -7.763 & -5.577 \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} -2.338 & -2.981 & -6.197 \\ -7.629 & -4.076 & -4.096 \\ -5.475 & -6.967 & -6.298 \end{bmatrix}, A_{32} = \begin{bmatrix} -3.912 & -3.988 & -1.043 \\ -4.867 & -1.407 & -1.981 \\ -4.844 & -7.212 & -3.992 \end{bmatrix}, A_{33} = \begin{bmatrix} 34.478 & -13.084 & -1.478 \\ -13.084 & 17.336 & -1.243 \\ -1.478 & -1.243 & 20.047 \end{bmatrix}$$

$$c^1 = c^2 = c^3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top .$$

This game has the following three Nash equilibria*1:

$$1: (\bar{x}^1, \bar{x}^2, \bar{x}^3) = ((0.490, 0.510, 0.000),\ (0.000, 0.688, 0.312),\ (0.195, 0.360, 0.443)).$$
$$2: (\bar{x}^1, \bar{x}^2, \bar{x}^3) = ((0.715, 0.011, 0.274),\ (1.000, 0.000, 0.000),\ (0.234, 0.501, 0.266)),$$
$$3: (\bar{x}^1, \bar{x}^2, \bar{x}^3) = ((0.671, 0.304, 0.025),\ (0.596, 0.208, 0.196),\ (0.208, 0.456, 0.335)),$$

Moreover, we consider the robust Nash equilibrium problems under Assumption 4 with parameters

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{21} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{bmatrix} = \begin{bmatrix} 0.01+k & 0.01 & 0.01 \\ 0.01 & 0.01+k & 0.01 \\ 0.01 & 0.01 & 0.01+k \end{bmatrix}, \quad \gamma_1 = \gamma_2 = \gamma_3 = 0,$$

where k is chosen as $k = 0.1,\ 0.5,\ 1.0,\ 1.1485,\ 1.5$. In order to obtain as many equilibria as possible, we solve the equivalent SOCCP with randomly generated 100 starting points*2. Table 1 shows the concrete values of obtained robust Nash equilibria. For $k = 0.1, 0.5, 1.0, 1.1485$, we obtain three robust Nash equilibria. However, for $k = 1.5$, we obtain only one robust Nash equilibrium. Figure 1 shows the trajectory of player 1's strategies at the robust Nash equilibria for each k*3, in which the vertical and horizontal axes denote the first and second components of the robust Nash equilibria, respectively. Figure 1 indicates that two of the three equilibria are getting closer to each other as k increases, and they almost coincide at $k = 1.1485$. Furthermore, at $k = 1.5$, the two equilibria disappear and only one equilibrium is obtained.

*1 We can find all Nash equilibria by using a branch and bound based approach.

*2 Since we employ an iterative method, we can choose an arbitrary starting point. Indeed, it is expected that a different starting point can lead to a different solution when the SOCCP has multiple solutions.

*3 We omit the other players' trajectories since they are similar to player 1's.

Table 1 Sizes of uncertainty sets and obtained robust Nash equilibria

k		robust Nash equilibria
0.1	1:	((0.490, 0.510, 0.000), (0.000, 0.685, 0.315), (0.198, 0.360, 0.442))
	2:	((0.708, 0.020, 0.272), (1.000, 0.000, 0.000), (0.234, 0.499, 0.267))
	3:	((0.667, 0.294, 0.039), (0.608, 0.200, 0.193), (0.210, 0.457, 0.333))
0.5	1:	((0.492, 0.508, 0.000), (0.000, 0.676, 0.324), (0.199, 0.363, 0.439))
	2:	((0.684, 0.057, 0.259), (0.949, 0.000, 0.051), (0.232, 0.491, 0.277))
	3:	((0.657, 0.252, 0.091), (0.660, 0.161, 0.179), (0.216, 0.460, 0.325))
1.0	1:	((0.493, 0.507, 0.000), (0.000, 0.666, 0.334), (0.201, 0.363, 0.436))
	2:	((0.658, 0.094, 0.249), (0.895, 0.000, 0.105), (0.231, 0.483, 0.286))
	3:	((0.650, 0.155, 0.195), (0.800, 0.059, 0.141), (0.226, 0.473, 0.301))
1.1485	1:	((0.494, 0.506, 0.000), (0.000, 0.664, 0.336), (0.202, 0.364, 0.435))
	2:	((0.6507, 0.1026, 0.2467), (0.8810, 0.0000, 0.1190), (0.2312, 0.4807, 0.2881))
	3:	((0.6507, 0.1027, 0.2466), (0.8809, 0.0001, 0.1190), (0.2312, 0.4807, 0.2881))
1.5	1:	((0.507, 0.493, 0.000), (0.052, 0.619, 0.329), (0.204, 0.372, 0.425))

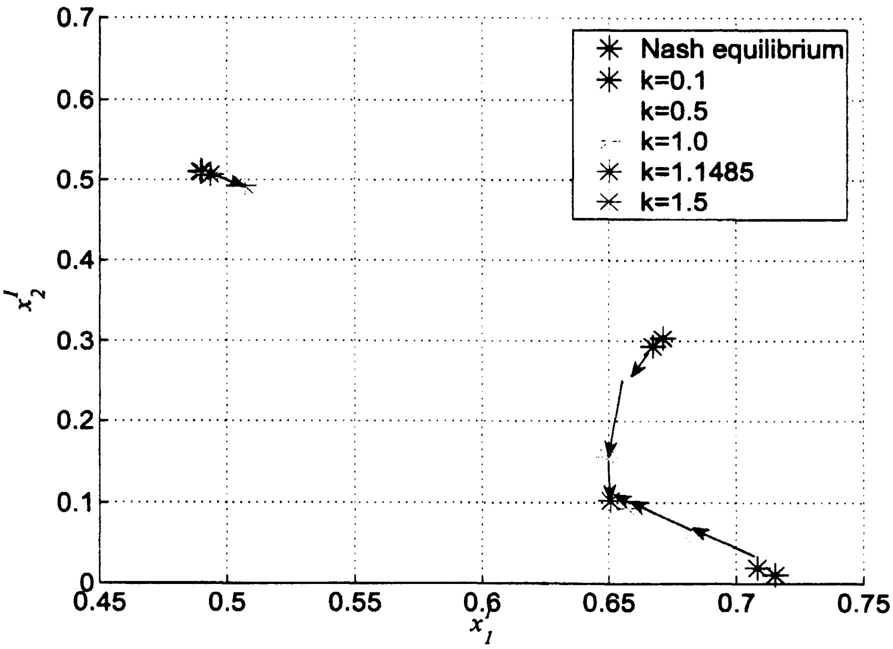

Figure 1 Trajectory of player 1's strategies at the robust Nash equilibria

7 Concluding remarks

In this paper, we have extended the concept of robust Nash equilibrium to N-person non-cooperative games with nonlinear cost functions, and derived sufficient conditions for existence and uniqueness of the robust Nash equilibria by means of the GVIP or VIP reformulation techniques. In addition, we have shown that the robust Nash equilibrium problems with quadratic cost functions and uncertainty sets can be reformulated as SOCCPs. We also solved some examples of the robust Nash equilibrium problem, and observed some numerical properties.

We still have some future issues to be addressed. One important issue is to weaken the sufficient conditions for uniqueness of the robust Nash equilibrium. In fact, the uniqueness conditions shown in the paper are rather restrictive, and there seems to remain much room for the improvement. Another issue is to consider the SOCCP reformulation for the robust Nash equilibrium problem in which both the cost function parameters and the opponents' strategies are uncertain. In this paper, we have only considered the case where either of them is uncertain. However, in the real situation, it would be natural to assume that both of them involve uncertainties.

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