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Upper triangular operators, SVEP and Browder, Weyl theorems

B. P. Duggal

Abstract

We show the important role played by SVEP, the single-valued extension property, and the polaroid property in relating the spectrum, and certain distinguished parts thereof, of the operators $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and $M_0$ for some Banach space operators $A, B$ and $C \in B(\mathcal{X})$.

1. Results

Let $B(\mathcal{X})$ denote the algebra of operators (equivalently, bounded linear transformations) on a Banach space $\mathcal{X}$. For $A, B, C \in B(\mathcal{X})$, let $M_C$ denote the upper triangular operator $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ and let $M_0 = A \oplus B$. The spectrum, and certain distinguished parts thereof, of the operators $M_C$ and $M_0$ has been studied by a number of authors in the recent past; see references. Of particular interest to us is the relationship between the spectral, the Fredholm, the Browder and the Weyl properties.

Most of our notation is standard. For an operator $T \in B(\mathcal{X})$:

\begin{align*}
\sigma(T) &= \text{spectrum of } T, \\
\sigma_a(T) &= \text{approximate point spectrum of } T, \\
\Phi_+(\mathcal{X}) &= \{ T \in B(\mathcal{X}) : T \text{ is upper semi-Fredholm} \}, \quad \text{and} \\
\Phi_-(\mathcal{X}) &= \{ T \in B(\mathcal{X}) : T \text{ is lower semi-Fredholm} \}.
\end{align*}

The Browder, the Weyl, the upper semi-Fredholm, the lower semi-Fredholm spectrum of $T$ are the sets

\begin{align*}
\sigma_b(T) &= \{ \lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{asc}(T - \lambda) \neq \text{dsc}(T - \lambda) \}, \\
\sigma_w(T) &= \{ \lambda \in \sigma(T) : T - \lambda \text{ is not Fredholm or } \text{ind}(T - \lambda) \not\leq 0 \}, \\
\sigma_{SF+}(T) &= \{ \lambda \in \sigma(T) : T - \lambda \not\in \Phi_+(\mathcal{X}) \}, \quad \text{and} \\
\sigma_{SF-}(T) &= \{ \lambda \in \mathbb{C} : T - \lambda \not\in \Phi_-(\mathcal{X}) \}.
\end{align*}

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respectively. The Browder essential approximate point spectrum $\sigma_{ab}(T)$, and the Weyl essential approximate point spectrum $\sigma_{aw}(T)$, of $T$ are the sets

$$\sigma_{ab}(T) = \{ \lambda \in \sigma_a(T) : T - \lambda \notin \Phi_+(\mathcal{X}) \text{ or } \text{asc}(T - \lambda) = \infty \},$$

and

$$\sigma_{aw}(T) = \{ \lambda \in \sigma_a(T) : T - \lambda \notin \Phi_+(\mathcal{X}) \text{ or } \text{ind}(T - \lambda) \leq 0 \}.$$

Let

$$\Xi(T) = \{ \lambda \in \mathbb{C} : T \text{ does not have SVEP at } \lambda \},$$

$$\Xi_+(T) = \{ T - \lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T) \},$$

and

$$\Xi_+(T^*) = \{ T - \lambda \in \Phi_+(\mathcal{X}) : \lambda \in \Xi(T^*) \}.$$

The following inclusions/equalities are either well known or are easily proved:

$$\sigma(M_C) \subseteq \sigma(A) \cup \sigma(B) = \sigma(M_0) = \sigma(M_C) \cup \{ \Xi(A^*) \cup \Xi(B) \},$$

$$\sigma_b(M_C) \subseteq \sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_0),$$

and

$$\sigma_{aw}(M_C) \subseteq \sigma_{aw}(M_0) \subseteq \sigma(A) \cup \sigma(B).$$

Furthermore, if we let $P = A$ and $Q = B$ or $P = A^*$ and $Q = B^*$, then:

$$\sigma_b(M_0) = \sigma_b(M_C) \cup \{ \Xi(A^*) \cup \Xi(B) \},$$

and

$$\sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{aw}(M_C) \cup \{ \Xi(P) \cup \Xi(Q) \}.$$

Consequently, if $\Xi(P) \cup \Xi(Q) = \emptyset$, then

$$\sigma_b(M_0) = \sigma_b(M_C) = \sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B).$$

For the spectra $\sigma_{ab}$ and $\sigma_{aw}$, one has:

$$\sigma_{ab}(M_C) \subseteq \sigma_{ab}(M_0) \subseteq \sigma_{ab}(M_C) \cup \{ \Xi_+(A) \cup \Xi_+(B) \},$$

and

$$\sigma_{aw}(M_0) \subseteq \sigma_{aw}(A) \cup \sigma_{aw}(B) \subseteq \sigma_{aw}(A) \cup \sigma_{ab}(B).$$

Recall that a necessary and sufficient condition for $M_0$ to satisfy Browder's theorem (or $Bt$), $\text{acc}_a(M_0) \subseteq \sigma_w(M_0) \iff \sigma_w(M_0) = \sigma_b(M_0)$ (resp., $a$-Browder's theorem (or $a-Bt$), $\text{acc}_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{aw}(M_0) = \sigma_{ab}(M_0)$), is that $A$ and $B$ have SVEP at points $\lambda \in \sigma(M_0)$ such that $A - \lambda, B - \lambda$ are Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) = 0$ (resp., $A$ and $B$ have SVEP at points $\lambda \in \sigma_a(M_0)$ such that $A - \lambda, B - \lambda$ are upper semi-Fredholm and $\text{ind}(A - \lambda) + \text{ind}(B - \lambda) \leq 0$). Similarly, a necessary and sufficient condition for $M_C$ to satisfy $Bt$, $\text{acc}_a(M_C) \subseteq \sigma_w(M_C) \iff \sigma_w(M_C) = \sigma_b(M_C)$ (resp., $a-Bt$, $\text{acc}_a(M_0) \subseteq \sigma_{aw}(M_0) \iff \sigma_{ab}(M_0) = \sigma_{aw}(M_C)$), is that $M_C$ has SVEP at points $\lambda \in \sigma(M_0) \setminus \sigma_{aw}(M_C)$ (resp., at points $\lambda \in \sigma_a(M_C) \setminus \sigma_{aw}(M_C)$). It is easily verified that if $M_0$ satisfies $Bt$ then $\sigma_w(M_0) = \sigma_a(M) \cup \sigma_w(B)$, and that if $M_0$ satisfies $a-Bt$ then $\sigma_{aw}(M_0) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$. A similar result, however, fails for the operator $M_C$, as follows from a consideration of the operator

$$\begin{pmatrix} U & 1 - UU^* \\ 0 & U^* \end{pmatrix},$$

where $U$ is the forward unilateral shift on a Hilbert space. Evidently, the complement of $\sigma_w(M_C)$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_w(M_0)$ in $\mathbb{C}$ with $\sigma_w(M_0) \setminus \sigma_w(M_C)$, and the complement of $\sigma_{aw}(M_C)$ in the complex plane $\mathbb{C}$ is the union of the complement of $\sigma_{aw}(M_0)$ in $\mathbb{C}$ with $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$; thus, if $M_0$ satisfies $Bt$ (resp., $a-Bt$) and $M_C$ has SVEP on $\sigma_w(M_0) \setminus \sigma_w(M_C)$ (resp., $\sigma_{aw}(M_0) \setminus \sigma_{aw}(M_C)$), then $M_C$ satisfies $Bt$ (resp., $a-Bt$). This may be achieved in a number of ways.
Upper triangular operator matrices

**Theorem 1.1** (a) If either (i) $A$ has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A)$ and $B$ has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF+}(B)$, or (ii) both $A$ and $A^*$ have SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A)$, or (iii) $A^*$ has SVEP at points $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A)$ and $B^*$ has SVEP at points $\mu \in \sigma_w(M_0) \setminus \sigma_{SF+}(B)$, then $M_0$ satisfies $Bt$ implies $M_C$ satisfies $Bt$.

(b) If (i) $A$ has SVEP on $\lambda \in \sigma_{aw}(M_0) \setminus \sigma_{SF-}(A)$ and $A^*$ has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF+}(A)$, or (ii) $A^*$ has SVEP on $\lambda \in \sigma_w(M_0) \setminus \sigma_{SF-}(A)$ and $B^*$ has SVEP on $\mu \in \sigma_w(M_0) \setminus \sigma_{SF+}(B)$, then $M_0$ satisfies $-Bt$ implies $M_C$ satisfies $-Bt$.

As a consequence one has:

**Corollary 1.2** (a) [4, Theorem 4.1] If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, then $M_0$ satisfies $Bt$ (resp., $a-Bt$) implies $M_C$ satisfies $Bt$ (resp., $a-Bt$).

(b) [2, Theorem 3.2] If either $\sigma_{aw}(A) = \sigma_{SF-}(B)$ or $\sigma_{SF-}(A) \cap \sigma_{SF+}(B) = \emptyset$, then $M_0$ satisfies $Bt$ (resp., $a-Bt$) implies $M_C$ satisfies $Bt$ (resp., $a-Bt$).

Both Theorem 1.1 and Corollary 1.2 are a particular case of the following theorem.

**Theorem 1.3** (i) If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $M_0$ satisfies $Bt$ if and only if $M_C$ satisfies $Bt$.

(ii) If $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$, then $M_0$ satisfies $a-Bt$ implies $M_C$ satisfies $a-Bt$. If in addition either $A^*$ or $B$ has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then $M_C$ satisfies $a-Bt$ if and only if $M_0$ satisfies $a-Bt$.

Here, we observe that if $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $\sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_0) = \sigma_w(M_C)$; the hypothesis $\sigma_{aw}(M_C) = \sigma_{aw}(A) \cup \sigma_{aw}(B)$ implies that $\sigma_{aw}(M_C) = \sigma_{aw}(M_0)$ and $\sigma(M_C) = \sigma(M_0)$. Observe also that if $B$ has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$ and $M_C$ satisfies $a-Bt$, then $A$ and $B$ have SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$; if $A^*$ has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then $\sigma(M_C) = \sigma_a(A) \cup \sigma_a(B)$.

Let $\pi_0(M_C) = \{\lambda \in \sigma_{aw}(M_C) : 0 < \dim \ker (M_C - \lambda)^{-1}(0) < \infty\}$, and let $\pi_0^a(M_C) = \{\lambda \in \sigma_{aw}(M_C) : 0 < \dim \ker (M_C - \lambda)^{-1}(0) < \infty\}$. An operator $T$ is said to be polaroid (resp., $a$-polaroid) at a points $\lambda \in \sigma_{aw}(T)$ (resp., $\lambda \in \sigma_{aw}(T)$) if $\lambda$ is a pole of the resolvent of $T$ (resp., $(T - \lambda)\mathcal{X}$ is closed and $\text{asc}(T - \lambda) < \infty$). Let $\mathcal{R}_0(T) = \{\lambda \in \sigma_{aw}(T) : \lambda$ is a finite rank pole of the resolvent of $T\}$ and $\mathcal{R}_0^a(T) = \{\lambda \in \sigma_{aw}(T) : T - \lambda \in \Phi_+(\mathcal{X}), \text{asc}(T - \lambda) < \infty\}$.

In common with current terminology, we say that $T$ satisfies Weyl's theorem, or $Wt$ (resp., $a$-Weyl's theorem, or $a-Wt$) if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ (resp., $\sigma_a(T) \setminus \sigma_{aw}(T) = \pi_0^a(T)$).

The following result is not difficult to prove:

**Theorem 1.4** (a). $M_C$ satisfies $Wt$ if and only if $M_C$ has SVEP at $\lambda \notin \sigma_w(M_C)$ and $M_C$ is polaroid at points $\mu \in \pi_0(M_C)$.

(b). $M_C$ satisfies $a-Wt$ if and only if $M_C$ has SVEP at $\lambda \notin \sigma_{aw}(M_C)$ and $M_C$ is $a$-polaroid at points $\mu \in \pi_0^a(M_C)$.

Combining Theorems 1.1 and 1.4, an additional well known argument, see [5] and [6], implies the following:

**Theorem 1.5** [5, Theorem 3.7] If either of the SVEP hypotheses (i), (ii) and (iii) of Theorem 1.1(a) is satisfied, then $M_C$ satisfies $Wt$ for every $C \in B(\mathcal{X})$ if and only if $M_0$ satisfies $Wt$ and $A$ is polaroid at $\lambda \in \pi_0(M_C)$.

Theorem 1.5 implies the following:

**Corollary 1.6** (a) [4, Theorem 4.2] If $\{\Xi(A) \cap \Xi(B^*)\} \cup \Xi(A^*) = \emptyset$, $A$ is polaroid at $\lambda \in \pi_0(M_C)$ (or $A$ is $a$-isoloid and satisfies $Wt$) and $M_0$ satisfies $Wt$, then $M_C$ satisfies $Wt$.

(b) [2, Theorem 3.2] If $\sigma_{aw}(A) = \sigma_{SF-}(B)$ or $\sigma_{SF-}(A) \cap \sigma_{SF+}(B) = \emptyset$, $A$ is polaroid at $\lambda \in \pi_0(M_C)$ (or $A$ is isoloid and satisfies $Wt$) and $M_0$ satisfies $Wt$, then $M_C$ satisfies $Wt$. 
Both Theorem 1.5 and Corollary 1.6 are subsumed by the following general result.

**Theorem 1.7** If $\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B)$, then $M_0$ satisfies $Wt \iff M_C$ satisfies $Wt$ if and only if $\mathcal{R}_0(M_0) = \pi_0(M_C)$.

The proof of the theorem is a straightforward consequence of the facts that $Wt \implies Bt$, and 

$\sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \implies \sigma_w(M_0) = \sigma_w(M_C)$ and $\sigma(M_C) = \sigma(M_0)$.

The result corresponding to Theorem 1.7 for $a-Wt$ is the following:

**Theorem 1.8** (i). If $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$, then $M_0$ satisfies $a-Wt$ implies $M_C$ satisfies $a-Wt$ if and only if $\pi_0^a(M_C) \subseteq \pi_0^a(M_0)$.

(ii). Conversely, if $\sigma_{aw}(M_0) = \sigma_{aw}(M_C)$ and $A^*$ has SVEP on $\sigma_a(M_C) \setminus \sigma_{aw}(M_C)$, then $M_C$ satisfies $a-Wt$ implies $M_0$ satisfies $a-Wt$ if and only if $\pi_0^a(M_0) \subseteq \pi_0^a(M_C)$.

Theorem 1.8 implies in particular that:

**Corollary 1.9** [5, Theorem 3.11] If $\Xi(A^*) \cup \Xi(B^*) = \emptyset$, $A$ is polaroid at $\lambda \in \pi_0^a(M_C)$ (or, $A$ is isoloid and satisfies $Wt$) and $B$ is polaroid at $\mu \in \pi_0^a(B)$, then $M_C$ satisfies $a-Wt$.

We note here that the hypothesis $\Xi(A^*) \cup \Xi(B^*) = \emptyset$ implies that $a$-poles of $A$ and $B$ are indeed poles of their respective resolvents.

It is well known that if a Banach space operator $T$ is such that $T^*$ has SVEP, then $T$ satisfies $Wt$ if and only if $T$ satisfies $a-Wt$. Observe that a sufficient condition for $M_0^*$ and $M_{C}^*$ to have SVEP is that both $A^*$ and $B^*$ have SVEP. More generally:

**Theorem 1.10** Let $M_X = M_0$ or $M_C$. If $A^*$ has SVEP on $\sigma(A) \setminus \sigma_{SF+}(A)$ and $B^*$ has SVEP on $\sigma(B) \setminus \sigma_{SF+}(B)$, then $M_X$ satisfies $Wt \iff M_X$ satisfies $a-Wt$.

Although $T$ has SVEP and satisfies $Wt$ does not guarantee $T^*$ satisfies $a-Wt$, we do have that if $T$ has SVEP and is polaroid, then $T$ satisfies $Wt$ and $T^*$ satisfies $a-Wt$. This leads us to: if $A$ and $B$ have SVEP and are polaroid, then $M_C$ satisfies $Wt$ and $M_C^*$ satisfies $a-Wt$.

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**References**


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