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Kyoto University
Viscous Shock Profiles for 2 × 2 Systems of Hyperbolic Conservation Laws with Quadratic Flux Functions

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1. INTRODUCTION

The purpose of this paper is to understand the shock wave structure of conservation laws that come from the extraction of petroleum.

An oil reservoir is a subsurface pool of hydrocarbons contained in porous rock formations. If the underground pressure of in the reservoir is sufficient, then the oil is naturally forced to the surface and extracted by valves on the well. This is called the primary recovery and usually about 20% of the oil in a oil reservoir can be extracted.

Over the lifetime of the well, the underground pressure will be insufficient to force the oil to the surface. Secondary recovery techniques increase the reservoir pressure by injecting water and gas (air or CO2). Generally 25% to 35% of the oil in a oil reservoir can be extracted by primary and secondary recovery together.

Figure 1: WAG Enhanced Oil Recovery (schematic picture)
Water-Alternating-Gas (WAG) Enhanced Oil Recovery: Although the water injection has good sweep efficiency, 40 to 60% of the original oil on-site is left behind at the end of the injection. The gas injection has good displacement efficiency but is an expensive operation. Hence the injection of gas after water followed by water and gas injection causes significant redistribution of fluids in the reservoir and will be more efficient than injection of water or gas alone.

Because of the gravity, three phases: oil, gas and water are separated from one another away from the WAG injector and it is only near the injector where three phase flow actually occurs. Mathematical structure of the three phase flows has been investigated by many authors (for example, Marchesin-Plohr [7], Medeiros [8], Schaeffer-Shearer [10]) and, in this paper, we shall confine ourselves particularly to their shock wave structure.

Stone’s Model: In order to simplify the three-phase flow in a porous medium, we neglect the gravity and assume that the medium is homogeneous and the flow is incompressible and immiscible. Let us denote:

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<th>water</th>
<th>gas</th>
<th>oil</th>
</tr>
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<tr>
<td>Volume Fractions:</td>
<td>$s_W = u$</td>
<td>$s_G = v$</td>
<td>$s_O = 1 - u - v$</td>
</tr>
<tr>
<td>Permeability Functions:</td>
<td>$k_W$</td>
<td>$k_G$</td>
<td>$k_O$</td>
</tr>
<tr>
<td>Fluid Viscosity:</td>
<td>$\mu_W$</td>
<td>$\mu_G$</td>
<td>$\mu_O$</td>
</tr>
<tr>
<td>Fluid Velocity:</td>
<td>$v_W$</td>
<td>$v_G$</td>
<td>$v_O$</td>
</tr>
<tr>
<td>Pressure:</td>
<td>$p_W$</td>
<td>$p_G$</td>
<td>$p_O$</td>
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The relationship between the flow rate and the pressure gradient is expressed by Darcy’s Law

$$v_i = -\frac{k_i}{\mu_i} \nabla p_i, \quad i = W, G, O.$$  

It is usually assumed that the water and gas permeability functions depend only on the water and gas volume fraction

$$k_W = k_W(u), \quad k_G = k_G(v)$$

which is called Stone’s assumption. We finally assume that the flow is one dimensional and the capillary pressure is negligible.

By using relative permeability functions

$$f(u) = \frac{k_W(u)}{\mu_W}, \quad g(v) = \frac{k_G(v)}{\mu_G}, \quad h(u, v) = \frac{k_O(u, v)}{\mu_W}$$

the mass conservation laws are expressed in the form

Water: \[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{f(u)}{f(u) + g(v) + h(u, v)} \right) = 0, \tag{1}
\]

Gas: \[
\frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \left( \frac{g(v)}{f(u) + g(v) + h(u, v)} \right) = 0 \tag{2}
\]
in \( \Omega : 0 < u + v < 1, \ u, \ v > 0 \) ([7],[8],[10]). These equations constitute a system of conservation laws that is discussed in this paper.

**Hyperbolicity:** We say that the system of equations (1) and (2) is hyperbolic, when the Jacobian matrix of the flux function has real eigenvalues \( \lambda_1(U), \lambda_2(U) \) for any \( U \in \Omega \). If, in particular, these eigenvalues are distinct: \( \lambda_1(U) < \lambda_2(U) \), the system is called strictly hyperbolic at \( U \). Corresponding right eigenvectors are denoted by \( R_1(U), R_2(U) \) respectively. A state \( U^* \in \Omega \) is called an umbilic point, if \( \lambda_1(U^*) = \lambda_2(U^*) \) and the Jacobian matrix is diagonalizable, hence a scalar matrix. Marchesin, Paes-Leme, Schaeffer and Shearer have shown in [10].

**Theorem 1 (Existence of Umbilic Point)** Assume that \( h(u, v) = h(1-u-v) \) and \( f(0) = g(0) = h(0) = 0, f''(u), g''(v), h''(w) > 0 \). Then the system of equations (1), (2) is hyperbolic and has a unique umbilic point in \( \Omega \).

After the change of unknown functions, we may assume that \( U^* = O \) and \( F(O) = O \). Thus we have the Taylor expansion of the flux function \( F(U) \) near \( U = O \):

\[
F(U) = \lambda^* U + Q(U) + O(1)|U|^3
\]

where \( \lambda^* = \lambda_1(U^*) = \lambda_2(U^*) \) and \( Q : \mathbb{R}^2 \to \mathbb{R}^2 \) is a homogeneous quadratic mapping. After the Galilean change of variables: \( x \to x - \lambda^*t \), we observe that the system of equations (1) and (2) is reduced to

\[
U_t + Q(U)_x = O, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+,
\]

modulo higher order terms. By a change of unknown functions \( V = S^{-1}U \) with a regular constant matrix \( S \), we have a new system of equations \( V_t + P(V)_x = 0 \) with \( P(V) = S^{-1}Q(SV) \). Hence we say that two quadratic mappings \( Q_1(U) \) and \( Q_2(U) \) are equivalent, if there is a constant matrix \( S \in GL_2(\mathbb{R}) \) such

\[
Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbb{R}^2.
\]

Schaeffer-Shearer [10] shows that every hyperbolic quadratic mapping \( Q(U) \) with an isolated umbilic point \( U = O \) is equivalent to

\[
Q(U) = \frac{1}{2} \begin{pmatrix}
a u^2 + 2 b u v + v^2 \\
b u^2 + 2 u v
\end{pmatrix} = \frac{1}{2} \nabla C(U),
\]

\[
C(U) = \frac{1}{3} a u^3 + b u^2 v + u v^2.
\]

where \( a \) and \( b \) are two real parameters satisfying \( a \neq 1 + b^2 \). For Stone's model, either

**Case I:** \( a < \frac{3}{4} b^2 \) or **Case II:** \( \frac{3}{4} b^2 < a < 1 + b^2 \).

A constant characteristic vector field \( \Xi = \xi(1, \xi) \) exists if and only if

\[
\xi^3 + 2 b \xi^2 + (a - 2) \xi - b = -\Xi^\perp \nabla Q(\Xi) \Xi = 0
\]

Three district (real) roots are denoted by \( \mu_1, \mu_2, \mu_3 \) and Medians are defined by \( M_j : v = \mu_j u, \ j = 1, 2, 3 \). Medians play an important role in the following discussion.
Gomes' Paper [4] and our aim: M. E. S. Gomes has proved the existence of viscous shock profiles for shock waves in Case I by topological methods and also shown an example of compressive shock wave without viscous shock profiles. The aim of this paper is to complete her results by using both topological and analytical methods: existence of viscous profiles in Case I and II and general condition for non-existence of viscous profiles. We shall show in this paper only outline of proof and details will be published in Asakura-Yamazaki [2].

2. UNDERCOMPRESSIVE AND OVERCOMPRESSIVE SHOCK WAVES

**Rankine-Hugoniot condition:** A jump discontinuity defined by

$$U(x, t) = \begin{cases} U_L & \text{for } x < st, \\ U_R & \text{for } x > st, \end{cases}$$

with a real constant $s$, is a piecewise constant weak solution to the the conservation laws (3), if and only if these quantities satisfy the Rankine-Hugoniot condition:

$$s(U_R - U_L) = Q(U_R) - Q(U_L).$$

The weak solution (5) satisfying (6) is often called a shock wave of speed $s$ joining the state $U_L$, on the left, to the state $U_R$, on the right.

**Compressive shock wave:** The shock wave is said to be a $j$-compressive ($j = 1, 2$) if the speed satisfies the Lax entropy conditions:

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R)$$

Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$.

Figure 2: Compressive Shock waves
Undercompressive shock wave: Undercompressive if $s$ satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L)$$

Overcompressive shock wave: Overcompressive if $s$ satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L)$$

Figure 3: Undercompressive Shock wave

Figure 4: Overcompressive Shock wave
Stability and Admissibility of Shock Waves: It is generally believed

- **Compressive shock waves** are generally stable and admissibility is independent of diffusion matrices in a generic class.

- **Undercompressive shock waves** are stable with additional (kinetic) condition and admissibility depends on diffusion matrices.

- **Overcompressive shock waves** are generally unstable.

Admissibility is defined in next section.

3. **Viscous Shock Profiles**

**Admissibility:** The jump discontinuity is said to be **admissible** if there exists a travelling wave solution \( U_t(x, t) = \hat{U}(\frac{x-st}{\epsilon}) \) to the parabolic system

\[
U_t + Q(U)_x = \epsilon U_{xx}, \quad \epsilon > 0 \tag{7}
\]

satisfying \( U_t(+\infty, t) = U_R, U_t(-\infty, t) = U_L \). The vector function \( \hat{U} = \hat{U}(\xi) \) is called a **viscous shock profile**.

**Differential Equations and Vector Field:** By integrating (7), \( \hat{U}(\xi) \) satisfies a system of nonlinear differential equations

\[
\frac{d\hat{U}}{d\xi} = -s(\hat{U} - U_L) + F(\hat{U}) - F(U_L) = X_s(U, U_L)
\]

Note that \( U_L \) is a critical point of \( X_s(U, U_L) \) and by Rankine-Hugoniot condition \( U_R \) is also a critical point. Since the flux functions has a potential \( C(U) \), by setting

\[
\phi_s(U_L, U) = C(U) - \nabla C(U_L) \cdot (U - U_L) - s|U - U_L|^2,
\]

the differential equations turn out to be

\[
\frac{d\hat{U}}{d\xi} = \frac{1}{2} \nabla \phi_s(U_L, \hat{U}). \tag{8}
\]

Hence the admissibility is equivalent to the existence of solution of this equations satisfying the boundary conditions at infinity:

\[
\lim_{\xi \to -\infty} \hat{U}(\xi) = U_L, \quad \lim_{\xi \to \infty} \hat{U}(\xi) = U_R
\]

or to finding flow connecting two **critical points** \( U_L \) and \( U_R \) of the vector field \( \nabla \phi_s(U_L, U) \) (or \( \phi_s \) equivalently).
4. EXISTENCE OF VISCOUS SHOCK PROFILES

Critical Points: Classification of compressive, undercompressive and overcompressive shock waves corresponds to that of critical points:

<table>
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<tr>
<th>shock wave</th>
<th>$U_L$</th>
<th>$U_R$</th>
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<tr>
<td>1-compressive</td>
<td>node (repeller)</td>
<td>saddle</td>
</tr>
<tr>
<td>2-compressive</td>
<td>saddle</td>
<td>node (attractor)</td>
</tr>
<tr>
<td>undercompressive</td>
<td>saddle</td>
<td>saddle</td>
</tr>
<tr>
<td>overcompressive</td>
<td>node (repeller)</td>
<td>node (attractor)</td>
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</table>

There are at most four critical points in the finite plane (intersection of two conics). In the four critical point case:

Case I: one node and three saddles [4], Case II: two nodes and two saddles [1]

Saddle-Saddle Connection: Flow of a saddle-saddle connection lies on $M_j$, $j = 1, 2, 3$ ([3],[4]). If $U_L \in M_j$, $j = 1, 2, 3$, the equation of viscous shock profile turns out to be the Burgers equation

$$\frac{du}{d\xi} = \frac{b + 2\mu_j}{2\mu_j}(u - u_1)(u - u_L), \quad u_1 = -u_L + \frac{2\mu_j}{b + 2\mu_j}s.$$ 

By direct computations we have

Theorem 2 ([2]) Undercompressive shocks with viscous profile exist only on $M_1 \cup M_2 \cup M_3$ in Case I and on $M_1 \cup M_3$ in Case II. Overcompressive shocks with viscous profile exist only on $M_2$ in Case II.

Existence of Viscous Profiles: If there are no saddle-saddle connection, the connection problem is settled as the following:

Theorem 3 ([2], Case I) If $U_L$ is a node, then for each single saddle point there exists a viscous shock profile between $U_L$ and the saddle point.

Theorem 4 ([2], Case II) Two nodes consist of one attractor and one repeller. If $U_L$ is a node, then for each of two saddle points, there exists a viscous shock profile shock profile between $U_L$ and the saddle point. Moreover there exist infinitely many viscous shock profiles from the repeller to the attractor.

Proof of the above both theorems is based on a generalization of the first theorem of Morse to non-compact level sets: if $|\nabla \phi_s(U, U_L)|^2 \geq m$ for any $U \in \phi_s^{-1}[p, q]$, then

$$\phi_s^{-1}[p, q] = \bigcup_{U_p \in \phi_s^{-1}(p)} I(U_p): \text{Morse foliation},$$
where $I(U_p)$: integral curves of the equation (8) connecting $U_p \in \phi_{s}^{-1}(p)$ and a certain point on the level set $\phi_{s}^{-1}(q)$.

**Case I:** We may assume that $U_L$ is a repeller. Figure 7 to 9 show nine level curves of $\phi_{s}$ for $a = 0.5, b = 1, s = -3.5$ and $U_L = '1(1, 1)$. Let $\epsilon$ be a positive small constant. The level set $\{\phi_{s} = \epsilon\}$ is composed of a small closed curve enclosing $U_L$ and three unbounded regular curves (Fig 7: $\phi_{s} = 10.00$). Suppose that a critical point $U_1$ exists on the level set $\{\phi_{s}(U) = p_1\}$, (Fig 7: $p_1 = 25.88$) such that there is no critical point in $\{\epsilon \leq \phi_{s}(U) \leq p_1 - \epsilon\}$. By the Morse lemma, we find that $\phi_{s}^{-1}[\epsilon, p_1 - \epsilon]$ is a Morse foliation. When the level curve meets a critical point for $\phi_{s}(U) = p_1$, an integral curve connects two critical points (Fig 7: $\phi_{s} = 25.88$). Repeating this argument, we have three trajectories connecting critical points (Fig 8, 9).

**Case II:** We may assume that $U_L$ is a repeller. Figure 10 to 12 show nine level curves of $\phi_{s}$ for $a = 1.5, b = 1, s = -1$ and $U_L = '1(1, 1)$. The level set $\{\phi_{s} = \epsilon : \text{small}\}$ is composed of a small closed curve enclosing $U_L$ and a single unbounded regular curves in this case (Fig 10: $\phi_{s} = 0.150$). Suppose that the first critical point $U_1$ exists on the level set $\{\phi_{s}(U) = p_1\}$, (Fig 10: $p_1 = 0.800$) such that there is no critical point in $\{\epsilon \leq \phi_{s}(U) \leq p_1 - \epsilon\}$. By the same argument as above, we find a trajectory connecting $U_L$ and the first critical point (Fig 10: $\phi_{s} = 0.800$). Repeating this argument (Fig 11: $\phi_{s} = 1.000$ to 12: $\phi_{s} = 20.00$), we have the second trajectory.
Above the second critical point, we have a closed curve and a single unbounded curve (Fig 12: $\phi_s = 21.00, 27.00$). Since the closed curve encloses an attractor, we conclude that there are infinitely many trajectories issuing from $U_L$ and drawn into the attractor.

Figure 7: $\phi_s = 10.00, 23.00, 25.88$

Figure 8: $\phi_s = 30.00, 65.00, 85.55$

Figure 9: $\phi_s = 100.0, 118.5, 140.0$
5. Compressive Shock without Viscous Shock Profile

**Liu-Oleinik Condition:** Let us denote $\mathcal{H}(U_L) : U = U(\xi; U_L)$ the Hugoniot curve issuing from $U_L; s(\xi) :$ the shock speed at $U(\xi); U_R = U(\xi_1)$. We say that $U_L$ and $U_R$ satisfy the (strict) **Liu-Oleinik condition** if $s(\xi_1) < s(\xi)$ for all $0 \leq \xi < \xi_1$ ([9]).

For strictly hyperbolic systems, as long as $U_R$ is sufficiently close to $U_L$, there exists a viscous shock profile connecting these states if and only if they satisfy the Liu-Oleinik condition ([6]).
State $U_L$ on a Median (Case I): In this case, the Hugoniot curves are composed of the median and a hyperbola, and their intersection points are $U_L$ (first bifurcation point) and $U_*$ (second bifurcation point). We can deduce by Theorem 2 that there is a saddle-saddle connection (Fig. 14: left).

**Theorem 5** ([2]) Suppose that the medians and the inflection curves intersect only at the origin $O$ and that $U_L \in M_j \setminus \{O\}$ ($j = 1, 2, 3$). Then there exists one branch $\mathcal{H}_*$ of the hyperbola $\mathcal{H}_j(U_L)$ issuing from $U_*$ such that the state $U_L$, on the left, can be joined to any state $\in \mathcal{H}_*$ sufficiently close to $U_*$, on the right, by an inadmissible, compressive Liu-Oleinik shock. In this case, there exists a saddle-saddle connection along $M_j$.

**Outline of proof:** Let $U_L \in M_1$ and $u_L > 0$. We find by direct computation that the 2-shock curve issuing from $U_L$ is composed of the segment $U_L U_*$ and one branch of the hyperbola $\mathcal{H}_j$ issuing from $U_*$. Since $s = \lambda_2$ and $s \not= \dot{\lambda}_2$ at $U_*$, one branch of hyperbola containing $U_*$ is a compressive branch of the 2-shock curve. Hence by choosing the shock speed $s$ close to $s_*$ we have a 2-compressive shock connecting $U_L$ and a state $U_R$ that is close to $U_*$. As we have noticed there is a saddle-saddle connection from $U_L$ to a certain state $U_1 \in M_1$.
which is close to $U_*$, hence close to $U_R$. Thus we conclude from the configuration of trajectories that it is impossible.

Figure 14 is the Hugoniot curves and the graph of shock speed for $a = 0.1$, $b = 1$, $U_L = (0.5, 0.5 \mu_1)$, $\mu_1 = -2.65004$, $\mu_2 = -0.369954$, $\mu_3 = 1.02$, $U_* = (-0.0484773, 0.128465)$, $s_* = 0.653999$; the parameter of the shock speed is $\xi = \frac{v-u}{u-u_i}$. In the graph of shock speed, $s$ decreases from $\lambda_2(U_L)$ to $\lambda_*$ along $\xi = \mu_1$. then the compressive blanch goes to the right. It is clear from this figure, the Liu-Oleinik condition actually holds. Figure 15 shows the level curves of the potential function for

![Figure 15](image)

Figure 15: $s = 0.646, \phi_s = 0.4730785$ (left), $\phi_s = 0.473086$ (right)

$s = 0.646$. The left figure: $\phi_s = 0.4730785$ shows the connection of $U_L$ and a certain state on the median $M_1$, hence the existence of an undercompressive shock wave. The right one shows a small closed level curve that encloses an attractor.

References


