A Hardy type inequality and application to the stability of degenerate stationary waves (Mathematical Analysis in Fluid and Gas Dynamics)

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A Hardy type inequality
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1 Introduction

This note is a survey of our joint paper [2] on the stability problem of degenerate stationary waves for viscous conservation laws in the half space $x > 0$:

\begin{align}
  u_t + f(u)_x &= u_{xx}, \\
  u(0, t) &= -1, \quad u(x, 0) = u_0(x).
\end{align}  \tag{1.1}

Here $u_0(x) \to 0$ as $x \to \infty$, and $f(u)$ is a smooth function satisfying

\begin{equation}
  f(u) = \frac{1}{q}(-u)^{q+1}(1 + g(u)), \quad f''(u) > 0 \text{ for } -1 \leq u < 0,
\end{equation}  \tag{1.2}

where $q$ is a positive integer (degeneracy exponent) and $g(u) = O(|u|)$ for $u \to 0$. Notice that $1 + g(u) > 0$ for $-1 \leq u \leq 0$. It is known that the corresponding stationary problem

\begin{align}
  \phi_x &= f(\phi), \\
  \phi(0) &= -1, \quad \phi(x) \to 0 \text{ as } x \to \infty,
\end{align}  \tag{1.3}
admits a unique solution $\phi(x)$ (called degenerate stationary wave) which verifies $\phi(x) \sim -(1 + x)^{-1/q}$. In particular, we have $\phi(x) = -(1 + x)^{-1/q}$ when $g(u) \equiv 0$.

To discuss the stability of the degenerate stationary wave $\phi(x)$, it is convenient to introduce the perturbation $v$ by $u(x, t) = \phi(x) + v(x, t)$ and rewrite the problem (1.1) as

\begin{align*}
u_t + (f(\phi + v) - f(\phi))_x &= v_{xx}, \\
v(0, t) = 0, \quad v(x, 0) &= v_0(x), \tag{1.4}
\end{align*}

where $v_0(x) = u_0(x) - \phi(x)$, and $v_0(x) \to 0$ as $x \to \infty$. The stability of degenerate stationary waves has been studied recently in [14, 2]. The paper [14] proved the following stability result: If the initial perturbation $v_0(x)$ is in the weighted space $L^2_\alpha$, then the perturbation $v(x, t)$ decays in $L^2$ at the rate $t^{-\alpha/4}$ as $t \to \infty$, provided that $\alpha < \alpha_*(q)$, where

$$\alpha_*(q) := (q + 1 + \sqrt{3q^2 + 4q + 1})/q.$$ 

The decay rate $t^{-\alpha/4}$ obtained in [14] would be optimal but the restriction $\alpha < \alpha_*(q)$ was not very sharp. This restriction has been relaxed to $\alpha < \alpha_c(q) := 3 + 2/q$ in our joint paper [2] by employing the space-time weighted energy method in [14] and by applying a Hardy type inequality with the best possible constant. Notice that $\alpha_*(q) < \alpha_c(q)$. This new stability result will be reviewed in this note.

It is interesting to note that a similar restriction on the weight is imposed also for the stability of degenerate shock profiles (see [9]). We remark that our stability result for degenerate stationary waves is completely different from those for non-degenerate case. In fact, for non-degenerate stationary waves, we have the better decay rate $t^{-\alpha/2}$ for the perturbation without any restriction on $\alpha$. See [4, 5, 13, 15] for the details. See also [6, 8, 10] for the related stability results for stationary waves.

To check the validity of our restriction $\alpha < \alpha_c(q) := 3 + 2/q$, it is important to discuss the dissipativity of the following linearized operator associated with (1.4):

$$Lv = v_{xx} - (f'(\phi)v)_x.$$ \hspace{1cm} \tag{1.5}

In a simpler situation including the case $g(u) \equiv 0$ in (1.2), we showed in [2] that the operator $L$ is uniformly dissipative in $L^2_\alpha$ for $\alpha < \alpha_c(q)$ but can not be dissipative for $\alpha > \alpha_c(q)$. This suggests that the exponent $\alpha_c(q)$ is the critical exponent of the stability problem of degenerate stationary waves. This result on the characterization of the dissipativity of $L$ is an improvement on the previous one in [14] and has been established again by using a Hardy
type inequality with the best possible constant. This result will be also reviewed in this note.

**Notations.** For $1 \leq p \leq \infty$ and a nonnegative integer $s$, $L^p$ and $W^{s,p}$ denote the usual Lebesgue space on $\mathbb{R}_+ = (0, \infty)$ and the corresponding Sobolev space, respectively. When $p = 2$, we write $H^s = W^{s,2}$. We introduce weighted spaces. Let $w = w(x) > 0$ be a weight function defined on $[0, \infty)$ such that $w \in C^0[0, \infty)$. Then, for $1 \leq p < \infty$, we denote by $L^p(w)$ the weighted $L^p$ space on $\mathbb{R}_+$ equipped with the norm

$$\|u\|_{L_p(w)} := \left( \int_0^\infty |u(x)|^p w(x) \, dx \right)^{1/p}. \quad (1.6)$$

The corresponding weighted Sobolev space $W^{s,p}(w)$ is defined by $W^{s,p}(w) = \{u \in L^p(w); \partial_x^k u \in L^p(w) \text{ for } k \leq s \}$ with the norm $\| \cdot \|_{W^{s,p}(w)}$. Also, we denote by $W^{1,p}_0(w)$ the completion of $C^\infty_0(\mathbb{R}_+)$ with respect to the norm

$$\|u\|_{W^{1,p}_0(w)} := \|\partial_x u\|_{L_p(w)} = \left( \int_0^\infty |\partial_x u(x)|^p w(x) \, dx \right)^{1/p}. \quad (1.7)$$

When $p = 2$, we write $H^s(w) = W^{s,2}(w)$ and $H^1_0(w) = W^{1,2}_0(w)$. In the special case where $w = (1 + x)^\alpha$ with $\alpha \in \mathbb{R}$, these weighted spaces are abbreviated as $L^p_\alpha, W^{s,p}_\alpha, W^{1,p}_0, H^s_\alpha$ and $H^{1,0}_\alpha$, respectively.

## 2 Hardy type inequality

Our Hardy type inequality used in [2] is a simple modification of the original Hardy’s inequality introduced in [1, 7] (see also [12]).

**Proposition 2.1.** Let $\psi \in C^1[0, \infty)$ and assume either

1. $\psi > 0$, $\psi_x > 0$ and $\psi(x) \to \infty$ for $x \to \infty$; or
2. $\psi < 0$, $\psi_x > 0$ and $\psi(x) \to 0$ for $x \to \infty$.

Then we have

$$\int_0^\infty v^2 \psi_x \, dx \leq 4 \int_0^\infty v_x^2 \psi^2/\psi_x \, dx \quad (2.1)$$

for $v \in C^\infty_0(\mathbb{R}_+)$ and hence for $v \in H^1_0(w)$ with $w = \psi^2/\psi_x$. Here $4$ is the best possible constant, and there is no function $v \in H^1_0(w)$, $v \neq 0$, which attains the equality in (2.1).

**Proof.** The proof is quite simple. Let $v \in C^\infty_0(\mathbb{R}_+)$. A simple calculation gives

$$v^2 \psi_x = v^2 \psi_x + 2uv \psi = \frac{1}{2}v^2 \psi_x + \frac{1}{2}(v + 2v \psi/\psi_x)^2 \psi_x - 2v_x^2 \psi^2/\psi_x. \quad (2.2)$$
Integrating (2.2) in \( x \), we obtain
\[
\int_{0}^{\infty} v^{2} \psi_{x} \, dx + \int_{0}^{\infty} (v + 2v_{x}\psi/\psi_{x})^{2} \, dx = 4 \int_{0}^{\infty} v_{x}^{2} \psi^{2} / \psi_{x} \, dx, \tag{2.3}
\]
which gives the desired inequality (2.1). It follows from (2.3) that the equality in (2.1) holds if and only if \( v + 2v_{x}\psi/\psi_{x} \equiv 0 \). But we find that such a \( v \) in \( H_{0}^{1}(w) \) must be \( v \equiv 0 \).

We show the best possibility of the constant 4 in (2.1). We consider the case (1). Let us fix \( a > 0 \). Let \( \epsilon > 0 \) be a small parameter and put
\[
v^{\epsilon}(x) = \begin{cases} 
0, & 0 \leq x < a, \\
(x - a)\psi(x)^{-1/2-\epsilon}, & a < x < a + 1, \\
\psi(x)^{-1/2-\epsilon}, & a + 1 < x.
\end{cases} \tag{2.4}
\]
Then we have after straightforward computations that
\[
\frac{\int_{0}^{\infty}(v^{\epsilon}_{x})^{2} \psi^{2} / \psi_{x} \, dx}{\int_{0}^{\infty}(v^{\epsilon})^{2} \psi_{x} \, dx} = \frac{O(1) + (1/2 + \epsilon)^{2} \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}}{O(1) + \frac{1}{2\epsilon} \psi(a + 1)^{-2\epsilon}} \rightarrow \frac{1}{4}
\]
for \( \epsilon \to 0 \). This shows that 4 in (2.1) is the best possible constant. The case (2) can be treated similarly if we take a test function \( v^{\epsilon}(x) \) as
\[
v^{\epsilon}(x) = \begin{cases} 
0, & 0 \leq x < a, \\
(x - a)(-\psi(x))^{-1/2-\epsilon}, & a < x < a + 1, \\
(-\psi(x))^{-1/2-\epsilon}, & a + 1 < x,
\end{cases}
\]
but we omit the details. This completes the proof of Proposition 2.1. \( \square \)

The \( L^{p} \) version of Proposition 2.1 is given as follows.

**Proposition 2.2.** Let \( \psi \) be the same as in Proposition 2.1. Let \( 1 < p < \infty \). Then we have
\[
\int_{0}^{\infty} |v|^{p} \psi_{x} \, dx \leq p^{p} \int_{0}^{\infty} |v_{x}|^{p} |\psi|^{p} / \psi_{x}^{p-1} \, dx \tag{2.5}
\]
for \( v \in C_{0}^{\infty}(\mathbb{R}_{+}) \) and hence for \( v \in W_{0}^{1,p}(w) \) with \( w = |\psi|^{p} / \psi_{x}^{p-1} \). Here \( p^{p} \) is the best possible constant, and there is no function \( v \in W_{0}^{1,p}(w) \), \( v \neq 0 \), which attains the equality in (2.5).
Proof. We only prove the inequality (2.5) and omit the other discussions. Let $1 < p < \infty$ and $v \in C_0^\infty(\mathbb{R}_+)$. A simple calculation gives

$$
(|v|^p \psi)_x = |v|^p \psi_x + p|v|^{p-2}vv_x \psi = \frac{1}{p}(|v|^p \psi_x - p^p |v_x|^p |\psi|^p/\psi_x^{p-1}) + R,
$$

(2.6)

where

$$
R = (1 - \frac{1}{p})|v|^p \psi_x + \frac{1}{p}p^p |v_x|^p |\psi|^{p-1} + p|v|^{p-2}vv_x \psi.
$$

Integrating (2.6) in $x$, we obtain

$$
\int_0^\infty |v|^p \psi_x \, dx + p \int_0^\infty R \, dx = p^p \int_0^\infty |v_x|^p |\psi|^p/\psi_x^{p-1} \, dx.
$$

(2.7)

By applying the Young inequality $AB \leq (1 - 1/p)A^{p/(p-1)} + (1/p)B^p$ for $A = |v|^{p-1} \psi_x^{(p-1)/p}$ and $B = p|v_x| |\psi|^{(p-1)/p}$, we find that $R \geq 0$, which together with (2.7) gives the desired inequality (2.5). $\square$

The following variant of Proposition 2.1 is useful in our application.

**Proposition 2.3.** Let $\phi \in C^1[0, \infty)$, $\phi < 0$, $\phi_x > 0$, and $\phi(x) \to 0$ for $x \to \infty$. Let $\sigma \in \mathbb{R}$ with $\sigma \neq 0$, and define the weight functions $w$ and $w_1$ by

$$
w = (-\phi)^{-\sigma+1}/\phi_x, \quad w_1 = (-\phi)^{-\sigma-1}\phi_x.
$$

(2.8)

Then we have

$$
\int_0^\infty v^2 w_1 \, dx \leq \frac{4}{\sigma^2} \int_0^\infty v_x^2 w \, dx
$$

(2.9)

for $v \in H_0^1(w)$. Here $4/\sigma^2$ is the best possible constant, and there is no function $v \in H_0^1(w), \, v \neq 0$, which attains the equality in (2.9).

**Proof.** We put $\psi = (-\phi)^{-\sigma}$ for $\sigma > 0$ and $\psi = -(-\phi)^{-\sigma}$ for $\sigma < 0$, and apply Proposition 2.1. This gives the desired conclusion. $\square$

As a simple corollary of Proposition 2.3, we have:

**Corollary 2.4.** Let $\alpha \in \mathbb{R}$ with $\alpha \neq 1$. Then we have

$$
\|v\|_{L^2_{\alpha-2}} \leq \frac{2}{|\alpha - 1|} \|v_x\|_{L^2_{\alpha}}
$$

(2.10)

for $v \in H^1_{\alpha,0}$. Here the constant $2/|\alpha - 1|$ is the best possible, and there is no function $v \in H^1_{\alpha,0}, \, v \neq 0$, which attains the equality in (2.10).

**Proof.** Let $\phi = -(1 + x)^{-1/q}$ with $q > 0$. We apply Proposition 2.3 for this $\phi$ and $\sigma = (\alpha - 1)q$. This gives the proof. $\square$
3 Dissipativity of the linearized operator

Following [2], we discuss the dissipativity of the operator $L$ defined by (1.5) in the weighted space $L^2(w)$, where $w$ is given by (2.8) with $\phi$ being the the degenerate stationary wave. Note that our degenerate stationary wave $\phi$ is a smooth solution of (1.3) and verifies

$$-1 \leq \phi(x) < 0, \quad \phi_x(x) > 0, \quad \phi(x) \to 0 \text{ for } x \to \infty, \quad (3.1)$$

$$c(1 + x)^{-1/q} \leq -\phi(x) \leq C(1 + x)^{-1/q}. \quad (3.2)$$

Now, letting $w > 0$ be a smooth weight function depending only on $x$, we calculate the inner product $\langle Lv, v \rangle_{L^2(w)}$ for $v \in C_0^\infty(\mathbb{R}_+)$, where

$$\langle u, v \rangle_{L^2(w)} := \int_0^\infty uvw \, dx. \quad (3.3)$$

We multiply (1.5) by $v$. Then a simple computation gives

$$(Lv)v = (vv_x - \frac{1}{2}f'(\phi)v^2)_x - v_x^2 - \frac{1}{2}f''(\phi)\phi_xv^2.$$  

Multiplying this equality by $w$, we obtain

$$(Lv)vw = \{ (vv_x - \frac{1}{2}f'(\phi)v^2)_x \, w - \frac{1}{2}v^2w_x \}_{x}$$

$$- v_x^2w + \frac{1}{2}v^2(w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x). \quad (3.4)$$

Now we choose the weight function $w$ and the corresponding $w_1$ in terms of our degenerate stationary wave $\phi$ by (2.8), where $\sigma \in \mathbb{R}$. Then we have $w = (-\phi)^{-\sigma+1}/f(\phi)$ and $w_1 = (-\phi)^{-\sigma-1}f(\phi)$ by $\phi_x = f(\phi)$. After straightforward computations, we find that

$$w_{xx} + w_xf'(\phi) - wf''(\phi)\phi_x = 2(c_1(\sigma) - r(\phi))w_1, \quad (3.5)$$

where

$$c_1(\sigma) := \sigma(\sigma - 1)/2 - q(q + 1),$$

$$r(u) := (-u)^2f''(u)/f(u) - q(q + 1). \quad (3.6)$$

Substituting (3.5) into (3.4) and integrating with respect to $x$, we get the following conclusion.
Claim 3.1. Let $\phi$ be the degenerate stationary wave and define the weight functions $w$ and $w_1$ by (2.8) with $\sigma \in \mathbb{R}$. Then the operator $L$ defined in (1.5) verifies

$$
\langle Lv, v \rangle_{L^2(w)} = -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)} - \int_0^\infty v^2 r(\phi) w_1 \, dx \tag{3.7}
$$

for $v \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ and hence for $v \in H_0^1(w)$, where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6).

The term $r(\phi)$ in (3.7) can be regarded as a small perturbation. In fact, a straightforward computation gives

$$
r(u) = (-u)((-u)g''(u) - 2(q+1)g'(u))/(1 + g(u)) \tag{3.8}
$$

which shows that $r(u) = O(|u|)$ for $u \to 0$. In particular, we have $r(u) \equiv 0$ if $g(u) \equiv 0$. With these preparations, we have the following result on the characterization of the dissipativity of $L$.

Theorem 3.2. Assume (1.2). Let $\phi$ be the degenerate stationary wave and $L$ be the operator defined in (1.5). Let $w$ and $w_1$ be the weight functions in (2.8) with the parameter $\sigma \in \mathbb{R}$. Then we have:

1. Let $-2q < \sigma < 2(q+1)$. Then, under the additional assumption that $r(u) \geq 0$ for $-1 \leq u \leq 0$, the operator $L$ is uniformly dissipative in $L^2(w)$. Namely, there is a positive constant $\delta$ such that

$$
\langle Lv, v \rangle_{L^2(w)} \leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \quad \text{for } v \in H_0^1(w). \tag{3.9}
$$

2. Let $\sigma > 2(q+1)$ or $\sigma < -2q$. Then the operator $L$ can not be dissipative in $L^2(w)$. Namely, we have $\langle Lv, v \rangle_{L^2(w)} > 0$ for some $v \in H_0^1(w)$ with $v \neq 0$.

Proof. The proof is based on the equality (3.7) in Claim 3.1 and the Hardy type inequality (2.9) in Proposition 2.3.

Let $-2q < \sigma < 2(q+1)$. This is equivalent to $c_1(\sigma) < \sigma^2/4$. Therefore we can choose $\delta > 0$ so small that $\delta(1 + \sigma^2/4) \leq \sigma^2/4 - c_1(\sigma)$. Since $r(\phi) \geq 0$ by the additional assumption on $r(u)$ and since $(\sigma^2/4)\|v\|^2_{L^2(w_1)} \leq \|v_x\|^2_{L^2(w)}$ by the Hardy type inequality (2.9), we have from (3.7) that

$$
\langle Lv, v \rangle_{L^2(w)} \leq -\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}
= -\delta\|v_x\|^2_{L^2(w)} - (1 - \delta)\|v_x\|^2_{L^2(w)} + c_1(\sigma)\|v\|^2_{L^2(w_1)}
\leq -\delta\|v_x\|^2_{L^2(w)} - \{(1 - \delta)\sigma^2/4 - c_1(\sigma)\}\|v\|^2_{L^2(w_1)}
\leq -\delta(\|v_x\|^2_{L^2(w)} + \|v\|^2_{L^2(w_1)}) \tag{3.10}
$$
for \( v \in C_{0}^{\infty}(\mathbb{R}_{+}) \) and hence for \( v \in H_{0}^{1}(w) \), where we have used the fact that 
\((1 - \delta)\sigma^{2}/4 - c_{1}(\sigma) \geq \delta \). This completes the proof of the uniform dissipative case \((1)\).

Next we consider the case where \( \sigma > 2(q + 1) \); the case \( \sigma < -2q \) can be treated similarly and we omit the argument in this latter case. When \( \sigma > 2(q + 1) \), we have \( c_{1}(\sigma) > \sigma^{2}/4 \). Then we choose \( \delta > 0 \) so small that \( c_{1}(\sigma) \geq \sigma^{2}/4 + 3\delta \). Since \( r(u) = O(|u|) \) for \( u \to 0 \) and \( \phi(x) \to 0 \) for \( x \to \infty \), we take \( a = a(\delta) > 0 \) so large that \( |r(\phi)| \leq \delta \) for \( x \geq a \). For this choice of \( a \) and for \( \epsilon > 0 \), we take a test function \( v^{\epsilon} \) as in \((2.4)\):

\[
 v^{\epsilon}(x) = \begin{cases} 
 0, & 0 \leq x < a, \\
 (x - a)(-\phi(x))^{(1/2+\epsilon)}, & a < x < a + 1, \\
 (-\phi(x))^{(1/2+\epsilon)}, & a + 1 < x. 
\end{cases} \tag{3.11}
\]

Then we have

\[
 \left| \int_{0}^{\infty} (v^{\epsilon})^{2} r(\phi) w_{1} \, dx \right| \leq \delta \int_{a}^{\infty} (v^{\epsilon})^{2} w_{1} \, dx = \delta \| v^{\epsilon} \|_{L^{2}(w_{1})}^{2},
\]

so that we have from \((3.7)\) that

\[
 \langle Lv^{\epsilon}, v^{\epsilon} \rangle_{L^{2}(w)} \geq -\| v_{x}^{\epsilon} \|_{L^{2}(w)}^{2} + (c_{1}(\sigma) - \delta) \| v^{\epsilon} \|_{L^{2}(w_{1})}^{2}. \tag{3.12}
\]

Also, by straightforward computations, we find that

\[
 \frac{\| v_{x}^{\epsilon} \|_{L^{2}(w)}^{2}}{\| v^{\epsilon} \|_{L^{2}(w_{1})}^{2}} = \frac{O(1) + \sigma^{2}(1/2 + \epsilon)^{2} \frac{1}{2\epsilon} (-\phi(a + 1))^{2\epsilon}}{O(1) + \frac{1}{2\epsilon} (-\phi(a + 1))^{2\epsilon}} \to \frac{\sigma^{2}}{4}
\]

for \( \epsilon \to 0 \). Thus we have \( \| v_{x}^{\epsilon} \|_{L^{2}(w)}^{2}/\| v^{\epsilon} \|_{L^{2}(w_{1})}^{2} \leq \sigma^{2}/4 + \delta \) for a suitably small \( \epsilon = \epsilon(\delta) > 0 \). Consequently, we have from \((3.12)\) that

\[
 \frac{\langle Lv^{\epsilon}, v^{\epsilon} \rangle_{L^{2}(w)}}{\| v^{\epsilon} \|_{L^{2}(w_{1})}^{2}} \geq \frac{\| v_{x}^{\epsilon} \|_{L^{2}(w)}^{2}}{\| v^{\epsilon} \|_{L^{2}(w_{1})}^{2}} + c_{1}(\sigma) - \delta \geq -\left( \frac{\sigma^{2}}{4} + \delta \right) + c_{1}(\sigma) - \delta \geq \delta.
\]

This completes the proof of the non-dissipative case \((2)\). Thus the proof of Theorem 3.2 is complete. \( \square \)
In the special case where $g(u) \equiv 0$ so that $f(u) = \frac{1}{q}(-u)^{q+1}$, we have $\phi = -(1 + x)^{-1/q}$ and the operator $L$ in (1.5) is reduced to

$$L_0v = v_{xx} + \frac{q + 1}{q} \left( \frac{v}{1 + x} \right)_x.$$  \hspace{1cm} (3.13)

In this simplest case, we have the complete characterization of the dissipativity of the operator $L_0$.

**Theorem 3.3.** Let $\alpha_c(q) := 3 + 2/q$. Then we have the complete characterization of the dissipativity of the operator $L_0$ given in (3.13):

1. Let $-1 < \alpha < \alpha_c(q)$. Then $L_0$ is uniformly dissipative in $L^2_\alpha$. Namely, there is a positive constant $\delta$ such that
   $$\langle L_0v, v \rangle_{L^2_{\alpha}} \leq -\delta(\|v_x\|_{L^2_{\alpha}}^2 + \|v\|_{L^2_{\alpha-2}}^2)$$ for $v \in H^1_{\alpha,0}$. \hspace{1cm} (3.14)

2. Let $\alpha = \alpha_c(q)$ or $\alpha = -1$. Then $L_0$ is strictly dissipative in $L^2_\alpha$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} < 0$ for $v \in H^1_{\alpha,0}$ with $v \neq 0$.

3. Let $\alpha > \alpha_c(q)$ or $\alpha < -1$. Then $L_0$ cannot be dissipative in $L^2_\alpha$. Namely, we have $\langle L_0v, v \rangle_{L^2_{\alpha}} > 0$ for some $v \in H^1_{\alpha,0}$ with $v \neq 0$.

**Proof.** In this case, we have $\phi = -(1 + x)^{-1/q}$, $L = L_0$ and $r(u) \equiv 0$. Therefore, (3.7) is reduced to

$$\langle L_0v, v \rangle_{L^2_{\alpha}(w)} = -\|v_x\|_{L^2_{\alpha}(w)}^2 + c_1(\sigma)\|v\|_{L^2_{\alpha}(w_1)}^2,$$ \hspace{1cm} (3.15)

where $w$ and $w_1$ are the weight functions defined in (2.8) with $\phi = -(1 + x)^{-1/q}$ and $\sigma = (\alpha - 1)q$. The desired conclusions easily follow from (3.15) by applying the same argument as in Theorem 3.2. We omit the details. \hspace{1cm} $\square$

\section{Nonlinear stability}

The following stability result for the nonlinear problem (1.4) was obtained in [2] as a refinement of the result in [14].

**Theorem 4.1.** Assume (1.2). Suppose that $v_0 \in L^2_\alpha \cap L^\infty$ for some $\alpha$ with $1 \leq \alpha < \alpha_c(q) := 3 + q/2$. Then there is a positive constant $\delta_1$ such that if $\|v_0\|_{L^2_\alpha} \leq \delta_1$, then the problem (1.4) has a unique global solution $v \in C^0([0, \infty); L^2_\alpha \cap L^p)$ for each $p$ with $2 \leq p < \infty$. Moreover, the solution verifies the decay estimate

$$\|v(t)\|_{L^p} \leq C(\|v_0\|_{L^2_\alpha} + \|v_0\|_{L^\infty})(1 + t)^{-\alpha/4 - \nu}$$ \hspace{1cm} (4.1)

for $t \geq 0$, where $2 \leq p < \infty$, $\nu = (1/2)(1/2 - 1/p)$, and $C$ is a positive constant.
Proof. A key to the proof of this theorem is to show the following space-time weighted energy inequality:

\[
(1 + t)^\gamma \|v(t)\|_{L_\beta}^2 + \int_0^t (1 + \tau)^\gamma (\|v_x(\tau)\|_{L_\beta}^2 + \|v(\tau)\|_{L_{\beta-2}}^2) d\tau \leq C\|v_0\|_{L_\beta}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \|v(\tau)\|_{L_\beta}^2 d\tau + CS_\beta^\gamma(t) \tag{4.2}
\]

for any \(\gamma \geq 0\) and \(\beta\) with \(0 \leq \beta \leq \alpha\), where \(1 \leq \alpha < \alpha_c(q) := 3 + 2/q\), \(C\) is a constant independent of \(\gamma\) and \(\beta\), and

\[
S_\beta^\gamma(t) = \int_0^t (1 + \tau)^\gamma \|v(\tau)\|_{L_{\beta-1}}^3 d\tau. \tag{4.3}
\]

Here we give an outline of the proof of (4.2) and omit the other discussions. We refer to [2, 14] for the complete proof of Theorem 4.1.

**Proof of (4.2) for \(\beta = 0\).** The proof is based on the time weighted \(L^2\) energy method. First we note that

\[
\|v(t)\|_{L^\infty} \leq M_\infty, \tag{4.4}
\]

where \(M_\infty = \|v_0\|_{L^\infty} + 2\). This is an easy consequence of the maximum principle (see [5] for the details). Now we multiply the equation (1.4) by \(v\). This yields

\[
\left(\frac{1}{2}v^2\right)_t + (F - vv_x)_x + v_x^2 + G = 0, \tag{4.5}
\]

where

\[
F = \left(f(\phi + v) - f(\phi)\right)v - \int_0^v (f(\phi + \eta) - f(\phi))d\eta,
\]

\[
G = \int_0^v (f'(\phi + \eta) - f'(\phi))d\eta \cdot \phi_x.
\]

We note that

\[
F = \frac{1}{2}f'(\phi)v^2 + O(|v|^3), \quad G = \frac{1}{2}f''(\phi)\phi_xv^2 + \phi_xO(|v|^3) \tag{4.7}
\]

for \(v \to 0\). Here, a careful computation, using (3.2) and (4.4), shows that

\[
G \geq c(1 + x)^{-2}v^2 - C(1 + x)^{-1-1/q}|v|^3 \tag{4.8}
\]

for any \(x \in \mathbb{R}_+\). We integrate (4.5) over \(\mathbb{R}_+\) and substitute (4.8) into the resulting equality, obtaining

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L_2}^2 + \|v_x\|_{L_2}^2 + c\|v\|_{L_{\beta-2}}^2 \leq C\|v\|_{L_{\beta-1}}^3.
\]
We multiply this inequality by $(1 + t)\gamma$ and integrate with respect $t$. This yields the desired inequality (4.2) for $\beta = 0$.

**Proof of (4.2) for $\beta > 0$.** We apply the space-time weighted energy method employed in [14, 2] (see also [3]). Let $w > 0$ be a smooth weight function depending only on $x$, which will be specified later. We multiply (4.5) by $w$, obtaining

$$\left(\frac{1}{2}v^2w\right)_t + \left\{(F - \mu vv_x)w + \frac{1}{2}v^2w_x\right\}_x + v_x^2w - \left(\frac{1}{2}v^2w_{xx} + Fw_x - Gw\right) = 0. \quad (4.9)$$

Here, using (4.7), we have

$$\frac{1}{2}v^2w_{xx} + Fw_x - Gw = \frac{1}{2}v^2(w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x) + R, \quad (4.10)$$

where $R = w_x O(|v|^3) - w\phi_x O(|v|^3)$ for $v \to 0$. Notice that the coefficient $w_{xx} + w_x f'(\phi) - wf''(\phi)\phi_x$ in (4.10) is just the same as that appeared in (3.4).

Now we choose the weight function $w$ and the corresponding $w_1$ by (2.8) with $\sigma = (\beta - 1)q$, where $0 \leq \beta \leq \alpha$ and $1 \leq \alpha < \alpha_c(q) := 3 + 2/q$. Then we have (3.5) with $\sigma = (\beta - 1)q$. Substituting these expressions into (4.9) and integrating over $\mathbb{R}_+$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|v\|_{L^2(w)}^2 + \|v_x\|_{L^2(w)}^2 - c_1(\sigma)\|v\|_{L^2(w_1)}^2 + \int_{0}^{\infty}v^2r(\phi)w_1 dx = \int_{0}^{\infty}R dx, \quad (4.11)$$

where $c_1(\sigma)$ and $r(\phi)$ are given in (3.6) with $\sigma = (\beta - 1)q$. Here our weight functions verify

$$w \sim (1 + x)^\beta, \quad w_1 \sim (1 + x)^{\beta-2}, \quad (4.12)$$

where the symbol $\sim$ means the equivalence. This implies that the norms $\|\cdot\|_{L^2(w)}$ and $\|\cdot\|_{L^2(w_1)}$ are equivalent to $\|\cdot\|_{L_\beta^2}$ and $\|\cdot\|_{L_{\beta-2}^2}$, respectively.

We estimate (4.11) similarly as in (1) of Theorem 3.2. To this end, we note that $\sigma_1 \leq \sigma \leq \sigma_2$, where $\sigma_1 = -q$ and $\sigma_2 = (\alpha - 1)q$. Since $c_1(\sigma) < \sigma^2/4$ for $-2q < \sigma < 2(q + 1)$ and since $-2q < \sigma_1 < \sigma_2 < 2(q + 1)$, we can choose $\delta > 0$ so small that

$$\delta \leq \min_{\sigma_1 \leq \sigma \leq \sigma_2} \frac{\sigma^2/4 - c_1(\sigma)}{2 + \sigma^2/4}.$$

Notice that this $\delta$ is independent of $\beta$. For this choice of $\delta$, we take $a = a(\delta) > 0$ so large that $|r(\phi)| \leq \delta$ for $x \geq a$. Then we have

$$\left|\int_{0}^{\infty}v^2r(\phi)w_1 dx\right| \leq \delta\|v\|_{L^2(w_1)}^2 + C\|v\|_{L_{\beta-2}^2}^2.$$
where \( C \) is a constant satisfying \( C \geq (1 + x)^2 |r(\phi)| w_1 \) for \( 0 \leq x \leq a \). Also, using the Hardy type inequality \((\sigma^2/4) \| v \|_{L^2(w_1)}^2 \leq \| v_x \|_{L^2(w)}^2\) in (2.9) and estimating similarly as in (3.10), we have
\[
\| v_x \|_{L^2(w)}^2 - c_1(\sigma) \| v \|_{L^2(w_1)}^2 \geq \delta \| v_x \|_{L^2(w)}^2 + 2\delta \| v \|_{L^2(w_1)}^2,
\]
where we have used the fact that \((1 - \delta)\sigma^2/4 - c_1(\sigma) \geq 2\delta\). On the other hand, using (4.4), we see that \(| R | \leq C (| w_x | + w\phi_x) | v |^3\). Moreover, a straightforward computation shows that \(| w_x | + w\phi_x \leq C (1 + x)^{\beta-1}\). Substituting all these estimates into (4.11), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| v \|_{L^2(w)}^2 + \delta (\| v_x \|_{L^2(w)}^2 + \| v \|_{L^2(w_1)}^2) \leq C \| v \|_{L^2_{\beta}}^2 + C \| v \|_{L^3_{\beta-1}}^3,
\]
where \( \delta \) and \( C \) are independent of \( \beta \). We multiply this inequality by \((1 + t)^\gamma\) and integrate with respect to \( t \). By virtue of (4.12), we have
\[
(1 + t)^\gamma \| v(t) \|_{L^2_{\beta}}^2 + \int_0^t (1 + \tau)^\gamma (\| v_x(\tau) \|_{L^2_{\beta}}^2 + \| v(\tau) \|_{L^2_{\beta-2}}^2) d\tau
\leq C \| v_0 \|_{L^2_{\beta}}^2 + \gamma C \int_0^t (1 + \tau)^{\gamma-1} \| v(\tau) \|_{L^2_{\beta}}^2 d\tau
\]
\[
+ C \int_0^t (1 + \tau)^\gamma \| v(\tau) \|_{L^2_{\beta-2}}^2 d\tau + CS_\beta^\gamma(t),
\]
where the constant \( C \) is independent of \( \gamma \) and \( \beta \). Here the third term on the right hand side of (4.14) was already estimated by (4.2) with \( \beta = 0 \). Hence we have proved (4.2) also for \( 0 < \beta \leq \alpha \). This completes the proof. \( \square \)

References


