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Product Possibility Space with Finitely Many Independent Fuzzy Vectors

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Abstract

This paper discusses about product possibility space and independent fuzzy variables.

Keywords: uncertainty theory, possibility measure, product possibility space, fuzzy variable, fuzzy vector, independence

1 Introduction

Fuzzy set was first introduced by Zadeh [24] in 1965. This notion has been very useful in human decision making under uncertainty. We can see lots of papers which use this fuzzy set theory in K.Iwamura and B.Liu [3] [4], B.Liu and K.Iwamura [18] [19] [20], X.Ji and K.Iwamura [9], X.Gao and K.Iwamura [2], G.Wang and K.Iwamura [22], M.Wen and K.Iwamura [23] and others. We also have some books on fuzzy decision making under fuzzy environments such as D.Dubois and H.Prade [1], H-J.Zimmermann [26], M.Sakawa [27], J.Kacprzyk [10], B.Liu and A.O.Esogbue [17].

Recently B.Liu has founded a frequentionist fuzzy theory with huge amount of applications in fuzzy mathematical programming. We see it in books such as B.Liu [12] [16] [13]. In his book [13] published in 2004, B.Liu [13] has succeeded in establishing an axiomatic foundation for uncertainty theory, where they proposed a notion of independent fuzzy variables.

In this paper, we discuss on possibility axioms, construct product possibility space and show that we not only have finitely many independent fuzzy variables[6] but also we have finitely many independent fuzzy vectors in a wide sense.

The rest of the paper is organized as follows. The next section provides a brief review on the results of possibility measure axioms with definitions of fuzzy variables, fuzzy vectors, independence of the two. Section 3 presents how we can get finitely many independent fuzzy vectors under possibility measures.

2 Possibility Measure and Product Possibility Space

We start with the axiomatic definition of possibility measure given by B. Liu [13] in 2004. Let Θ be an arbitrary nonempty set, and let $\mathcal{P}(\Theta)$ be the power set of Θ.
The three axioms for possibility measure are listed as follows:

Axiom 1. $\mathrm{Pos}\{\Theta\} = 1$.

Axiom 2. $\mathrm{Pos}\{\emptyset\} = 0$.

Axiom 3. $\mathrm{Pos}\{\cup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$ *for any collection* $\{A_i\}$ *in* $\mathcal{P}(\Theta)$.

We call Pos a possibility measure over Θ if it satisfies these three axioms. We call the triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ a possibility space.

Theorem 2.1 *(B. Liu, 2004).* Let Θ_i be nonempty sets on which $\mathrm{Pos}_i\{\cdot\}$ satisfy the three axioms, $i = 1, 2, \cdots, n$, respectively, and let $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. Define $\mathrm{Pos}\{\cdot\}$ by

$$\mathrm{Pos}\{A\} = \sup_{(\theta_1,\theta_2,\cdots,\theta_n)\in A} \mathrm{Pos}_1\{\theta_1\} \wedge \mathrm{Pos}_2\{\theta_2\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \tag{1}$$

for each $A \in \mathcal{P}(\Theta)$. Then $\mathrm{Pos}\{\cdot\}$ satisfies the three axioms for possibility measure.

Therefore, a newly defined triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ is a possibility space. $\mathrm{Pos}\{\cdot\}$ is a possibility measure on $\mathcal{P}(\Theta)$. We call it product possibility measure derived from $(\Theta_i, \mathrm{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i)), i = 1, \cdots, n$. We call $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ the product possibility space derived from $(\Theta_i, \mathrm{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i)), i = 1, \cdots, n$.

Lemma 2.1 *Let* $0 \le a_i \le 1$ *and let* $\epsilon > 0$, *for* $1 \le i \le n$. *Then we get*

$$(\epsilon + a_1) \wedge (\epsilon + a_2) \wedge \cdots \wedge (\epsilon + a_n) \le \epsilon + a_1 \wedge a_2 \wedge \cdots \wedge a_n \tag{2}$$

Lemma 2.2 *Let* $A_i \in \mathcal{P}(\Theta_i)$ *for* $1 \le i \le n$. *Then we get* $A_1 \times \cdots \times A_n \in \mathcal{P}(\Theta)$ *and*

$$\mathrm{Pos}\{A_1 \times \cdots \times A_n\} = \mathrm{Pos}_1\{A_1\} \wedge \mathrm{Pos}_2\{A_2\} \wedge \cdots \wedge \mathrm{Pos}_n\{A_n\}. \tag{3}$$

Lemma 2.3 $\mathrm{Pos}\{\}$ *on* $\mathcal{P}(\Theta)$ *at (1) is monotone with respect to set inclusion, i.e., for any sets* $A, B \in \mathcal{P}(\Theta)$ *with* $A \subset B$, *we get*

$$\mathrm{Pos}\{A\} \le \mathrm{Pos}\{B\}. \tag{4}$$

A fuzzy variable is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ to the set of reals $\mathcal{R}$. An $n-$dimensional fuzzy vector is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ to an $n-$dimensional Eucledian space $\mathcal{R}^n$. Let ξ be a fuzzy variable defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$.

Then its membership function is derived through the possibility measure Pos by

$$\mu(x) = \mathrm{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\}, \quad x \in \Re. \tag{5}$$

Theorem 2.2 *(B. Liu, 2004). Let* $\mu : \Re \to [0,1]$ *be a function with* $\sup \mu(x) = 1$. *Then there is a fuzzy variable whose membership function is* μ.

The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if and only if

$$\mathrm{Pos}\{\xi_i \in B_i, i = 1, 2, \cdots, m\} = \min_{1 \le i \le m} \mathrm{Pos}\{\xi_i \in B_i\}$$

for any subsets $B_1, B_2, \cdots, B_m$ of the set of reals $\mathcal{R}$. The fuzzy vectors $\boldsymbol{\xi}_i (1 \le i \le m)$ are said to be independent if and only if

$$\mathrm{Pos}\{\boldsymbol{\xi}_i \in B_i, i = 1, 2, \cdots, m\} = \min_{1 \le i \le m} \mathrm{Pos}\{\boldsymbol{\xi}_i \in B_i\}$$

for any subsets $B_i \in \mathcal{R}^{n_i} (1 \le i \le m)$. Here after we use $a \wedge b$ in place of $\min\{a, b\}$.

Note 2.1 *: We have fuzzy variables* ξ_1, ξ_2 *which are not independent(Liu[13]). Let* $\Theta = \{\theta_1, \theta_2\}, \mathrm{Pos}\{\theta_1\} = 1, \mathrm{Pos}\{\theta_2\} = 0.8$ *and define* ξ_1, ξ_2 *by*

$$\xi_1(\theta) = \begin{cases} 0, & \text{if } \theta = \theta_1 \\ 1, & \text{if } \theta = \theta_2, \end{cases} \qquad \xi_2(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_1 \\ 0, & \text{if } \theta = \theta_2 \end{cases}$$

Then we have $\mathrm{Pos}\{\xi_1 = 1, \xi_2 = 1\} = \mathrm{Pos}\{\emptyset\} = 0 \neq 0.8 \wedge 1 = \mathrm{Pos}\{\xi_1 = 1\} \wedge \mathrm{Pos}\{\xi_2 = 1\}$.

3 Finitely Many Independent Fuzzy Vectors

Let $\boldsymbol{\xi}_i$ be a fuzzy vector from a possibility space $(\Theta_i, \mathcal{P}(\Theta_i), \mathrm{Pos}_i)$ to the l_i-th dimensional Eucledian space $\mathcal{R}^{l_i}$, for $i = 1, 2, \cdots, n$. Define Θ by

$$\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n \tag{6}$$

and $\widetilde{\boldsymbol{\xi}}_i$ on Θ by

$$\widetilde{\boldsymbol{\xi}}_i(\theta) = \xi_i(\theta_i) \quad \text{for any} \quad \theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \Theta, 1 \le i \le n. \tag{7}$$

For any subset B_i of $\mathcal{R}^{l_i}$, we get([8])

Theorem 3.1 *The fuzzy vectors $\widetilde{\boldsymbol{\xi}}_i(1 \le i \le n)$ given above are independent fuzzy vectors, i.e.,*

$$\mathrm{Pos}\{\widetilde{\boldsymbol{\xi}}_i(\theta) \in B_i(1 \le i \le n)\} = \mathrm{Pos}\{\widetilde{\boldsymbol{\xi}}_1 \in B_1\} \wedge \mathrm{Pos}\{\widetilde{\boldsymbol{\xi}}_2 \in B_2\} \wedge \cdots \wedge \mathrm{Pos}\{\widetilde{\boldsymbol{\xi}}_n \in B_n\} \quad (8)$$

4 Conclusion

We have shown that Axiom. 4 in B.Liu([13]) can be proved through Axiom. 1 ,2 and 3. We have proved the fact that there exists finitely many independent fuzzy vectors. Through these proofs we have shown that for possibility measures existence of finitely many independent fuzzy vectors depends on product possibility space. Although the notion of independence was discovered by L.A.Zadeh and others([25]) under the term of "noninteractiveness" or "unrelatedness", our notion of independence through product possibility space have brought about a grand world of fuzzy process, hybrid process and uncertain process of B.Liu([15]).

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