Upper triangular operators, SVEP and Browder, Weyl theorems

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Abstract

We show the important role played by SVEP, the single-valued extension property, and the polaroid property in relating the spectrum, and certain distinguished parts thereof, of the operators $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $M_0$ for some Banach space operators $A, B$ and $C \in B(\mathcal{X})$.

1. Results

Let $B(\mathcal{X})$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space $\mathcal{X}$. For $A, B, C \in B(\mathcal{X})$, let $M_C$ denote the upper triangular operator $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and let $M_0 = A \oplus B$. The spectrum, and certain distinguished parts thereof, of the operators $M_C$ and $M_0$ has been studied by a number of authors in the recent past; see references. Of particular interest to us is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Most of our notation is standard. For an operator $T \in B(\mathcal{X})$:

\begin{align*}
\sigma(T) &= \text{spectrum of } T, \\
\sigma_a(T) &= \text{approximate point spectrum of } T, \\
\Phi_+(\mathcal{X}) &= \{ T \in B(\mathcal{X}) : T \text{ is upper semi-Fredholm} \}, \text{ and} \\
\Phi_-(\mathcal{X}) &= \{ T \in B(\mathcal{X}) : T \text{ is lower semi-Fredholm} \}.
\end{align*}

The Browder, the Weyl, the upper semi-Fredholm, the lower semi-Fredholm spectrum of $T$ are the sets

\begin{align*}
\sigma_b(T) &= \{ \lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{asc}(T - \lambda) \neq \text{dsc}(T - \lambda) \}, \\
\sigma_w(T) &= \{ \lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{ind}(T - \lambda) \not\leq 0 \}, \\
\sigma_{SF_+}(T) &= \{ \lambda \in \sigma(T) : T - \lambda \not\in \Phi_+(\mathcal{X}) \}, \text{ and} \\
\sigma_{SF_-}(T) &= \{ \lambda \in \mathbb{C} : T - \lambda \not\in \Phi_-(\mathcal{X}) \},
\end{align*}

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respectively. The Browder essential approximate point spectrum $\sigma_{ab}(T)$, and the Weyl essential approximate point spectrum $\sigma_{aw}(T)$, of $T$ are the sets

$$\sigma_{ab}(T) = \{ \lambda \in \sigma_{a}(T) : T - \lambda \notin \Phi_{+}(\mathcal{X}) \text{ or } \text{asc}(T - \lambda) = \infty \}, \text{ and}$$

$$\sigma_{aw}(T) = \{ \lambda \in \sigma_{a}(T) : T - \lambda \notin \Phi_{+}(\mathcal{X}) \text{ or } \text{ind}(T - \lambda) \leq 0 \}.$$ 

Let

$$\Xi(T) = \{ \lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda \},$$

$$\Xi_{+}(T) = \{ T - \lambda \in \Phi_{+}(\mathcal{X}) : \lambda \in \Xi(T) \}, \text{ and}$$

$$\Xi_{+}^{*}(T) = \{ T - \lambda \in \Phi_{+}(\mathcal{X}) : \lambda \in \Xi(T^{*}) \}.$$ 

The following inclusions/equalities are either well known or are easily proved:

$$\sigma(M_{C}) \subseteq \sigma(A) \cup \sigma(B) = \sigma(M_{0}) = \sigma(M_{C}) \cup \{ \Xi(A^{*}) \cup \Xi(B) \},$$

$$\sigma_{b}(M_{C}) \subseteq \sigma_{b}(A) \cup \sigma_{b}(B) = \sigma_{b}(M_{0}), \text{ and}$$

$$\sigma_{w}(M_{C}) \subseteq \sigma_{w}(M_{0}) \subseteq \sigma_{w}(A) \cup \sigma_{w}(B).$$

Furthermore, if we let $P = A$ and $Q = B$ or $P = A^{*}$ and $Q = B^{*}$, then:

$$\sigma_{b}(M_{0}) = \sigma_{b}(A) \cup \{ \Xi(A^{*}) \cup \Xi(B) \}, \text{ and}$$

$$\sigma_{w}(A) \cup \sigma_{w}(B) \subseteq \sigma_{w}(M_{C}) \cup \{ \Xi(P) \cup \Xi(Q) \}. $$

Consequently, if $\Xi(P) \cup \Xi(Q) = \emptyset$, then

$$\sigma_{b}(M_{0}) = \sigma_{w}(M_{0}) = \sigma_{b}(M_{C}) = \sigma_{w}(M_{C}) = \sigma_{w}(A) \cup \sigma_{w}(B).$$

For the spectra $\sigma_{ab}$ and $\sigma_{aw}$, one has:

$$\sigma_{ab}(M_{C}) \subseteq \sigma_{ab}(M_{0}) \subseteq \sigma_{ab}(M_{C}) \cup \{ \Xi_{+}(A) \cup \Xi_{+}(B) \}, \text{ and}$$

$$\sigma_{aw}(M_{0}) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{aw}(A) \cup \sigma_{ab}(B)$$

$$\sigma_{ab}(M_{0}) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \cup \{ \Xi_{+}(A) \cup \Xi_{+}(B) \}.$$ 

Recall that a necessary and sufficient condition for $M_{0}$ to satisfy Browder’s theorem (or, $Bt$), $\text{acc}_{w}(M_{0}) \subseteq \sigma_{w}(M_{0}) \iff \sigma_{w}(M_{0}) = \sigma_{b}(M_{0})$ (resp., $a$-Browder’s theorem (or, $a - Bt$), $\text{acc}_{a}(M_{0}) \subseteq \sigma_{aw}(M_{0}) \iff \sigma_{aw}(M_{0}) = \sigma_{ab}(M_{0})$), is that $A$ and $B$ have SVEP at points $\lambda \in \sigma(M_{0})$ such that $A - \lambda, B - \lambda$ are Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) = 0$ (resp., $A$ and $B$ have SVEP at points $\lambda \in \sigma(A_{0})$ such that $A - \lambda, B - \lambda$ are upper semi-Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0$). Similarly, a necessary and sufficient condition for $M_{C}$ to satisfy $Bt$, $\text{acc}_{w}(M_{C}) \subseteq \sigma_{aw}(M_{C}) \iff \sigma_{w}(M_{C}) = \sigma_{b}(M_{C})$ (resp., $a - Bt$, $\text{acc}_{a}(M_{0}) \subseteq \sigma_{aw}(M_{0}) \iff \sigma_{ab}(M_{0}) = \sigma_{aw}(M_{C})$), is that $M_{C}$ has SVEP at points $\lambda \in \sigma(M_{0}) \setminus \sigma_{aw}(M_{C})$ (resp., at points $\lambda \in \sigma_{a}(M_{C}) \setminus \sigma_{aw}(M_{C})$). It is easily verified that if $M_{0}$ satisfies $Bt$ then $\sigma_{w}(M_{0}) = \sigma_{w}(A) \cup \sigma_{w}(B)$, and that if $M_{0}$ satisfies $a - Bt$ then $\sigma_{aw}(M_{0}) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. A similar result, however, fails for the operator $M_{C}$, as follows from a consideration of the operator

$$\left( \begin{array}{cc} U & 1 - UU^{*} \\ 0 & U^{*} \end{array} \right),$$

where $U$ is the forward unilateral shift on a Hilbert space. Evidently, the complement of $\sigma_{w}(M_{C})$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_{w}(M_{0})$ in $\mathbb{C}$ with $\sigma_{w}(M_{0}) \setminus \sigma_{w}(M_{C})$, and the complement of $\sigma_{aw}(M_{C})$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_{aw}(M_{0})$ in $\mathbb{C}$ with $\sigma_{aw}(M_{0}) \setminus \sigma_{aw}(M_{C})$; thus, if $M_{0}$ satisfies $Bt$ (resp., $a - Bt$) and $M_{C}$ has SVEP on $\sigma_{w}(M_{0}) \setminus \sigma_{w}(M_{C})$ (resp., $\sigma_{aw}(M_{0}) \setminus \sigma_{aw}(M_{C})$), then $M_{C}$ satisfies $Bt$ (resp., $a - Bt$). This may be achieved in a number of ways.
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Theorem 1.1 (a) If either (i) \( A \) has SVEP at points \( \lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A) \) and \( B \) has SVEP at points \( \mu \in \sigma_w(M_0) \setminus \sigma_{SF-}(B) \), or (ii) both \( A \) and \( A^* \) have SVEP at points \( \lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A) \), or (iii) \( A^* \) has SVEP at points \( \lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A) \) and \( B^* \) has SVEP at points \( \mu \in \sigma_w(M_0) \setminus \sigma_{SF-}(B) \), then \( M_0 \) satisfies \( Bt \) implies \( M_C \) satisfies \( Bt \).

(b) If (i) \( A \) has SVEP on \( \lambda \in \sigma_{aw}(M_0) \setminus \sigma_{SF-}(A) \) and \( A^* \) has SVEP on \( \mu \in \sigma_w(M_0) \setminus \sigma_{SF-}(A) \), or (ii) \( A^* \) has SVEP on \( \lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A) \) and \( B^* \) has SVEP on \( \mu \in \sigma_w(M_0) \setminus \sigma_{SF-}(B) \), then \( M_0 \) satisfies \( Bt \) implies \( M_C \) satisfies \( Bt \).

As a consequence one has:

Corollary 1.2 (a) [4, Proposition 4.1] If \( \Xi(A) \cap \Xi(B^*) \cup \Xi(A^*) = \emptyset \), then \( M_0 \) satisfies \( Bt \) (resp., \( a-Bt \)) implies \( M_C \) satisfies \( Bt \) (resp., \( a-Bt \)).

(b) [2, Theorem 3.2] If either \( \sigma_{aw}(A) = \sigma_{SF-}(B) \) or \( \sigma_{SF-}(A) \cap \sigma_{SF-}(B) = \emptyset \), then \( M_0 \) satisfies \( Bt \) (resp., \( a-Bt \)) implies \( M_C \) satisfies \( Bt \) (resp., \( a-Bt \)).

Both Theorem 1.1 and Corollary 1.2 are a particular case of the following theorem.

Theorem 1.3 (i) If \( \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \), then \( M_0 \) satisfies \( Bt \) if and only if \( M_C \) satisfies \( Bt \).

(ii) If \( \sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \), then \( M_0 \) satisfies \( a-Bt \) implies \( M_C \) satisfies \( a-Bt \).

If in addition either \( A^* \) or \( B \) has SVEP on \( \sigma_a(M_C) \setminus \sigma_{aw}(M_C) \), then \( M_C \) satisfies \( a-Bt \) if and only if \( M_0 \) satisfies \( a-Bt \).

Here, we observe that if \( \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \), then \( \sigma_w(M_0) = \sigma_w(M_C) \) and \( \sigma(M_C) = \sigma(M_0) \); the hypothesis \( \sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B) \) implies that \( \sigma_{aw}(M_C) = \sigma_{aw}(M_0) \) (and \( \sigma(M_C) = \sigma(M_0) \)). Observe also that if \( B \) has SVEP on \( \sigma_a(M_C) \setminus \sigma_{aw}(M_C) \) and \( M_C \) satisfies \( a-Bt \), then \( A \) and \( B \) have SVEP on \( \sigma_a(M_C) \setminus \sigma_{aw}(M_C) \); if \( A^* \) has SVEP on \( \sigma_a(M_C) \setminus \sigma_{aw}(M_C) \), then \( \sigma_a(M_C) = \sigma_a(M_0) \).

Let \( \pi_0(M_C) = \{ \lambda \in \sigma_{aw}(M_C) : 0 < \dim \ker(M_C - \lambda)^{-1}(0) < \infty \} \), and let \( \pi^0_0(M_C) = \{ \lambda \in \sigma_{aw}(M_C) : 0 < \dim \ker(M_C - \lambda)^{-1}(0) < \infty \} \). An operator \( T \) is said to be polaroid (resp., \( a \)-polaroid) at a points \( \lambda \in \sigma_{aw}(T) \) (resp., \( \lambda \in \sigma_w(T) \)) if \( \lambda \) is a pole of the resolvent of \( T \) (resp., \( T - \lambda \)). \( \mathcal{X} \) is closed and \( \text{asc}(T - \lambda) < \infty \). Let \( \mathcal{R}(T) = \{ \lambda \in \sigma_{aw}(T) : \lambda \in \Phi_+(\mathcal{X}), \text{asc}(T - \lambda) < \infty \} \). In common with current terminology, we say that \( T \) satisfies Weyl's theorem, or \( Wt \) (resp., \( a \)-Weyl's theorem, or \( a-Wt \)) if \( \sigma(T) \setminus \sigma_{aw}(T) = \pi_0(T) \) (resp., \( \sigma_a(T) \setminus \sigma_{aw}(T) = \pi_0^a(T) \)).

The following result is not difficult to prove:

Theorem 1.4 (a) \( M_C \) satisfies \( Wt \) if and only if \( M_C \) has SVEP at \( \lambda \notin \sigma_w(M_C) \) and \( M_C \) is polaroid at points \( \mu \in \pi_0(M_C) \).

(b) \( M_C \) satisfies \( a-Wt \) if and only if \( M_C \) has SVEP at \( \lambda \notin \sigma_{aw}(M_C) \) and \( M_C \) is \( a \)-polaroid at points \( \mu \in \pi_0^a(M_C) \).

Combining Theorems 1.1 and 1.4, an additional well known argument, see [5] and [6], implies the following:

Theorem 1.5 [5, Theorem 3.7] If either of the SVEP hypotheses (i), (ii) and (iii) of Theorem 1.1(a) is satisfied, then \( M_C \) satisfies \( Wt \) for every \( C \in B(\mathcal{X}) \) if and only if \( M_0 \) satisfies \( Wt \) and \( A \) is polaroid at \( \lambda \in \pi_0(M_C) \).

Theorem 1.5 implies the following:

Corollary 1.6 (a) [4, Theorem 4.2] If \( \Xi(A) \cap \Xi(B^*) \cup \Xi(A^*) = \emptyset \), \( A \) is polaroid at \( \lambda \in \pi_0(M_C) \) (or \( A \) is isoloid and satisfies \( Wt \)) and \( M_0 \) satisfies \( Wt \), then \( M_C \) satisfies \( Wt \).

(b) [2, Theorem 3.3] If \( \sigma_{aw}(A) = \sigma_{SF-}(B) \) or \( \sigma_{SF-}(A) \cap \sigma_{SF-}(B) = \emptyset \), \( A \) is polaroid at \( \lambda \in \pi_0(M_C) \) (or \( A \) is isoloid and satisfies \( Wt \)) and \( M_0 \) satisfies \( Wt \), then \( M_C \) satisfies \( Wt \).
Both Theorem 1.5 and Corollary 1.6 are subsumed by the following general result.

Theorem 1.7 If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $M_0$ satisfies $Wt$ if and only if $\mathcal{R}_0(M_0) = \pi_0(M_C)$.

The proof of the theorem is a straightforward consequence of the facts that $Wt \implies Bt$, and $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \implies \sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$.

The result corresponding to Theorem 1.7 for $a - Wt$ is the following:

Theorem 1.8 (i). If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, then $M_0$ satisfies $a - Wt$ if and only if $\pi_0^a(M_C) \subseteq \pi_0^a(M_0)$.

(ii). Conversely, if $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and $A^*$ has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then $M_C$ satisfies $a - Wt$ implies $M_0$ satisfies $a - Wt$ if and only if $\pi_0^a(M_0) \subseteq \pi_0^a(M_C)$.

Theorem 1.8 implies in particular that:

Corollary 1.9 [5, Theorem 3.11] If $\Xi(A^*) \cup \Xi(B^*) = \emptyset$, $A$ is polaroid at $\lambda \in \pi_0^a(M_C)$ (or, $A$ is isoloid and satisfies $Wt$) and $B$ is polaroid at $\mu \in \pi_0^a(B)$, then $M_C$ satisfies $a - Wt$.

We note here that the hypothesis $\Xi(A^*) \cup \Xi(B^*) = \emptyset$ implies that $a$-poles of $A$ and $B$ are indeed poles of their respective resolvents.

It is well known that if a Banach space operator $T$ is such that $T^*$ has SVEP, then $T$ satisfies $Wt$ if and only if $T^*$ satisfies $Wt$. Observe that a sufficient condition for $M_0$ and $M_C$ to have SVEP is that both $A^*$ and $B^*$ have SVEP. More generally:

Theorem 1.10 Let $M_X = M_0$ or $M_C$. If $A^*$ has SVEP on $\sigma(A) \setminus \sigma_{SF_+}(A)$ and $B^*$ has SVEP on $\sigma(B) \setminus \sigma_{SF_+}(B)$, then $M_X$ satisfies $Wt$ if and only if $M_X$ satisfies $a - Wt$.

Although $T$ has SVEP and satisfies $Wt$ does not guarantee $T^*$ satisfies $a - Wt$, we do have that if $T$ has SVEP and is polaroid, then $T$ satisfies $Wt$ if and $T^*$ satisfies $a - Wt$. This leads us to: if $A$ and $B$ have SVEP and are polaroid, then $M_C$ satisfies $Wt$ and $M_C^*$ satisfies $a - Wt$.

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References


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