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On $L(n)$-hyponormal operators

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Abstract

We introduce a new notion of L(n)-hyponormality in order to provide a bridge between subnormality and paranormality, since 1950s, two notions of operator theorists are receiving considerable attention. And L(n)-hyponormality is offered to a criterion. Relationships to other hyponormality notions are discussed in the context of weighted shift and composition operators.

1. Introduction

This is based on the joint work with I. Jung and J. Stochel and was talked at RIMS Workshop: Application of Geometry to Operator Theory, which was held at Kyoto University on October 29-31 in 2008. Some additional results with the detailed proofs will be appeared in some other journal.

Let $\mathcal{H}$ be a separable, infinite dimension, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Recall that $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j=0}^{n} \langle T^{*j}T^{i}f_j, f_i\rangle \geq 0$ for all $f_0, f_1, \cdots, f_n \in \mathcal{H}$ and all $n \in \mathbb{N}$ ([Bra], [Hal]). To construct a bridge between subnormal and hyponormal operators on $\mathcal{H}$, R. Curto ([Cu1]) introduced a n-hyponormality as following.

Definition 1.1. An operator $T \in \mathcal{L}(\mathcal{H})$ is *n-hyponormal* if $\sum_{i,j=0}^{n} \langle T^{*j}T^{i}f_j, f_i\rangle \geq 0$ for all $f_0, f_1, \cdots, f_n \in \mathcal{H}$.

As a parellel definition on Embry's characterization for subnormal operator (cf., [Em]), the $E(n)$-hyponormality was given in [JLP] as following.

Definition 1.2. An operator T is *$E(n)$-hyponormal* if $\sum_{i,j=0}^{n} \langle T^{*i+j}T^{i+j}f_i, f_j\rangle \geq 0$ for all $f_0, f_1, \cdots, f_n \in \mathcal{H}$.

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Note that the $E(n)$-hyponormality is weaker than the n-hyponormality ([MP2]). In particular, if T is $E(1)$-hyponormal, then T is of class A operator, i.e., $|T^2| \geq |T|^2$ ([Fur]). In [JLP] they characterized the $E(n)$-hyponormality for composition operators on L^2 via Radon-Nikodym derivatives. Hence $E(n)$-hyponormality is a bridge between subnormal and class A operators. An operator $T \in \mathcal{L}(\mathcal{H})$ is *paranormal* if $||T^2h|| \geq ||Th||^2$ for an unit vector $h \in \mathcal{H}$. It is well-known that every class A operator is paranormal. In [Lam] or [Sto] it was proved that T is subnormal if and only if $\sum_{i,j=0}^{n} \langle T^{*i+j}T^{i+j}f, f\rangle \lambda_i\bar{\lambda}_j \geq 0$ for all $\lambda_i, \lambda_j \in \mathbb{C}$, $0 \leq i,j \leq n$, $f \in \mathcal{H}$, $n \in \mathbb{N}$. Hence we may give the following defintion.

Definition 1.3. An operator $T \in \mathcal{L}(\mathcal{H})$ is *$L(n)$-hyponormal* if for all $\lambda_i, \lambda_j \in \mathbb{C}$, $0 \leq i,j \leq n$, $f \in \mathcal{H}$, $\sum_{i,j=0}^{n} ||T^{i+j}f||^2\lambda_i\bar{\lambda}_j \geq 0$.

Proposition 1.4. An operator $T \in \mathcal{L}(\mathcal{H})$ is $L(1)$-hyponormal operator if and only if T is paranormal.

Proof. Since T is $L(1)$-hyponormal operator, using definition, we get

$$\begin{pmatrix} ||h||^2 & ||Th||^2 \\ ||Th||^2 & ||T^2h||^2 \end{pmatrix} \geq 0, \qquad \text{for all } h \in \mathcal{H}.$$

This is equivalent to $||T^2h|| \geq ||Th||^2$ for an unit vector $h \in \mathcal{H}$. So T is paranormal. ∎

From above Remark, the notion of $L(n)$-hyponormalities provides a bridge between subnormal and paranormal operators.

This note will be organized four sections. In Section 2 we discuss the $L(n)$-hyponormality of weighted shifts and prove that every $L(n)$-hyponormal weighted shift is n-hyponormal. In Section 3 we give a characterization for composition operator C_ϕ on L^2 corresponding to a nonsingular measurable transformation ϕ whose proof is much simpler than that of [JLP, Theorem 2.3].

2. $L(n)$-hyponormal weighted shifts

Let W_α be a weighted shift with weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers. Let $\gamma_0 = 1$, $\gamma_n := \alpha_0^2\alpha_1^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$. (The values γ_n are called sometimes *moments* because this sequence $\{\gamma_n\}$ satisfies Stieltjes moment equation when W_α is subnormal.) Then it follows from [MP2] and [Cu2] that W_α is n-hyponormal if and only if W_α is $E(n)$-hyponormal, also it is equivalent to that the $(n+1) \times (n+1)$ matrix $[\gamma_{i+j+k}]_{0 \leq i+j \leq n} \geq 0$ for all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Proposition 2.1. *Let W_α be a weighted shift with weight sequence $\alpha := \{\alpha_n\}_{n=0}^{\infty}$. Then the following assertions are equivalent:*

(i) *W_α is n-hyponormal;*

(ii) *W_α is $L(n)$-hyponormal.*

Proof. Since the implication (i) $\Rightarrow$ (ii) is obvious by definitions, we will prove only the reverse implication. To do so, we suppose that W_α is $L(n)$-hyponormal. For a standard orthonormal basis $\{e_k\}_{k=0}^{\infty}$ in l^2, $\sum_{i,j=0}^{n} \|W_\alpha^{i+j} e_k\|^2 \lambda_i \bar{\lambda}_j \geq 0$ for all $\lambda_i, \lambda_j \in \mathbb{C}$, $0 \leq i, j \leq n$, i.e.,

$$\begin{pmatrix} 1 & \alpha_k^2 & \alpha_k^2\alpha_{k+1}^2 & \cdots & \alpha_k^2\cdots\alpha_{k+n-1}^2 \\ \alpha_k^2 & \alpha_k^2\alpha_{k+1}^2 & \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 & \cdots & \alpha_k^2\cdots\alpha_{k+n}^2 \\ \alpha_k^2\alpha_{k+1}^2 & \alpha_k^2\alpha_{k+1}^2\alpha_{k+2}^2 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \\ \alpha_k^2\cdots\alpha_{k+n-1}^2 & \alpha_k^2\cdots\alpha_{k+n}^2 & \cdots & & \alpha_k^2\cdots\alpha_{k+2n}^2 \end{pmatrix} \geq 0,$$

which is equivalent to

$$\begin{pmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n} & \gamma_{k+n+1} & \cdots & \gamma_{k+2n-1} \end{pmatrix} \geq 0$$

for all $n \in \mathbb{N}$. By [Cu1], W_α is n-hyponormal. ■

Let W_α be a subnormal weighted shift with a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and let $\alpha(x) : x, \alpha_1, \alpha_2, \cdots$ with variable $x > 0$. Let

$$\mathcal{LH}(n) = \{x \in (0,\infty) : W_{\alpha(x)} \text{ is } L(n)\text{-hyponormal}\}, \qquad n \in \mathbb{N} \cup \{\infty\}.$$

Then obviously $\mathcal{LH}(\infty)$ is the set of $x \in (0,\infty)$ such that $W_{\alpha(x)}$ is subnormal operator. By Proposition 2.1, the $L(n)$-hyponormality of W_α is equivalent to the n-hyponormality. Hence we have the following example.

Example 2.2 ([JL]). Let W_α be a subnormal weighted shift whose corresponding Berger measure has infinite support. Then by [JL], $\mathcal{LH}(n)\backslash\mathcal{L}(n+1) \neq \varnothing$ for all $n \in \mathbb{N} \cup \{\infty\}$. Especially, let W_α be the Bergmann shift with a weight sequence $\alpha = \{\sqrt{(n+1)/(n+2)}\}_{n=0}^{\infty}$ and let $\alpha(x) : \sqrt{x}, \sqrt{1/2}, \sqrt{2/3}, \cdots$. Then $\mathcal{LH}(1) = (0, \sqrt{2/3}]$, $\mathcal{LH}(2) = (0, 3/4]$, $\mathcal{LH}(3) = (0, \sqrt{8/15}]$, $\mathcal{LH}(4) = (0, \sqrt{25/48}], \cdots$,etc., and $\mathcal{LH}(\infty) = (0, \sqrt{1/2}]$.

3. $L(n)$-composition operators

Let $(X, \mathcal{A}, \mu)$ be a σ finite measure space and let $\phi : X \to X$ be a measurable nonsingular transformation (i.e., $\phi^{-1}\mathcal{A} \subset \mathcal{A}$ and $\mu \circ \phi^{-1} \ll \mu$). We assume that the Radon-Nikodym derivative $h = d\mu \circ \phi^{-1}/d\mu$ is in L^∞ and we define $h_n = d\mu \circ T^{-n}/d\mu$. The composition operator C_ϕ acting on $L^2 := L^2(X, \mathcal{A}, \mu)$ is defined by $C_\phi f = f \circ \phi$. The condition $h \in L^\infty$ assures that C_ϕ is bounded.

Theorem 3.1. *Let C_ϕ be a composition operator on L^2 and suppose $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) C_ϕ *is* $E(n)$*-hyponormal;*
(ii) C_ϕ *is* $L(n)$*-hyponormal;*
(iii) $(n+1)\times(n+1)$ *matrix* $[h_{i+j}(x)]_{i,j=0}^{n}\geq 0$ *for* μ*-almost every* $x\in X$.

Proof. (i) $\Rightarrow$ (ii) It is obvious.
(ii) $\Rightarrow$ (iii) For $f\in L^2$ and for any $\lambda_0,\lambda_1,\cdots,\lambda_n\in\mathbb{C}$, we have

$$\begin{aligned}0 &\leq \sum_{i,j=0}^{n}\|C_\phi^{i+j}f\|^2\lambda_i\bar{\lambda}_j=\left(\sum_{i,j}^{n}\int_X |f|^2\,\phi^{i+j}(x)d\mu\right)\lambda_i\bar{\lambda}_j\\ &= \left(\sum_{i,j}^{n}\int_X |f|^2\,(x)d\mu\circ\phi^{-(i+j)}\right)\lambda_i\bar{\lambda}_j=\left(\sum_{i,j=0}^{n}\int_X |f|^2\,h_{i+j}(x)d\mu(x)\right)\lambda_i\bar{\lambda}_j.\end{aligned}$$

For $\sigma\in\mathcal{A}$ with $\mu(\sigma)<\infty$, if we consider $f=\chi_\sigma\in L^2$, then

$$\int_\sigma\sum_{i,j=0}^{n}h_{i+j}(x)\lambda_i\bar{\lambda}_j d\mu(x)\geq 0 \tag{3.1}$$

for all $\boldsymbol{\lambda}=(\lambda_0,\lambda_1,\cdots,\lambda_n)\in\mathbb{C}^{n+1}$. Since X is σ-finite, we may write $X=\cup_{n\in\mathbb{N}}X_n$ with $\mu(X_n)<\infty$. For brevity, we let

$$H_\lambda(x)=\sum_{i,j=0}^{n}h_{i+j}(x)\lambda_i\bar{\lambda}_j,\qquad \boldsymbol{\lambda}=(\lambda_0,\lambda_1,\cdots,\lambda_n)\in\mathbb{C}^{n+1}$$

and

$$\Omega_\lambda:=\{x\in X: H_\lambda(x)\geq 0\}.$$

Since (3.1) for all $\sigma\in\mathcal{A}$ with $\sigma\subset X_n$ was arbitrary, we have that $H_\lambda(x)\geq 0$, a.e. $[\mu]$ on X_n, and also $\Omega_\lambda\in\mathcal{A}$ such that $\mu(X\backslash\Omega_\lambda)=0$ for all $\lambda\in\mathbb{C}^{n+1}$. Consider a countable dense subset $\mathcal{Z}$ of $\mathbb{C}^{n+1}$. And set $\Omega:=\cap_{\lambda\in\mathcal{Z}}\Omega_\lambda$. Then

$$\mu(X\backslash\Omega)=\mu(\cup_{\lambda\in\mathcal{Z}}(X\backslash\Omega_\lambda))\leq\sum_{\lambda\in\mathcal{Z}}\mu(X\backslash\Omega_\lambda)=0.$$

For $\lambda\in\mathbb{C}^{n+1}$, there exists a sequence $\{\lambda^{(k)}\}_{k=1}^{\infty}$ in $\mathbb{C}^{n+1}$ with $\lambda^{(k)}\in\mathcal{Z}$, say $\lambda^{(k)}=(\lambda_0^{(k)},\cdots,\lambda_n^{(k)})\in\mathbb{C}^{n+1}$. Since $H_{\lambda^{(k)}}(x)\geq 0$ for all $x\in\Omega$ and $k\in\mathbb{N}$, obviously $H_\lambda(x)\geq 0$ for all $x\in\Omega$ and $\lambda\in\mathbb{C}^{n+1}$. Hence $H_\lambda(x)\geq 0$ for all $\lambda\in\mathbb{C}^{n+1}$ and x a.e. in X, which implies that $[h_{i+j}(x)]_{i,j=0}^{n}\geq 0$ for μ a.e. $x\in X$.

(iii) $\Rightarrow$ (i) See [JLP, Th. 2.3]. ■

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