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<td>Kawashima, Shuichi; Kurata, Kazuhiro</td>
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Kyoto University
1 Introduction

This note is a survey of our joint paper [2] on the stability problem of degenerate stationary waves for viscous conservation laws in the half space $x > 0$:

$$u_t + f(u)_x = u_{xx},$$
$$u(0, t) = -1, \quad u(x, 0) = u_0(x). \quad (1.1)$$

Here $u_0(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(u)$ is a smooth function satisfying

$$f(u) = \frac{1}{q}(-u)^{q+1}(1 + g(u)), \quad f''(u) > 0 \text{ for } -1 \leq u < 0, \quad (1.2)$$

where $q$ is a positive integer (degeneracy exponent) and $g(u) = O(|u|)$ for $u \rightarrow 0$. Notice that $1 + g(u) > 0$ for $-1 \leq u \leq 0$. It is known that the corresponding stationary problem

$$\phi_x = f(\phi),$$
$$\phi(0) = -1, \quad \phi(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad (1.3)$$
admits a unique solution \( \phi(x) \) (called *degenerate stationary wave*) which verifies \( \phi(x) \sim -(1 + x)^{-1/q} \). In particular, we have \( \phi(x) = -(1 + x)^{-1/q} \) when \( g(u) \equiv 0 \).

To discuss the stability of the degenerate stationary wave \( \phi(x) \), it is convenient to introduce the perturbation \( v \) by \( u(x, t) = \phi(x) + v(x, t) \) and rewrite the problem (1.1) as

\[
\begin{align*}
  v_t + (f(\phi + v) - f(\phi))_x &= v_{xx}, \\
  v(0, t) &= 0, \quad v(x, 0) = v_0(x),
\end{align*}
\]

(1.4)

where \( v_0(x) = u_0(x) - \phi(x) \), and \( v_0(x) \to 0 \) as \( x \to \infty \). The stability of degenerate stationary waves has been studied recently in [14, 2]. The paper [14] proved the following stability result: If the initial perturbation \( v_0(x) \) is in the weighted space \( L^2_\alpha \), then the perturbation \( v(x, t) \) decays in \( L^2 \) at the rate \( t^{-\alpha/4} \) as \( t \to \infty \), provided that \( \alpha < \alpha_*(q) \), where

\[
\alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.
\]

The decay rate \( t^{-\alpha/4} \) obtained in [14] would be optimal but the restriction \( \alpha < \alpha_*(q) \) was not very sharp. This restriction has been relaxed to \( \alpha < \alpha_c(q) := 3 + 2/q \) in our joint paper [2] by employing the space-time weighted energy method in [14] and by applying a Hardy type inequality with the best possible constant. Notice that \( \alpha_*(q) < \alpha_c(q) \). This new stability result will be reviewed in this note.

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [9]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate \( t^{-\alpha/2} \) for the perturbation without any restriction on \( \alpha \). See [4, 5, 13, 15] for the details. See also [6, 8, 10] for the related stability results for stationary waves.

To check the validity of our restriction \( \alpha < \alpha_c(q) := 3 + 2/q \), it is important to discuss the dissipativity of the following linearized operator associated with (1.4):

\[
Lv = v_{xx} - (f'(\phi)v)_x.
\]

(1.5)

In a simpler situation including the case \( g(u) \equiv 0 \) in (1.2), we showed in [2] that the operator \( L \) is uniformly dissipative in \( L^2_\alpha \) for \( \alpha < \alpha_c(q) \) but can not be dissipative for \( \alpha > \alpha_c(q) \). This suggests that the exponent \( \alpha_c(q) \) is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of \( L \) is an improvement on the previous one in [14] and has been established again by using a Hardy
type inequality with the best possible constant. This result will be also reviewed in this note.

**Notations.** For $1 \leq p \leq \infty$ and a nonnegative integer $s$, $L^p$ and $W^{s,p}$ denote the usual Lebesgue space on $\mathbb{R}_+ = (0, \infty)$ and the corresponding Sobolev space, respectively. When $p = 2$, we write $H^s = W^{s,2}$. We introduce weighted spaces. Let $w = w(x) > 0$ be a weight function defined on $[0, \infty)$ such that $w \in C^0[0, \infty)$. Then, for $1 \leq p < \infty$, we denote by $L^p(w)$ the weighted $L^p$ space on $\mathbb{R}_+$ equipped with the norm

$$
\|u\|_{L^p(w)} := \left( \int_0^\infty |u(x)|^p w(x) \, dx \right)^{1/p}.
$$

The corresponding weighted Sobolev space $W^{s,p}(w)$ is defined by $W^{s,p}(w) = \{ u \in L^p(w); \partial_x^k u \in L^p(w) \text{ for } k \leq s \}$ with the norm $\| \cdot \|_{W^{s,p}(w)}$. Also, we denote by $W^{1,p}_0(w)$ the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$
\|u\|_{W^{1,p}_0(w)} := \|\partial_x u\|_{L^p(w)} = \left( \int_0^\infty |\partial_x u(x)|^p w(x) \, dx \right)^{1/p}.
$$

When $p = 2$, we write $H^s(w) = W^{s,2}(w)$ and $H^1_0(w) = W^{1,2}_0(w)$. In the special case where $w = (1 + x)^\alpha$ with $\alpha \in \mathbb{R}$, these weighted spaces are abbreviated as $L^p_\alpha$, $W^{s,p}_\alpha$, $W^{1,p}_0$, $H^s_\alpha$ and $H^{1,0}_\alpha$, respectively.

## 2 Hardy type inequality

Our Hardy type inequality used in [2] is a simple modification of the original Hardy's inequality introduced in [1, 7] (see also [12]).

**Proposition 2.1.** Let $\psi \in C^1[0, \infty)$ and assume either

1. $\psi > 0$, $\psi_x > 0$ and $\psi(x) \to \infty$ for $x \to \infty$; or
2. $\psi < 0$, $\psi_x > 0$ and $\psi(x) \to 0$ for $x \to \infty$.

Then we have

$$
\int_0^\infty v^2 \psi_x \, dx \leq 4 \int_0^\infty v_x^2 \psi^2/\psi_x \, dx
$$

for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$ with $w = \psi^2/\psi_x$. Here 4 is the best possible constant, and there is no function $v \in H^1_0(w)$, $v \neq 0$, which attains the equality in (2.1).

**Proof.** The proof is quite simple. Let $v \in C_0^\infty(\mathbb{R}_+)$. A simple calculation gives

$$
(v^2 \psi)_x = v^2 \psi_x + 2uv_x \psi
$$

$$
= \frac{1}{2}v^2 \psi_x + \frac{1}{2} (v + 2v_x \psi/\psi_x)^2 \psi_x - 2v_x^2 \psi^2/\psi_x.
$$
Integrating (2.2) in $x$, we obtain
\[\int_{0}^{\infty} v^2 \psi_{x} \, dx + \int_{0}^{\infty} (v + 2v_{x} \psi/\psi_{x})^2 \, dx = 4 \int_{0}^{\infty} v_{x}^2 \psi^2 / \psi_{x} \, dx, \quad (2.3)\]
which gives the desired inequality (2.1). It follows from (2.3) that the equality in (2.1) holds if and only if $v + 2v_{x} \psi/\psi_{x} \equiv 0$. But we find that such a $v$ in $H_0^1(w)$ must be $v \equiv 0$.

We show the best possibility of the constant $4$ in (2.1). We consider the case (1). Let us fix $a > 0$. Let $\epsilon > 0$ be a small parameter and put
\[v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)\psi(x)^{-1/2-\epsilon}, & a < x < a + 1, \\ \psi(x)^{-1/2-\epsilon}, & a + 1 < x. \end{cases} \quad (2.4)\]
Then we have after straight-forward computations that
\[\frac{\int_{0}^{\infty}(v_{x}^\epsilon)^2 \psi^2 / \psi_{x} \, dx}{\int_{0}^{\infty}(v^\epsilon)^2 \psi_{x} \, dx} = \frac{O(1) + (1/2 + \epsilon)^2 \psi(a + 1)^{-2\epsilon}}{O(1) + \psi(a + 1)^{-2\epsilon}} \rightarrow \frac{1}{4}\]
for $\epsilon \rightarrow 0$. This shows that $4$ in (2.1) is the best possible constant. The case (2) can be treated similarly if we take a test function $v^\epsilon(x)$ as
\[v^\epsilon(x) = \begin{cases} 0, & 0 \leq x < a, \\ (x - a)(-\psi(x))^{-1/2-\epsilon}, & a < x < a + 1, \\ (-\psi(x))^{-1/2-\epsilon}, & a + 1 < x, \end{cases}\]
but we omit the details. This completes the proof of Proposition 2.1. \(\square\)

The $L^p$ version of Proposition 2.1 is given as follows.

**Proposition 2.2.** Let $\psi$ be the same as in Proposition 2.1. Let $1 < p < \infty$. Then we have
\[\int_{0}^{\infty} |v|^p \psi_{x} \, dx \leq p^p \int_{0}^{\infty} |v_{x}|^p |\psi|^p / \psi_{x}^{p-1} \, dx \quad (2.5)\]
for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in W_0^{1,p}(w)$ with $w = |\psi|^p / \psi_{x}^{p-1}$. Here $p^p$ is the best possible constant, and there is no function $v \in W_0^{1,p}(w)$, $v \neq 0$, which attains the equality in (2.5).
Proof. We only prove the inequality (2.5) and omit the other discussions. Let \( 1 < p < \infty \) and \( v \in C_0^\infty(\mathbb{R}_+) \). A simple calculation gives

\[
(|v|^p \psi)_x = |v|^p \psi_x + p|v|^{p-2}vv_x \psi = \frac{1}{p}(|v|^p \psi_x - p^p|v_x|^p|\psi|^p/\psi_x^{p-1}) + R,
\]

where

\[
R = (1 - \frac{1}{p})|v|^p \psi_x + \frac{1}{p}p^p|v_x|^p|\psi|^p/\psi_x^{p-1} + p|v|^{p-2}vv_x \psi.
\]

Integrating (2.6) in \( x \), we obtain

\[
\int_0^\infty |v|^p \psi_x \, dx + p \int_0^\infty R \, dx = p^p \int_0^\infty |v_x|^p|\psi|^p/\psi_x^{p-1} \, dx.
\]  

(2.7)

By applying the Young inequality \( AB \leq (1 - 1/p)A^{p/(p-1)} + (1/p)B^p \) for \( A = |v|^{p-1}\psi^{(p-1)/p} \) and \( B = p|v_x||\psi|^p/\psi_x^{(p-1)/p} \), we find that \( R \geq 0 \), which together with (2.7) gives the desired inequality (2.5).

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** Let \( \phi \in C^1[0, \infty), \phi < 0, \phi_x > 0 \), and \( \phi(x) \to 0 \) for \( x \to \infty \). Let \( \sigma \in \mathbb{R} \) with \( \sigma \neq 0 \), and define the weight functions \( w \) and \( w_1 \) by

\[
w = (-\phi)^{-\sigma+1}/\phi_x, \quad w_1 = (-\phi)^{-\sigma-1}\phi_x.
\]  

(2.8)

Then we have

\[
\int_0^\infty v^2 w_1 \, dx \leq \frac{4}{\sigma^2} \int_0^\infty v_x^2 w \, dx
\]

(2.9)

for \( v \in H_0^1(w) \). Here \( 4/\sigma^2 \) is the best possible constant, and there is no function \( v \in H_0^1(w), v \neq 0 \), which attains the equality in (2.9).

**Proof.** We put \( \psi = (-\phi)^{-\sigma} \) for \( \sigma > 0 \) and \( \psi = -(-\phi)^{-\sigma} \) for \( \sigma < 0 \), and apply Proposition 2.1. This gives the desired conclusion.

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** Let \( \alpha \in \mathbb{R} \) with \( \alpha \neq 1 \). Then we have

\[
\|v\|_{L_{\alpha-2}^2} \leq \frac{2}{|\alpha - 1|} \|v_x\|_{L_2^2}
\]

(2.10)

for \( v \in H_{\alpha,0}^1 \). Here the constant \( 2/|\alpha - 1| \) is the best possible, and there is no function \( v \in H_{\alpha,0}^1, v \neq 0 \), which attains the equality in (2.10).

**Proof.** Let \( \phi = -(1 + x)^{-1/q} \) with \( q > 0 \). We apply Proposition 2.3 for this \( \phi \) and \( \sigma = (\alpha - 1)q \). This gives the proof.
3 Dissipativity of the linearized operator

Following [2], we discuss the dissipativity of the operator $L$ defined by (1.5) in the weighted space $L^2(w)$, where $w$ is given by (2.8) with $\phi$ being the the degenerate stationary wave. Note that our degenerate stationary wave $\phi$ is a smooth solution of (1.3) and verifies

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \to 0 \text{ for } x \to \infty,$$

$$c(1 + x)^{-1/q} \leq -\phi(x) \leq C(1 + x)^{-1/q}. \quad (3.2)$$

Now, letting $w > 0$ be a smooth weight function depending only on $x$, we calculate the inner product $\langle Lv, v \rangle_{L^2(w)}$ for $v \in C_0^\infty(\mathbb{R}_+)$, where

$$\langle u, v \rangle_{L^2(w)} := \int_0^\infty uvw \, dx. \quad (3.3)$$

We multiply (1.5) by $v$. Then a simple computation gives

$$(Lv)v = (vv_x - \frac{1}{2}f'(\phi)v^2)_x - v_x^2 - \frac{1}{2}f''(\phi)\phi_xv^2.$$

Multiplying this equality by $w$, we obtain

$$(Lv)vw = \left\{ \left( vv_x - \frac{1}{2}f'(\phi)v^2 \right)w - \frac{1}{2}v^2w_x \right\}_x$$

$$- v_x^2w + \frac{1}{2}v^2(w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x). \quad (3.4)$$

Now we choose the weight function $w$ and the corresponding $w_1$ in terms of our degenerate stationary wave $\phi$ by (2.8), where $\sigma \in \mathbb{R}$. Then we have $w = (-\phi)^{-\sigma+1}/f(\phi)$ and $w_1 = (-\phi)^{-\sigma-1}f(\phi)$ by $\phi_x = f(\phi)$. After straightforward computations, we find that

$$w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x = 2(c_1(\sigma) - r(\phi))w_1, \quad (3.5)$$

where

$$c_1(\sigma) := \sigma(\sigma - 1)/2 - q(q + 1), \quad (3.6)$$

$$r(u) := (-u)^2f''(u)/f(u) - q(q + 1).$$

Substituting (3.5) into (3.4) and integrating with respect to $x$, we get the following conclusion.
Claim 3.1. Let $\phi$ be the degenerate stationary wave and define the weight functions $w$ and $w_1$ by (2.8) with $\sigma \in \mathbb{R}$. Then the operator $L$ defined in (1.5) verifies

$$\langle Lv, v \rangle_{L^2(w)} = -\|v_x\|_{L^2(w)}^2 + c_1(\sigma)\|v\|_{L^2(w_1)}^2 - \int_0^\infty v^2 r(\phi) w_1 \, dx \quad (3.7)$$

for $v \in C_0^\infty(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$, where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6).

The term $r(\phi)$ in (3.7) can be regarded as a small perturbation. In fact, a straightforward computation gives

$$r(u) = (-u)\{(u)g''(u) - 2(q + 1)g'(u)\}/(1 + g(u)) \quad (3.8)$$

which shows that $r(u) = O(|u|)$ for $u \to 0$. In particular, we have $r(u) \equiv 0$ if $g(u) \equiv 0$. With these preparations, we have the following result on the characterization of the dissipativity of $L$.

Theorem 3.2. Assume (1.2). Let $\phi$ be the degenerate stationary wave and $L$ be the operator defined in (1.5). Let $w$ and $w_1$ be the weight functions in (2.8) with the parameter $\sigma \in \mathbb{R}$. Then we have:

1. Let $-2q < \sigma < 2(q + 1)$. Then, under the additional assumption that $r(u) \geq 0$ for $-1 \leq u \leq 0$, the operator $L$ is uniformly dissipative in $L^2(w)$. Namely, there is a positive constant $\delta$ such that

$$\langle Lv, v \rangle_{L^2(w)} \leq -\delta(\|v_x\|_{L^2(w)}^2 + \|v\|_{L^2(w_1)}^2) \quad (3.9)$$

for $v \in H^1_0(w)$.

2. Let $\sigma > 2(q + 1)$ or $\sigma < -2q$. Then the operator $L$ can not be dissipative in $L^2(w)$. Namely, we have $\langle Lv, v \rangle_{L^2(w)} > 0$ for some $v \in H^1_0(w)$ with $v \neq 0$.

Proof. The proof is based on the equality (3.7) in Claim 3.1 and the Hardy type inequality (2.9) in Proposition 2.3.

Let $-2q < \sigma < 2(q + 1)$. This is equivalent to $c_1(\sigma) < \sigma^2/4$. Therefore we can choose $\delta > 0$ so small that $\delta(1 + \sigma^2/4) \leq \sigma^2/4 - c_1(\sigma)$. Since $r(\phi) \geq 0$ by the additional assumption on $r(u)$ and since $(\sigma^2/4)\|v\|^2_{L^2(w)} \leq \|v_x\|^2_{L^2(w)}$ by the Hardy type inequality (2.9), we have from (3.7) that

$$\langle Lv, v \rangle_{L^2(w)} \leq -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}$$

$$= -\delta\|v_x\|^2_{L^2(w)} - (1 - \delta)\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}$$

$$\leq -\delta\|v_x\|^2_{L^2(w)} - \{(1 - \delta)\sigma^2/4 - c_1(\sigma)\}\|v\|^2_{L^2(w_1)}$$

$$\leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \quad (3.10)$$
for \( v \in C_0^\infty(\mathbb{R}_+) \) and hence for \( v \in H_0^1(w) \), where we have used the fact that 
\[(1 - \delta)\sigma^2/4 - c_1(\sigma) \geq \delta.\]
This completes the proof of the uniform dissipative case (1).

Next we consider the case where \( \sigma > 2(q + 1) \); the case \( \sigma < -2q \) can be treated similarly and we omit the argument in this latter case. When \( \sigma > 2(q + 1) \), we have \( c_1(\sigma) > \sigma^2/4 \). Then we choose \( \delta > 0 \) so small that 
\[c_1(\sigma) \geq \sigma^2/4 + 3\delta.\]
Since \( r(u) = O(|u|) \) for \( u \to 0 \) and \( \phi(x) \to 0 \) for \( x \to \infty \), we take \( a = a(\delta) > 0 \) so large that \(|r(\phi)| \leq \delta \) for \( x \geq a \). For this choice of \( a \) and for \( \epsilon > 0 \), we take a test function \( v^\epsilon \) as in (2.4):

\[
v^\epsilon(x) = \begin{cases} 
0, & 0 \leq x < a, \\
(x - a)(-\phi(x))^{\sigma(1/2 + \epsilon)}, & a < x < a + 1, \\
(-\phi(x))^{\sigma(1/2 + \epsilon)}, & a + 1 < x.
\end{cases}
\]

Then we have

\[
\left| \int_0^\infty (v^\epsilon)^2 r(\phi)w_1 \, dx \right| \leq \delta \int_a^\infty (v^\epsilon)^2 w_1 \, dx = \delta \|v^\epsilon\|^2_{L^2(w_1)},
\]
so that we have from (3.7) that

\[
\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)} \geq -\|v^\epsilon_x\|^2_{L^2(w)} + (c_1(\sigma) - \delta)\|v^\epsilon\|^2_{L^2(w_1)}. \tag{3.12}
\]

Also, by straightforward computations, we find that

\[
\frac{\|v^\epsilon_x\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} = \frac{O(1) + \sigma^2(1/2 + \epsilon)^2\frac{1}{2\sigma\epsilon}(-\phi(a + 1))^{2\sigma\epsilon}}{O(1) + \frac{1}{2\sigma\epsilon}(-\phi(a + 1))^{2\sigma\epsilon}} \to \frac{\sigma^2}{4}
\]
for \( \epsilon \to 0 \). Thus we have \( \|v^\epsilon_x\|^2_{L^2(w)}/\|v^\epsilon\|^2_{L^2(w_1)} \leq \sigma^2/4 + \delta \) for a suitably small \( \epsilon = \epsilon(\delta) > 0 \). Consequently, we have from (3.12) that

\[
\frac{\langle Lv^\epsilon, v^\epsilon \rangle_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} \geq -\frac{\|v^\epsilon_x\|^2_{L^2(w)}}{\|v^\epsilon\|^2_{L^2(w_1)}} + c_1(\sigma) - \delta \geq -(\sigma^2/4 + \delta) + c_1(\sigma) - \delta \geq \delta.
\]

This completes the proof of the non-dissipative case (2). Thus the proof of Theorem 3.2 is complete. \qed
In the special case where \( g(u) \equiv 0 \) so that \( f(u) = \frac{1}{q}(-u)^{q+1} \), we have \( \phi = -(1 + x)^{-1/q} \) and the operator \( L \) in (1.5) is reduced to
\[
L_0v = v_{xx} + \frac{q+1}{q} \left( \frac{v}{1+x} \right)_x.
\] (3.13)

In this simplest case, we have the complete characterization of the dissipativity of the operator \( L_0 \).

**Theorem 3.3.** Let \( \alpha_c(q) := 3 + 2/q \). Then we have the complete characterization of the dissipativity of the operator \( L_0 \) given in (3.13):

1. Let \(-1 < \alpha < \alpha_c(q)\). Then \( L_0 \) is uniformly dissipative in \( L^2_\alpha \). Namely, there is a positive constant \( \delta \) such that
\[
\langle L_0v, v \rangle_{L^2_\alpha} \leq -\delta (\|v_x\|_{L^2_\alpha}^2 + \|v\|_{L^2_{\alpha-2}}^2)
\] for \( v \in H^1_{\alpha,0} \).

2. Let \( \alpha = \alpha_c(q) \) or \( \alpha = -1 \). Then \( L_0 \) is strictly dissipative in \( L^2_\alpha \). Namely, we have \( \langle L_0v, v \rangle_{L^2_\alpha} < 0 \) for \( v \in H^1_{\alpha,0} \) with \( v \neq 0 \).

3. Let \( \alpha > \alpha_c(q) \) or \( \alpha < -1 \). Then \( L_0 \) can not be dissipative in \( L^2_\alpha \). Namely, we have \( \langle L_0v, v \rangle_{L^2_\alpha} > 0 \) for some \( v \in H^1_{\alpha,0} \) with \( v \neq 0 \).

**Proof.** In this case, we have \( \phi = -(1 + x)^{-1/q} \), \( L = L_0 \) and \( r(u) \equiv 0 \). Therefore, (3.7) is reduced to
\[
\langle L_0v, v \rangle_{L^2_\alpha} = -\|v_x\|_{L^2_\alpha}^2 + c_1(\sigma)\|v\|_{L^2_{\alpha-1}}^2,
\] (3.15)
where \( w \) and \( w_1 \) are the weight functions defined in (2.8) with \( \phi = -(1 + x)^{-1/q} \) and \( \sigma = (\alpha - 1)q \). The desired conclusions easily follow from (3.15) by applying the same argument as in Theorem 3.2. We omit the details. \( \Box \)

**4 Nonlinear stability**

The following stability result for the nonlinear problem (1.4) was obtained in [2] as a refinement of the result in [14].

**Theorem 4.1.** Assume (1.2). Suppose that \( v_0 \in L^2_\alpha \cap L^\infty \) for some \( \alpha \) with \( 1 \leq \alpha < \alpha_c(q) := 3 + q/2 \). Then there is a positive constant \( \delta_1 \) such that if \( \|v_0\|_{L^2_\alpha} \leq \delta_1 \), then the problem (1.4) has a unique global solution \( v \in C^0([0, \infty); L^2_\alpha \cap L^p) \) for each \( p \) with \( 2 \leq p < \infty \). Moreover, the solution verifies the decay estimate
\[
\|v(t)\|_{L^p} \leq C(\|v_0\|_{L^2_\alpha} + \|v_0\|_{L^\infty})(1 + t)^{-\alpha/4 - \nu}
\] (4.1)
for \( t \geq 0 \), where \( 2 \leq p < \infty \), \( \nu = (1/2)(1/2 - 1/p) \), and \( C \) is a positive constant.
Proof. A key to the proof of this theorem is to show the following space-time weighted energy inequality:

\[
(1 + t)^\gamma \|v(t)\|_{L_\beta}^2 + \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|_{L_\beta}^2 + \|v(\tau)\|_{L_{\beta-2}}^2) d\tau 
\leq C\|v_0\|_{L_\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma - 1} \|v(\tau)\|_{L_\beta}^2 d\tau + CS_\beta^\gamma(t) 
\]

(4.2)

for any \(\gamma \geq 0\) and \(\beta\) with \(0 \leq \beta \leq \alpha\), where \(1 \leq \alpha < \alpha_c(q) := 3 + 2/q\), \(C\) is a constant independent of \(\gamma\) and \(\beta\), and

\[
S_\beta^\gamma(t) = \int_0^t (1 + \tau)^\gamma \|v(\tau)\|_{L_{\beta-1}}^3 d\tau. 
\]

(4.3)

Here we give an outline of the proof of (4.2) and omit the other discussions. We refer to [2, 14] for the complete proof of Theorem 4.1.

Proof of (4.2) for \(\beta = 0\). The proof is based on the time weighted \(L^2\) energy method. First we note that

\[
\|v(t)\|_{L^\infty} \leq M_{\infty}, 
\]

(4.4)

where \(M_{\infty} = \|v_0\|_{L^\infty} + 2\). This is an easy consequence of the maximum principle (see [5] for the details). Now we multiply the equation (1.4) by \(v\). This yields

\[
\left(\frac{1}{2}v^2\right)_t + (F - vv_x)_x + v_x^2 + G = 0, 
\]

(4.5)

where

\[
F = (f(\phi + v) - f(\phi))v - \int_0^v (f(\phi + \eta) - f(\phi)) d\eta, 
\]

(4.6)

\[
G = \int_0^v (f'(\phi + \eta) - f'(\phi)) d\eta \cdot \phi_x. 
\]

We note that

\[
F = \frac{1}{2}f'(\phi)v^2 + O(|v|^3), \quad G = \frac{1}{2}f''(\phi)\phi_x v^2 + \phi_x O(|v|^3) 
\]

(4.7)

for \(v \to 0\). Here, a careful computation, using (3.2) and (4.4), shows that

\[
G \geq c(1 + x)^{-2}v^2 - C(1 + x)^{-1-1/q}|v|^3 
\]

(4.8)

for any \(x \in \mathbb{R}_+\). We integrate (4.5) over \(\mathbb{R}_+\) and substitute (4.8) into the resulting equality, obtaining

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L_2}^2 + \|v_x\|_{L_2}^2 + c\|v\|_{L_{-2}}^2 \leq C\|v\|_{L_{-1}}^3. 
\]
We multiply this inequality by \((1 + t)^\gamma\) and integrate with respect \(t\). This yields the desired inequality (4.2) for \(\beta = 0\).

**Proof of (4.2) for \(\beta > 0\).** We apply the space-time weighted energy method employed in [14, 2] (see also [3]). Let \(w > 0\) be a smooth weight function depending only on \(x\), which will be specified later. We multiply (4.5) by \(w\), obtaining

\[
\left(\frac{1}{2}v^2w\right)_t + \left\{ (F - \mu vv_x)w + \frac{1}{2}v^2w_x \right\}_x + v_x^2w - \left(\frac{1}{2}v^2w_{xx} + Fw_x - Gw \right) = 0.
\]

Here, using (4.7), we have

\[
\frac{1}{2}v^2w_{xx} + Fw_x - Gw = \frac{1}{2}v^2(2v_{xx} + v_{x}f' (\phi) - v_1 f''(\phi)\phi_x) + R,
\]

where \(R = w_x O(|v|^3) - w_x O(|v|^3)\) for \(v \to 0\). Notice that the coefficient \(w_{xx} + v_{x} f'(\phi) - v_{1} f''(\phi)\phi_x\) in (4.10) is just the same as that appeared in (3.4). Now we choose the weight function \(w\) and the corresponding \(w_1\) by (2.8) with \(\sigma = (\beta - 1)q\), where \(0 \leq \beta \leq \alpha\) and \(1 \leq \alpha < \alpha_c(q) := 3 + 2/q\). Then we have (3.5) with \(\sigma = (\beta - 1)q\). Substituting these expressions into (4.9) and integrating over \(\mathbb{R}_+\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \|v_x\|_{L^2(w)}^2 - c_1(\sigma) \|v\|_{L^2(w_1)}^2 + \int_0^\infty v^2 r(\phi)w_1 \, dx = \int_0^\infty R \, dx,
\]

where \(c_1(\sigma)\) and \(r(\phi)\) are given in (3.6) with \(\sigma = (\beta - 1)q\). Here our weight functions verify

\[
w \sim (1 + x)^\beta, \quad w_1 \sim (1 + x)^{\beta - 2},
\]

where the symbol \(\sim\) means the equivalence. This implies that the norms \(\| \cdot \|_{L^2(w)}\) and \(\| \cdot \|_{L^2(w_1)}\) are equivalent to \(\| \cdot \|_{L_{\beta}^2}\) and \(\| \cdot \|_{L_{\beta - 2}^2}\), respectively.

We estimate (4.11) similarly as in (1) of Theorem 3.2. To this end, we note that \(\sigma_1 \leq \sigma \leq \sigma_2\), where \(\sigma_1 = -q\) and \(\sigma_2 = (\alpha - 1)q\). Since \(c_1(\sigma) < \sigma^2/4\) for \(-2q < \sigma < 2(q + 1)\) and since \(-2q < \sigma_1 < \sigma_2 < 2(q + 1)\), we can choose \(\delta > 0\) so small that

\[
\delta \leq \min_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.
\]

Notice that this \(\delta\) is independent of \(\beta\). For this choice of \(\delta\), we take \(a = a(\delta) > 0\) so large that \(|r(\phi)| \leq \delta\) for \(x \geq a\). Then we have

\[
\left| \int_0^\infty v^2 r(\phi)w_1 \, dx \right| \leq \delta \|v\|_{L^2(w_1)}^2 + C \|v\|_{L_{\beta - 2}^2}^2,
\]

where
where $C$ is a constant satisfying $C \geq (1 + x)^2 |r(\phi)| w_1$ for $0 \leq x \leq a$. Also, using the Hardy type inequality $(\sigma^2/4) \|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ in (2.9) and estimating similarly as in (3.10), we have
\[
\|v_x\|^2_{L^2(w)} - c_1(\sigma) \|v\|^2_{L^2(w_1)} \geq \delta \|v_x\|^2_{L^2(w)} + 2\delta \|v\|^2_{L^2(w_1)},
\]
where we have used the fact that $(1 - \delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta$. On the other hand, using (4.4), we see that $|R| \leq C(|w_x| + w\phi_x) |v|^3$. Moreover, a straightforward computation shows that $|w_x| + w\phi_x \leq C(1 + x)^{\beta-1}$. Substituting all these estimates into (4.11), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{L^2(w)} + \delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \leq C \|v\|^2_{L^2_\gamma} + C \|v\|^3_{L^3_{\beta-1}},
\]
(4.13)
where $\delta$ and $C$ are independent of $\beta$. We multiply this inequality by $(1 + t)^\gamma$ and integrate with respect to $t$. By virtue of (4.12), we have
\[
(1 + t)^\gamma \|v(t)\|^2_{L^2_\beta} + \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|^2_{L^2_\beta} + \|v(\tau)\|^2_{L^2_{\beta-2}}) d\tau 
\leq C \|v_0\|^2_{L^2_\beta} + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|v(\tau)\|^2_{L^2_\beta} d\tau 
+ C \int_0^t (1 + \tau)^\gamma \|v(\tau)\|^2_{L^2_\gamma} d\tau + CS^\gamma_{\beta}(t),
\]
(4.14)
where the constant $C$ is independent of $\gamma$ and $\beta$. Here the third term on the right hand side of (4.14) was already estimated by (4.2) with $\beta = 0$. Hence we have proved (4.2) also for $0 < \beta \leq \alpha$. This completes the proof. 

\section*{References}


