数理解析研究所講究録
第 1629 巻 2009 年 164-173

Metric-Preserving Reduction of Earth Mover's Distance

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Abstract

We prove that the earth mover's distance problem reduces to a problem with half the number of constraints regardless of the ground distance, and propose a further reduced formulation when the ground distance comes from a graph with a homogeneous neighborhood structure.

1 Introduction

Earth mover's distance (EMD in short) proposed by Rubner et al. [4] is a mathematical measure of the dissimilarity between two distributions. In a recent issue Ling and Okada [3] proposed a new formulation EMD-L_1 to compute EMD when the L_1 ground distance is used. It significantly simplifies the original formulation of EMD. Motivated by their work, we propose in this paper, a reduced EMD formulation and prove its equivalence to the original EMD problem via the flow decomposition theorem regardless of the ground distance employed. We also show that the number of variables of the reduced EMD formulation is reduced from $O(m^2)$ to $O(m)$ for a histogram with m locations when the ground distance is derived from a graph with a homogeneous neighborhood structure.

2 Earth Mover's Distance

Let us consider two histograms $\{\, p_{(i,j)} \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2 \,\}$ and $\{\, q_{(i,j)} \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2 \,\}$ defined on the two-dimensional coordinate system. Histogram is a mapping from a set of grid locations (i, j) to the set of non-negative weights $p_{(i,j)}$ or $q_{(i,j)}$, which can be seen a mass of earth (supply) and a collection of holes (demand), respectively. For example, digital imaging can be seen as a histogram if luminosity of each pixel corresponds to the weights. Then, by measuring the least distance to fill the holes with earth, EMD provides the dissimilarity of the two histograms. With the assumption that the total supply and demand are equal, i.e.,

$$\sum_{(i,j)\in\mathcal{N}} p_{(i,j)} = \sum_{(i,j)\in\mathcal{N}} q_{(i,j)}, \tag{2.1}$$

where $\mathcal{N} := \{\, (i,j) \mid 1 \le i \le m_1, 1 \le j \le m_2 \,\}$, EMD is computed as an optimal value of the following well-known transportation problem of Hitchcock type:

$$
(\text{EMD}) \quad \left|\begin{array}{ll}
\text{minimize} & \displaystyle\sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}} d_{(i,j)(k,l)} f_{(i,j)(k,l)} \\
\text{subject to} & \displaystyle\sum_{(k,l)\in\mathcal{N}} f_{(i,j)(k,l)} = p_{(i,j)} \quad \text{for all } (i,j) \in \mathcal{N} \\
& \displaystyle\sum_{(k,l)\in\mathcal{N}} f_{(k,l)(i,j)} = q_{(i,j)} \quad \text{for all } (i,j) \in \mathcal{N} \\
& f_{(i,j)(k,l)} \ge 0 \quad \text{for all } (i,j), (k,l) \in \mathcal{N},
\end{array}\right.
$$

where $f_{(i,j)(k,l)}$ is a flow from location (i,j) to location (k,l). The objective function coefficient $d_{(i,j)(k,l)}$ is a distance between location (i,j) and location (k,l), and referred to as the *ground distance*. Let $m = m_1 \times m_2$. For $k = 1, 2, \ldots, m$ let E_k be the $m \times m$ zero matrix with its kth row replaced by the m-dimensional row vector $\boldsymbol{e} := (1, 1, \ldots, 1)$. Let A denote the $m \times m^2$ matrix $[\,E_1 \,|\, E_2 \,|\, \ldots \,|\, E_m\,]$ and B denote the matrix $[\,I \,|\, I \,|\, \ldots \,|\, I\,]$ of the same size, where I is the $m \times m$ identity matrix. By an appropriate definition of row vector $\boldsymbol{d}$, column vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, and variable column vector $\boldsymbol{f}$, problem (EMD) is rewritten as follows:

$$
(\text{EMD}) \quad \left|\begin{array}{ll}
\text{minimize} & \boldsymbol{df} \\
\text{subject to} & A\boldsymbol{f} = \boldsymbol{p} \\
& B\boldsymbol{f} = \boldsymbol{q} \\
& \boldsymbol{f} \ge \boldsymbol{0}.
\end{array}\right.
$$

In the sequel we consider

$$
(\text{R}) \quad \left|\begin{array}{ll}
\text{minimize} & \boldsymbol{dg} \\
\text{subject to} & (A - B)\boldsymbol{g} = \boldsymbol{p} - \boldsymbol{q} \\
& \boldsymbol{g} \ge \boldsymbol{0},
\end{array}\right.
$$

which we call problem (R), standing for the reduced (EMD), and we denote the optimal value of a problem by $v(\,\cdot\,)$.

Lemma 2.1.

$$v(EMD) \ge v(R).$$

Proof. Straightforward from the fact that a feasible solution of (EMD) is a feasible solution of (R). □

3 Equivalence of the Two Problems

First note that the matrix $A - B$ is of the form

$$[\,E_1 - I \,|\, E_2 - I \,|\, \ldots \,|\, E_m - I\,],$$

and that this is the incidence matrix of a complete directed graph without a self loop on node set $\mathcal{N}$. We denote its arc set by $\mathcal{D}$. We classify the nodes according to the sign of $p_{(i,j)} - q_{(i,j)}$, namely

$$\begin{aligned}
\mathcal{N}_+ &:= \{ (i,j) \in \mathcal{N} \mid p_{(i,j)} - q_{(i,j)} > 0 \} \\
\mathcal{N}_0 &:= \{ (i,j) \in \mathcal{N} \mid p_{(i,j)} - q_{(i,j)} = 0 \} \\
\mathcal{N}_- &:= \{ (i,j) \in \mathcal{N} \mid p_{(i,j)} - q_{(i,j)} < 0 \}.
\end{aligned}$$

Following the convention of network flow theory (see for example [1]), we refer to a node in each set as *deficit node*, *balanced node* and *excess node*, respectively. Problem (R) is known as an arc flow formulation of network flow problem and a feasible solution $\boldsymbol{g}$ of (R) is called an *arc flow*. Another formulation, a path-and-cycle flow formulation, of the network flow problem starts with enumerating all directed paths between any pair of nodes and all directed cycles. The decision variables are the flow value on each path and cycle.

Theorem 3.1 (Theorem 3.5 (Flow Decomposition Theorem), [1]). *Every arc flow can be represented as a path-and-cycle flow (though not necessarily uniquely) such that every directed path with positive flow connects a deficit node to an excess node.*

Let Π and Γ be the set of all directed paths and the set of all directed cycles of the network $(\mathcal{N}, \mathcal{D})$, respectively. Applying the above theorem to problem (R), we obtain the following corollary.

Corollary 3.2. *Let $\boldsymbol{g}$ be a feasible solution of (R). Then for each directed path $\pi \in \Pi$ there is a non-negative path flow value $f(\pi)$, and for each directed cycle $\gamma \in \Gamma$ there is a non-negative cycle flow value $f(\gamma)$ with the following two properties:*

1. *For every arc $((i,j)(k,l)) \in \mathcal{D}$ it holds that*

$$g_{(i,j)(k,l)} = \sum_{\pi:((i,j)(k,l))\in\pi\in\Pi} f(\pi) + \sum_{\gamma:((i,j)(k,l))\in\gamma\in\Gamma} f(\gamma). \tag{3.1}$$

2. *$f(\pi)$ is positive only when path π connects a node in $\mathcal{N}_+$ to a node in $\mathcal{N}_-$.*

The *arc-path incidence vector* of a directed path π is the vector $\boldsymbol{\delta}(\pi)$ of components

$$\delta_{(i,j)(k,l)}(\pi) := \begin{cases} 1 & \text{when } ((i,j)(k,l)) \in \pi \\ 0 & \text{otherwise.} \end{cases}$$

The *arc-cycle incidence vector* of a directed cycle γ, denoted by $\boldsymbol{\delta}(\gamma)$, is defined in the same way. Then (4.4) is rewritten as

$$\boldsymbol{g} = \sum_{\pi\in\Pi} f(\pi)\boldsymbol{\delta}(\pi) + \sum_{\gamma\in\Gamma} f(\gamma)\boldsymbol{\delta}(\gamma).$$

Let

$$\boldsymbol{g}' = \sum_{\pi\in\Pi} f(\pi)\boldsymbol{\delta}(\pi). \tag{3.2}$$

Lemma 3.3. *If $\boldsymbol{g}$ is a feasible solution of (R), the following statements hold.*

1. *$\boldsymbol{g}'$ is a feasible solution of (R),*
2. *$\boldsymbol{dg}' \leq \boldsymbol{dg}$.*

Proof. Straightforward from the fact that $(A - B)\boldsymbol{\delta}(\gamma) = \mathbf{0}$ for every $\gamma \in \Gamma$, $\boldsymbol{d} \geq \mathbf{0}$ and the construction (3.2) of $\boldsymbol{g}'$. □

Take a pair of nodes $(i,j) \in \mathcal{N}_+$ and $(k,l) \in \mathcal{N}_-$ and let $\Pi((i,j)(k,l))$ be the set of all directed paths connecting (i,j) to (k,l), i.e., starting at (i,j) and ending at (k,l). Let $\boldsymbol{g}''$ be the vector of components

$$g''_{(i,j)(k,l)} := \begin{cases} \displaystyle\sum_{\pi \in \Pi((i,j)(k,l))} f(\pi) & \text{when } (i,j) \in \mathcal{N}_+ \text{ and } (k,l) \in \mathcal{N}_- \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Figure 1 shows the node set $\mathcal{N}$ and some path-flows and a cycle-flow. The broad arrow from (i,j) to (k,l) shows $g''_{(i,j)(k,l)}$.

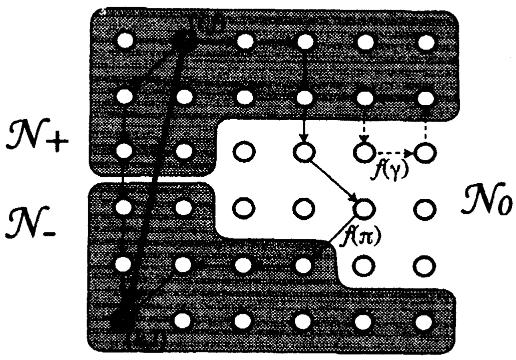

Figure 1: Reduction procedure

Lemma 3.4. *If $\boldsymbol{g}$ is a feasible solution of (R), the following statements hold.*

1. *$\boldsymbol{g}''$ is a feasible solution of (R),*
2. *$g''_{(k,l)(i,j)} = 0$ for all $(i,j) \in \mathcal{N}_+$ and $(k,l) \in \mathcal{N}$,*
3. *$g''_{(i,j)(k,l)} = g''_{(k,l)(i,j)} = 0$ for all $(i,j) \in \mathcal{N}_0$ and $(k,l) \in \mathcal{N}$,*
4. *$g''_{(i,j)(k,l)} = 0$ for all $(i,j) \in \mathcal{N}_-$ and $(k,l) \in \mathcal{N}$, and*
5. *$\boldsymbol{dg}'' \leq \boldsymbol{dg}'$.*

Proof. The first four claims are readily seen by Corollary 3.2 (2) and the construction (3.3) of g''. Let $s(\pi)$ and $t(\pi)$ denote the starting node and the terminal node of path π, respectively. The last claim is seen as follows.

$$\begin{aligned}
dg' &= \sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}} d_{(i,j)(k,l)} g'_{(i,j)(k,l)} \\
&= \sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}} d_{(i,j)(k,l)} \sum_{\pi:((i,j)(k,l))\in\pi\in\Pi} f(\pi) \\
&= \sum_{\pi\in\Pi} f(\pi) \sum_{((i,j)(k,l))\in\pi} d_{(i,j)(k,l)} \\
&\geq \sum_{\pi\in\Pi} f(\pi) d_{s(\pi)\,t(\pi)} \\
&= \sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}} d_{(i,j)(k,l)} \sum_{\pi\in\Pi((i,j)(k,l))} f(\pi) \\
&= \sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}} d_{(i,j)(k,l)} g''_{(i,j)(k,l)} \\
&= dg'',
\end{aligned}$$

where the inequality is due to the triangle inequality of distance $d_{(i,j)(k,l)}$. □

By the above lemma and the equality constraint of (R)

$$\sum_{(k,l)\in\mathcal{N}} g_{(i,j)(k,l)} - \sum_{(k,l)\in\mathcal{N}} g_{(k,l)(i,j)} = p_{(i,j)} - q_{(i,j)}$$

we see

$$\sum_{(k,l)\in\mathcal{N}} g''_{(i,j)(k,l)} = p_{(i,j)} - q_{(i,j)} \quad \text{for } (i,j) \in \mathcal{N}_+ \tag{3.4}$$

$$\sum_{(k,l)\in\mathcal{N}} g''_{(i,j)(k,l)} = \sum_{(k,l)\in\mathcal{N}} g''_{(k,l)(i,j)} = 0 \quad \text{for } (i,j) \in \mathcal{N}_0 \tag{3.5}$$

$$\sum_{(k,l)\in\mathcal{N}} g''_{(k,l)(i,j)} = -p_{(i,j)} + q_{(i,j)} \quad \text{for } (i,j) \in \mathcal{N}_-. \tag{3.6}$$

Finally add $q_{(i,j)}$ flow to $g''_{(i,j)(i,j)}$ for $(i,j) \in \mathcal{N}_+$, $p_{(i,j)}$ flow to $g''_{(i,j)(i,j)}$ for $(i,j) \in \mathcal{N}_-$, and $p_{(i,j)} = q_{(i,j)}$ flow to $g''_{(i,j)(i,j)}$ for $(i,j) \in \mathcal{N}_0$ to make g'''. Since $d_{(i,j)(i,j)} = 0$, we obtain the following lemma.

Lemma 3.5. *If g is a feasible solution of (R), the following statements hold.*

1. *g''' is a feasible solution of (EMD),*
2. *$dg''' = dg''$.*

Combining the above lemmas, we have the following inequality.

Lemma 3.6.

$$v(EMD) \leq v(R).$$

By Lemma 2.1 and 3.6 we see that problem (R) yields the same optimal objective function value as problem (EMD) does.

Theorem 3.7.

$$v(EMD) = v(R).$$

Note that this equality holds no matter what distance $d_{(i,j)(k,l)}$ is postulated on $\mathcal{N}$.

4 Problem Reduction Based on Homogeneous Neighborhood Structure

Suppose we are given a connected undirected graph, denoted by $\mathcal{G}$, with node set $\mathcal{N}$ and edge set $\mathcal{E}$ without a self-loop. The edge connecting nodes (i,j) and (k,l) is denoted by $[(i,j)(k,l)]$ and is assigned a positive value $\ell_{[(i,j)(k,l)]}$ called *length.*

For each pair of nodes (i,j) and (k,l) let $d^{\ell}_{(i,j)(k,l)}$ be the shortest length of paths between the pair. It is known and easily seen that $d^{\ell}_{(i,j)(k,l)}$ provides a distance defined on $\mathcal{N}$.

For each node $(i,j) \in \mathcal{N}$ we define

$$\mathcal{N}_{\mathcal{G}}(i,j) := \{\, (k,l) \in \mathcal{N} \mid [(i,j)(k,l)] \in \mathcal{E} \,\}, \tag{4.1}$$

and refer to $\mathcal{N}_{\mathcal{G}}(i,j)$ as node (i,j)'s *neighborhood* on $\mathcal{G}$.

Definition 4.1. Let $\mathcal{H}$ be a finite subset of integer grid points of $\mathbb{R}^2$ without $(0,0)$ and $\ell^{\mathcal{H}}_{(i',j')}$ be a positive number for $(i',j') \in \mathcal{H}$. Graph $\mathcal{G} = (\mathcal{N},\mathcal{E},\ell)$ is said to have the *homogeneous neighborhood structure* of $(\mathcal{H},\ell^{\mathcal{H}})$ when

1. $\mathcal{N}_{\mathcal{G}}(i,j) = \mathcal{N} \cap \{\, (i+i',j+j') \mid (i',j') \in \mathcal{H} \,\}$ for all $(i,j) \in \mathcal{N}$, and
2. $\ell_{[(i,j)(k,l)]} = \ell^{\mathcal{H}}_{(k-i,l-j)}$ for all $(k,l) \in \mathcal{N}_{\mathcal{G}}(i,j)$ and $(i,j) \in \mathcal{N}$.

Two graphs together with corresponding homogeneous neighborhood structures are shown in Figure 2.

The distance d^{ℓ} defined by the upper graph $\mathcal{G}$, Manhattan graph, with the neighborhood structure $\mathcal{H} = \{(-1,0),(0,-1),(0,1),(1,0)\}$ and

$$\ell^{\mathcal{H}}_{(i',j')} = 1 \text{ for all } (i',j') \in \mathcal{H} \tag{4.2}$$

is the L_1 distance on $\mathcal{N}$, while the other graph, Union Jack graph, with the neighborhood structure $\mathcal{H} = \{(-1,0),(-1,-1),(0,-1),(1,-1),(1,0),(1,1),(0,1),(-1,1)\}$ and $\ell^{\mathcal{H}}_{(i',j')} = 1$ for all $(i',j') \in \mathcal{H}$ defines the L_∞ distance. Bertsimas et al. [2] proposed the D-norm for $\boldsymbol{y} \in \mathbb{R}^n$ and $\rho \in [1,n]$ as the optimal value of the linear program

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{n} u_j |y_j| \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} u_j \le \rho; \quad 0 \le u_j \le 1 \quad \text{for } j = 1,\dots,n. \end{array}\right.$$

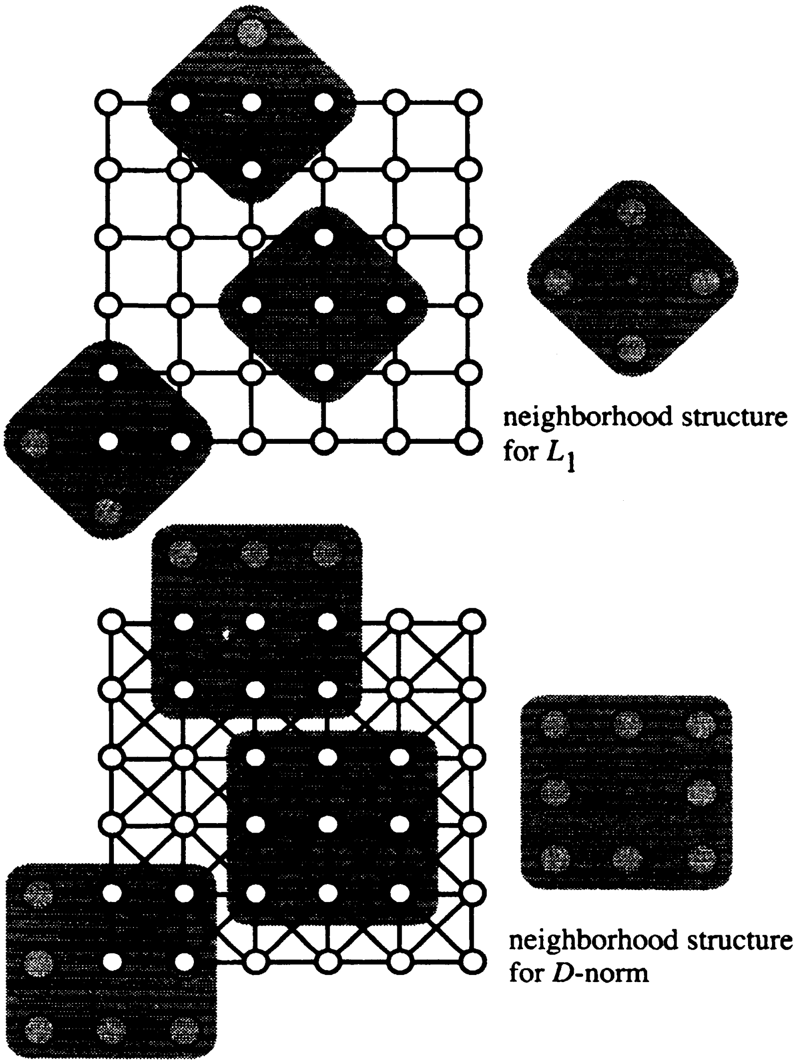

Figure 2: Graph and neighborhood structure defining a distance on $\mathcal{N}$

The Union Jack graph with

$$\ell^{\mathcal{H}}_{(i',j')} = \begin{cases} 1 & \text{for } (i',j') \in \{(-1,0),(0,-1),(1,0),(0,1)\} \\ \rho & \text{for } (i',j') \in \{(-1,-1),(1,-1),(1,1),(-1,1)\} \end{cases} \tag{4.3}$$

defines the D-norm, which gives, by setting the parameter ρ appropriately (e.g. $\rho = \sqrt{2}$), an in-between of L_1 and L_2.

Suppose the ground distance $d_{(i,j)(k,l)}$ among locations of $\mathcal{N}$ is given as the distance $d^{\ell}_{(i,j)(k,l)}$ for a graph $\mathcal{G}$ with a homogeneous neighborhood structure. Then for two distinct locations (i,j) and (k,l) there is an undirected path of edges $[(i_0,j_0)(i_1,j_1)], [(i_1,j_1)(i_2,j_2)], \ldots, [(i_{n-1},j_{n-1})(i_n,j_n)]$ such that $(i_0,j_0) = (i,j)$, $(i_n,j_n) = (k,l)$,

$$(i_{r+1},j_{r+1}) \in \mathcal{N}_{\mathcal{G}}(i_r,j_r) \quad \text{for } r = 0,\ldots,n-1$$

and

$$d_{(i,j)(k,l)} = \sum_{r=0}^{n-1} d_{(i_r,j_r)(i_{r+1},j_{r+1})} = \sum_{r=0}^{n-1} \ell^{\mathcal{H}}_{(i_{r+1}-i_r,j_{r+1}-j_r)}. \tag{4.4}$$

Add the constraints

$$g_{(i,j)(k,l)} = 0 \quad \text{for all } (i,j) \in \mathcal{N} \text{ and } (k,l) \notin \mathcal{N}_{\mathcal{G}}(i,j)$$

to problem (R) and denote it by ($\bar{\text{R}}$), i.e.,

$$(\bar{\text{R}}) \quad \left| \begin{array}{ll} \text{minimize} & \boldsymbol{dg} \\ \text{subject to} & (A-B)\boldsymbol{g} = \boldsymbol{p} - \boldsymbol{q} \\ & \boldsymbol{g} \geq \boldsymbol{0} \\ & g_{(i,j)(k,l)} = 0 \quad \text{for all } (i,j) \in \mathcal{N} \text{ and } (k,l) \notin \mathcal{N}_{\mathcal{G}}(i,j), \end{array} \right.$$

or equivalently

$$(\bar{\text{R}}) \quad \left| \begin{array}{ll} \text{minimize} & \displaystyle\sum_{(i,j)\in\mathcal{N}} \sum_{(k,l)\in\mathcal{N}_{\mathcal{G}}(i,j)} \ell^{\mathcal{H}}_{(k-i,l-j)} g_{(i,j)(k,l)} \\ \text{subject to} & \displaystyle\sum_{(k,l)\in\mathcal{N}_{\mathcal{G}}(i,j)} g_{(i,j)(k,l)} - \sum_{(k,l)\in\mathcal{N}_{\mathcal{G}}(i,j)} g_{(k,l)(i,j)} = p_{(i,j)} - q_{(i,j)} \\ & \qquad\qquad\qquad\qquad \text{for all } (i,j) \in \mathcal{N} \\ & g_{(i,j)(k,l)} \geq 0 \qquad \text{for all } (i,j) \in \mathcal{N},\ (k,l) \in \mathcal{N}_{\mathcal{G}}(i,j). \end{array} \right.$$

We see that problem ($\bar{\text{R}}$) is equivalent to problem (R).

Lemma 4.2. *Suppose that the graph $\mathcal{G}$ has the homogeneous neighborhood structure $(\mathcal{H}, \ell^{\mathcal{H}})$ and the ground distance $d_{(i,j)(k,l)}$ is given as the shortest length of paths in $\mathcal{G}$. Then every optimal solution of problem ($\bar{R}$) is an optimal solution of problem (R), and*

$$v(\bar{R}) = v(R).$$

Proof. Let $((i,j)(k,l))$ be an arc of $\mathcal{D}$. Since the ground distance is given as the shortest length of paths in $\mathcal{G}$, there is a series of arcs $((i_0,j_0)(i_1,j_1)), ((i_1,j_1)(i_2,j_2)), \ldots, ((i_{n-1},j_{n-1})(i_n,j_n))$ such that $(i_0,j_0) = (i,j)$, $(i_n,j_n) = (k,l)$, $(i_{r+1},j_{r+1}) \in \mathcal{N}_{\mathcal{G}}(i_r,j_r)$ for $r = 0,1,\ldots,n-1$, and also the equality (4.4) holds.

Now suppose we are given a feasible flow $\boldsymbol{g}$ of problem (R). The above observation implies that replacing the arc flow of $g_{(i,j)(k,l)}$ on arc $((i,j)(k,l))$ by the path-flow along $((i_0,j_0)(i_1,j_1))$, $((i_1,j_1)(i_2,j_2)), \ldots, ((i_{n-1},j_{n-1})(i_n,j_n))$ does not change the objective function value. Repeating this procedure if necessary, we will obtain a feasible flow satisfying the additional equality constraints

$$g_{(i,j)(k,l)} = 0 \text{ for all } (i,j) \in \mathcal{N} \text{ and } (k,l) \notin \mathcal{N}_{\mathcal{G}}(i,j)$$

of ($\bar{\text{R}}$) without changing the objective function value. This completes the proof. □

Theorem 4.3. *When the graph $\mathcal{G}$ has the homogeneous neighborhood structure $(\mathcal{H}, \ell^{\mathcal{H}})$ and problem (EMD) employs the shortest length of paths in $\mathcal{G}$ as the ground distance. Then*

$$v(\bar{R}) = v(EMD).$$

Proof. Straightforward from Theorem 3.7 and Lemma 4.2. □

Let h denote the size of $\mathcal{H}$, which is four for the Manhattan graph and eight for the Union Jack graph. Then comparing ($\bar{\text{R}}$) with (EMD), the number of variables reduces from m^2 to mh. This will greatly lighten the computational burden.

5 Experimental Results

We will report on some experimental results to demonstrate the usefulness of EMD and the computational effectiveness of our proposed formulation. The images we used are three sequential images of a swimmer in Figure 3 each of which consists of 32×32 pixels. By letting weight $p_{(i,j)}$

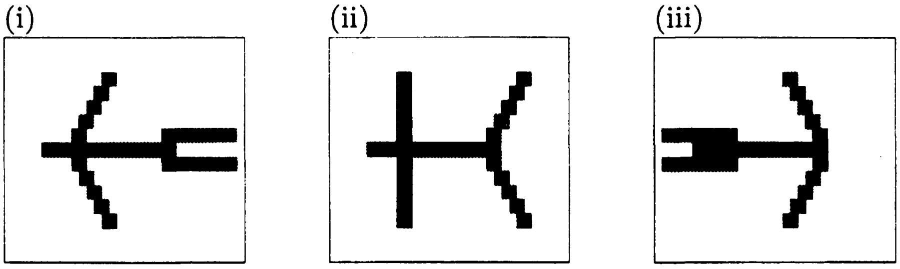

Figure 3: Sequential ((iii) → (ii) → (i)) swimmer images

be 1 when the grid location (i,j) corresponds to a colored pixel and 0 otherwise, we construct the three different 32×32 histograms and compute the dissimilarity among these histograms by

(a) Frobenius norm (i.e., $\sqrt{\sum_{i=1}^{m_1}\sum_{j=1}^{m_2}(p_{(i,j)} - q_{(i,j)})^2}$),

(b) (EMD) with L_2 ground distance,

(c) ($\bar{\text{R}}$) with Manhattan graph and (4.2), and

(d) ($\bar{\text{R}}$) with Union Jack graph and (4.3) with $\rho = 1.3$.

All computations are conducted on a personal computer with Core2 CPU (2.66GHz) and 4GB memory. Problems (b), (c) and (d) are solved by using CPLEX 10.1, OPL Studio 5.1.

The column *Time* of Table 1 shows the average time for computing the three values of dissimilarity, and the columns *#Var* and *#Const* show the number of variables and the number of constraints of each problem, respectively.

Noteworthy points are in order. Firstly, Frobenius norm (a) provides almost the same value of dissimilarity to all pairs of histograms, while (b), (c) and (d) give relatively large value of dissimilarity to the pair (i)↔(iii) and successfully reflect the sequential nature of the images. Secondly, the values of dissimilarity given by (d) are very close to those by (b). This supports that Manhattan graph with $\rho = 1.3$ sufficiently approximates the L_2 ground distance. Thirdly, because of the remarkable reduction of problem size (see the columns #Var and #Const), (c) and (d) reduce the computational time sharply in contrast to (b). This reduction would be especially valuable when applied to image retrieval systems that need to compute dissimilarity of a large number of pairs of images.

Table 1: Dissimilarity and computational time of Frobenius norm, (EMD) and ($\bar{\mathrm{R}}$)

	Dissimilarity			Time	#Var	#Const
	(i)↔(ii)	(ii)↔(iii)	(i)↔(iii)	(sec)		
(a) Frobenius norm	11.0	11.3	11.3	—	—	—
(b) (EMD) with L_2	259.0	386.0	573.0	32.0	1,048,576	2,048
(c) ($\bar{\mathrm{R}}$) with Manhattan	320.0	480.0	716.0	1.0	3,968	1,024
(d) ($\bar{\mathrm{R}}$) with Union Jack	264.0	368.0	576.0	1.0	7,812	1,024

6 Conclusion

We have proved that the earth mover's distance problem reduces to a problem with half the number of constraints regardless of the ground distance. Furthermore, we have proposed a further reduced formulation when the ground distance comes from a graph with a homogeneous neighborhood structure. The preliminary experiment has shown that the reduction helps compute the earth mover's distance efficiently. In this paper we have assumed that the location has two coordinates such as (i, j), however, it can be generalized to a higher dimensional coordinate system with a slight modification.

Acknowledgement

The authors thank Maiko Shigeno, University of Tsukuba for stimulating discussion, and Jun-ya Gotoh, Chuo University, and Akiko Takeda, Keio University for drawing their attention to D-norm. This research is supported in part by the Grant-in-Aid for Scientific Research (B) 18310101 of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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